Weak Solution of the Euler-Poisson-Darboux Equation for n=4.

Article *in* Applied Mathematical Sciences · January 2013

DOI: 10.12988/ams.2013.13028

Alfred Wanyama Manyonge
Maseno University

Kweyu David
Kenyatta University

Haron Nyambane
Independent Researcher

Applied Mathematical Sciences, Vol. 7, 2013, no. 7, 315 - 325

Weak Solution of the Singular Cauchy Problem of Euler-Poisson-Darboux Equation for $n = 4$

A. Manyonge[1], D. Kweyu[2], J. Bitok[3], H. Nyambane[3] and J. Maremwa[3]

[1]Department of Pure and Applied Mathematics
Maseno University, Maseno, Kenya
wmanyonge@gmail.com

[2]Department of Mathematics
Kenyatta University, Nairobi, Kenya
kweyudavid@yahoo.com

[3]Department of Mathematics and Computer Science
Chepkoilel University College, Eldoret, Kenya
nyamabane.haron@yahoo.com, jmaremwa@gmail.com

Abstract

We consider the singular Cauchy problem of Euler-Poisson-Darboux equation(EPD) of the form

$$u_{tt} + \frac{k}{t}u_t = \nabla^2 u$$

$$u(x, 0) = f(x), u_t(x, 0) = 0$$

where ∇^2 is the Laplacian operator in $\mathbb{R}^n$, n will refer to dimension and k a real parameter. The EPD equation finds applications in geometry, applied mathematics, physics etc. Various authors have solved this problem for different values of n and k using various techniques since the time of Euler[1]. In this paper, we shall take the Fourier transform of the EPD equation with respect to the space coordinate. The equation so obtained is transformed into a Bessel differential equation. We solve this equation and obtain the inverse Fourier transform of its solution. Finally on using the convolution theorem, we obtain the weak solution of the EPD equation for $n = 4$. The case for $n = 1$ is an interesting one for this problem as the solution will be that of a $1 - D$ wave equation. We therefore deduce the weak solution for the $1 - D$ wave equation as well.

Mathematics Subject Classification: 35Q05

Keywords: Singular Cauchy problem, EPD, Bessel function, weak solution

1 Introduction

The study of singular Cauchy problems of the Euler-Poisson-Darboux (EPD) equation is concerned with the equation

$$u_{tt} + \frac{k}{t}u_t = \nabla^2 u \tag{1}$$

$$u(x,0) = f(x), u_t(x,0) = 0 \tag{2}$$

where ∇^2 is the Laplacian operator in $\mathbb{R}^n$, n will refer to dimension, $x = x_1, x_2, \ldots, x_n$ is a point in $\mathbb{R}^n$, k is a real parameter and t is the time variable. Problems of type (1) will be called singular if the coefficient $\frac{k}{t} \to \infty$ as $t \to 0$ and degenerate if $\frac{k}{t} \to 0$ as $t \to 0$ in such a way as not to change the type of problem. In this paper, we shall study singular Cauchy problem of the EPD type. There has been several approaches by different people to the question of obtaining the solution of the singular Cauchy problems for the EPD equation. Our main objective is to obtain a weak solution of the EPD equation for a general n and deduce solution for $n = 4$. The case for $n = 1$ is an interesting one for this problem as the solution will be that of a $1 - D$ wave equation. We therefore further deduce the weak solution of the $1 - D$ wave equation. The EPD equation for special values of k and n has occurred in many classical problems in geometry, applied mathematics and physics for over two centuries. Euler[1] first considered the EPD equation for $n = 1$. Using the Riemann method, Martin[2] gave the solution of (1) for $n = 1$ and $k = -1, -2, -3 \ldots$. Diaz and Weinberger[3] obtained solutions of $((1) - (2))$ for $k = n, n+1, n+2 \ldots$ from the known solution for $k = n-1$. They directly verified that the resulting formula gives a solution of the problem for any k with $Re(k) > n - 1$. Blum[4] obtained solution of the EPD for exceptional cases $k = 1, -3, -5 \ldots$. Weinstein[5] gave a complete solution of singular Cauchy problem covering all values of k and n. Blum's solution differes from the solution of Weinstein in that the function f is required to have fewer continuous derivatives i.e. it is sufficient for f to have derivatives of order of at least $\frac{(n-k+3)}{2}$. Fusaro[6] solved the boundary value problem of the singular Cauchy problem of the EPD by separation of variables. Dernek[7], by using a series, obtained the solution of the EPD of the form

$$u_{tt} + (at + \frac{b}{t})u_t = \nabla^2 u, \quad u(x,0) = f(x), \quad u_t(x,0) = 0$$

where a and b are real constants and $f(x)$ is an initial function.Uchikoshi[8] studied local Cauchy problems in a complex domain for the EPD of an incompressible fluid. In this paper, our work considers the weak solution of the singular Cauchy problem of the Euler-poisson-Darboux equation for $n-4$ and deduce a weak solution of the $1-D$ wave equation.

2 Mathematical Methodology

2.1 Weak solution for n=4

In$(1)-(2)$ let $k = n-1$. This choice of k ensures that we are dealing only with singular Cauchy problems. For this, the EPD assumes the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{n-1}{t}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Definition 1. The Fourier transform of a function $f(x)$ on $\mathbb{R}^n$ denoted $\mathfrak{F}$ is defined by

$$F(\xi) = \mathfrak{F}[f(x)] = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

Taking the Fourier transform with respect to space coordinate x for $x \in \mathbb{R}$ of both sides of (3) and assuming that $u(x,t)$ and $\frac{\partial u}{\partial t}$ both $\to 0$ as $x \to -\infty$ we find

$$\frac{d^2\mathfrak{F}[u]}{dt^2} + \frac{n-1}{t}\frac{d\mathfrak{F}[u]}{dt} + |\xi|^2 \mathfrak{F}[u] = 0 \tag{4}$$

Let

$$\mathfrak{F}[u] = \mathfrak{F}[u](\xi, t) = t^{-(\frac{n-2}{n})} V(\theta) \tag{5}$$

where $\theta = |\xi| t$. Now

$$\frac{d\mathfrak{F}[u]}{dt} = t^{-(\frac{n-2}{n})} \left\{ |\xi| \frac{dV}{d\theta} - (\frac{n-2}{2t})V \right\}$$

and

$$\frac{d^2\mathfrak{F}[u]}{dt^2} = t^{-(\frac{n-2}{n})} \left\{ |\xi|^2 \frac{d^2 V}{d\theta^2} - (\frac{n-2}{t})|\xi| \frac{dV}{d\theta} + (\frac{n-2}{2})(\frac{n}{2})\frac{V}{t^2} \right\}$$

On substitution of the above expressions into (4) we find

$$|\xi|^2\frac{d^2V}{d\theta^2}+\frac{|\xi|}{t}\frac{dV}{d\theta}+\left(|\xi|^2-\frac{(n-2)^2}{4t^2}\right)V=0$$

or

$$\frac{d^2V}{d\theta^2}+\frac{1}{|\xi|t}\frac{dV}{d\theta}+\left(1-\left(\frac{n-2}{2|\xi|t}\right)^2\right)V=0$$

or

$$\frac{d^2V}{d\theta^2}+\frac{1}{\theta}\frac{dV}{d\theta}+\left(1-\left(\frac{n-2}{2\theta}\right)^2\right)V=0$$

or

$$\frac{d^2V}{d\theta^2}+\frac{1}{\theta}\frac{dV}{d\theta}+\left(1-\frac{(\frac{n-2}{2})^2}{\theta^2}\right)V=0 \tag{6}$$

Equation (6) amounts to a weak formulation of the original EPD, its solution will therefore be a weak solution. we notice that (6) is Bessel differential equation of order $\frac{n-2}{2}$ whose solution is given by

$$V(\theta)=AJ_{\frac{n-2}{2}}(\theta)+BY_{\frac{n-2}{2}}(\theta) \tag{7}$$

where A and B are constants and $J_{\frac{n-2}{2}}(\theta)$ and $Y_{\frac{n-2}{2}}(\theta)$ are Bessel functions of order $\frac{n-2}{2}$ of first and second kind respectively. Now function $Y_{\frac{n-2}{2}}(\theta)$ is singular at the origin i.e. $Y_{\frac{n-2}{2}}(\theta)\to\infty$ as $\theta\to 0$. Hence we choose $B=0$ so that

$$V(|\xi|t)=AJ_{\frac{n-2}{2}}(|\xi|t) \tag{8}$$

Now in general, $J_n(\xi t)$ is given by

$$J_n(\xi t)=\sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)}\left(\frac{\xi t}{2}\right)^{2r+n} \tag{9}$$

from which we find the series expansion of $J_n(|\xi|t)$ as

$$J_n(|\xi|t)=\frac{1}{\Gamma(n+1)}\left(\frac{|\xi|t}{2}\right)^n-\frac{1}{\Gamma(n+2)}\left(\frac{|\xi|t}{2}\right)^{n+2}+\frac{1}{2!}\frac{1}{\Gamma(n+3)}\left(\frac{|\xi|t}{2}\right)^{n+4}-\ldots$$

or

$$J_{\frac{n-2}{2}}(|\xi|t)=\frac{1}{\Gamma(\frac{n}{2})}\left(\frac{|\xi|t}{2}\right)^{\frac{n-2}{2}}-\frac{1}{\Gamma(\frac{n+2}{2})}\left(\frac{|\xi|t}{2}\right)^{\frac{n+2}{2}}+\frac{1}{2!}\frac{1}{\Gamma(\frac{n+4}{2})}\left(\frac{|\xi|t}{2}\right)^{\frac{n+6}{2}}-\ldots$$

Thus

$$\mathfrak{F}[u] = t^{-(\frac{n-2}{2})}V(\theta) = At^{-(\frac{n-2}{2})}\left\{\frac{1}{\Gamma(\frac{n}{2})}\left(\frac{|\xi|t}{2}\right)^{\frac{n-2}{2}} - \frac{1}{\Gamma(\frac{n+2}{2})}\left(\frac{|\xi|t}{2}\right)^{\frac{n+2}{2}} + \dots\right\}$$

or

$$\mathfrak{F}[u](\xi,t) = A\left\{\frac{1}{\Gamma(\frac{n}{2})}\left(\frac{|\xi|}{2}\right)^{\frac{n-2}{2}} - \frac{1}{\Gamma(\frac{n+2}{2})}\left(\frac{|\xi|}{2}\right)^{\frac{n+2}{2}} t^2 + \dots\right\}$$

Then as $t \to 0$ we have that

$$\mathfrak{F}[u](\xi,0) = A\left\{\frac{1}{\Gamma(\frac{n}{2})}\left(\frac{|\xi|}{2}\right)^{\frac{n-2}{2}}\right\} = F(|\xi|) \quad \text{by} \quad (2)$$

Thus

$$A = \frac{F(|\xi|)\Gamma(\frac{n}{2})}{\left(\frac{|\xi|}{2}\right)^{\frac{n-2}{2}}}$$

Hence from (5),

$$\mathfrak{F}[u](\xi,t) = t^{-(\frac{n-2}{2})}\frac{F(|\xi|)\Gamma(\frac{n}{2})}{\left(\frac{|\xi|}{2}\right)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(|\xi|t)$$

or

$$\mathfrak{F}[u](\xi,t) = 2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})F(|\xi|)\,(|\xi|t)^{-(\frac{n-2}{2})}\,J_{\frac{n-2}{2}}(|\xi|t) \tag{10}$$

Definition 2. The inverse Fourier transform of $F(\xi)$ in $\mathbb{R}^n$ denoted $\mathfrak{F}^{-1}$ is defined by

$$f(x) = \mathfrak{F}^{-1}[F(\xi)] = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} F(\xi)e^{i\xi x}d\xi$$

The solution of (10) will involve taking the inverse Fourier transform of both sides and in particular the inverse Fourier transform of the quantity $((|\xi|t)^{-(\frac{n-2}{2})}\,J_{\frac{n-2}{2}}(|\xi|t))$ since for $\mathfrak{F}[f(x)] = F(|\xi|)$. We state the following theorem without proof, for details see [9]

Theorem 3. *Bessel function of order n of first kind may be written in trigonometric form as*

$$(\xi t)^{-n}\,J_n(\xi t) = \frac{2^{1-n}}{\Gamma(n+\frac{1}{2})\sqrt{\pi}}\int_0^{\frac{\pi}{2}} \cos(\xi t\cos\theta)\sin^{2n}\theta d\theta$$

Hence from the theorem we have that

$$(|\xi|t)^{-(\frac{n-2}{2})} J_{\frac{n-2}{2}}(|\xi|t) = \frac{2^{\frac{4-n}{2}}}{\Gamma(\frac{n-1}{2})\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos(|\xi|t\cos\theta)\sin^{(n-2)}\theta d\theta \tag{11}$$

In this equation, let $z = |\xi|t$ and

$$\mathfrak{F}[Q] = (z)^{-(\frac{n-2}{2})} J_{\frac{n-2}{2}}(z) = \frac{2^{\frac{4-n}{2}}}{\Gamma(\frac{n-1}{2})\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos(z\cos\theta)\sin^{(n-2)}\theta d\theta \tag{12}$$

Take the inverse Fourier transform of (12) for $x \in \mathbb{R}$ on both sides to find

$$Q(x,t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \mathfrak{F}[Q]e^{i\xi x}d\xi$$

or

$$Q(x,t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{2^{\frac{4-n}{2}}}{\Gamma(\frac{n-1}{2})\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos(z\cos\theta)\sin^{(n-2)}\theta e^{i\xi x}d\theta d\xi$$

or

$$Q(x,t) = \frac{2^{-(\frac{n-3}{2})}}{\Gamma(\frac{n-1}{2})\pi} \int_{-\infty}^{\infty} \int_0^{\frac{\pi}{2}} \cos(z\cos\theta)\sin^{(n-2)}\theta e^{i\xi x}d\theta d\xi$$

or

$$Q(x,t) = \frac{2^{-(\frac{n-3}{2})}}{\Gamma(\frac{n-1}{2})\pi} \int_0^{\frac{\pi}{2}} \int_{-\infty}^{\infty} \left(\frac{e^{i(z\cos\theta+\xi x)} + e^{-i(z\cos\theta-\xi x)}}{2}\right)\sin^{(n-2)}\theta d\theta d\xi$$

or

$$Q(x,t) = \frac{2^{-(\frac{n-3}{2})}}{\Gamma(\frac{n-1}{2})\pi} \int_0^{\frac{\pi}{2}} \int_{-\infty}^{\infty} \left(\frac{e^{i(|\xi|t\cos\theta+x\xi)} + e^{-i(|\xi|t\cos\theta-x\xi)}}{2}\right)\sin^{(n-2)}\theta d\theta d\xi$$

Let

$$E = \frac{e^{i(|\xi|t\cos\theta+x\xi)} + e^{-i(|\xi|t\cos\theta-x\xi)}}{2} \tag{13}$$

so that

$$Q(x,t) = \frac{2^{-(\frac{n-3}{2})}}{\Gamma(\frac{n-1}{2})\pi} \int_0^{\frac{\pi}{2}} \left(\int_{-\infty}^{0} E d\xi + \int_0^{\infty} E d\xi\right)\sin^{(n-2)}\theta d\theta \tag{14}$$

or

$$Q(x,t) = \frac{2^{-(\frac{n-1}{2})}}{\Gamma(\frac{n-1}{2})\pi} \int_0^{\frac{\pi}{2}} \left(\int_{-\infty}^{0} e^{i(t\cos\theta+x)\xi}d\xi + \int_0^{\infty} e^{-i(t\cos\theta-x)\xi}d\xi\right)\sin^{(n-2)}\theta d\theta$$

or

$$Q(x,t) = \frac{2^{-(\frac{n-1}{2})}}{\Gamma(\frac{n-1}{2})\pi i} \int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta}{t^2\cos^2\theta - x^2} \right) \sin^{(n-2)}\theta d\theta \tag{15}$$

We now construct the weak solution of the EPD for $n = 4$. (15) now becomes

$$Q(x,t) = \frac{2^{-(\frac{3}{2})}}{\Gamma(\frac{3}{2})\pi i} \int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta\sin^2\theta}{t^2\cos^2\theta - x^2} \right) d\theta$$

or

$$Q(x,t) = \frac{1}{2^{\frac{3}{2}}\sqrt{\pi^3} i} \int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta\sin^2\theta}{t^2\cos^2\theta - x^2} \right) d\theta \tag{16}$$

To evaluate the integral in (16) we transform as follows: Let $z = e^i\theta$ where $0 \leq \theta \leq 2\pi$. This means that

$$d\theta = \frac{dz}{iz}, \quad \sin^\theta = \frac{(z\sin^2 - 1)^2}{(2iz)^2}, \quad \cos\theta = \frac{z\theta^2 + 1}{2z}, \quad \cos^\theta = \frac{(z^2+1)^2}{(2z)^2}$$

from which we find

$$t\cos\theta\sin^2\theta = \frac{t(z^2+1)(z^2-1)^2}{-8z^3} \text{and} \quad t^2\cos^2\theta - x^2 = \frac{t^2(z^2+1)^2 - 4x^2z^2}{4z^2}$$

inserting these quantities in the integral we find

$$\int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta}{t^2\cos^2\theta - x^2} \right) \sin^2\theta d\theta = -\frac{1}{2it} \int_c \frac{(z^2+1)(z^2-1)^2 dz}{z^2(z^4 + (2 - \frac{4x^2}{t^2})z^2 + 1)}$$

where c is a unit circle with centre at the origin. The denominator of the right hand side may be written as

$$\int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta}{t^2\cos^2\theta - x^2} \right) \sin^2\theta d\theta = -\frac{1}{2it} \int_c \frac{(z^2+1)(z^2-1)^2 dz}{z^2(z-\sqrt{\alpha})(z+\sqrt{\alpha})(z-\sqrt{\beta})(z+\sqrt{\beta})}$$

where

$$\sqrt{\alpha} = \sqrt{\left\{ \frac{(2x^2 - t^2) + 2x\sqrt{x^2 - t^2}}{t^2} \right\}}, \quad -\sqrt{\alpha} = -\sqrt{\left\{ \frac{(2x^2 - t^2) + 2x\sqrt{x^2 - t^2}}{t^2} \right\}}$$

$$\sqrt{\beta} = \sqrt{\left\{ \frac{(2x^2 - t^2) - 2x\sqrt{x^2 - t^2}}{t^2} \right\}}, \quad -\sqrt{\beta} = -\sqrt{\left\{ \frac{(2x^2 - t^2) - 2x\sqrt{x^2 - t^2}}{t^2} \right\}}$$

are the roots of the equation

$$z^4 + (2 - \frac{4x^2}{t^2})z^2 + 1$$

$z = 0$ obviously lies within c, for the roots $\sqrt{\beta}$ and $-\sqrt{\beta}$ for $x > t$, it can be shown that they lie in c. Let

$$f(z) = \frac{(z^2+1)(z^2-1)^2 dz}{z^2(z^4 + (2 - \frac{4x^2}{t^2})z^2 + 1)}$$

We examine the residues of $f(z)$ within c. For residue of $f(z)$ at $z = 0$, this is a pole of order 2 given by

$$\lim_{z \to 0} \frac{1}{1!} \frac{d}{dz} \left\{ \frac{(z-0)^2(z^2+1)(z^2-1)^2}{z^2(z-\sqrt{\alpha})(z+\sqrt{\alpha})(z-\sqrt{\beta})(z+\sqrt{\beta})} \right\}$$

With a little simplification, we find

$$\text{Residue of f(z) at z=0} = -\frac{2\sqrt{\alpha}}{\alpha^2\beta} \tag{17}$$

For residue of $f(z)$ at $z = \sqrt{\beta}$ we have

$$\lim_{z \to \sqrt{\beta}} \frac{(z^2+1)(z^2-1)^2}{z^2(z-\sqrt{\alpha})(z+\sqrt{\alpha})(z+\sqrt{\beta})} = \frac{(\beta+1)(\beta-1)^2}{2\beta\sqrt{\beta}(\beta-\alpha)} \tag{18}$$

and for residue of $f(z)$ at $z = -\sqrt{\beta}$ we have

$$\lim_{z \to -\sqrt{\beta}} \frac{(z^2+1)(z^2-1)^2}{z^2(z-\sqrt{\alpha})(z+\sqrt{\alpha})(z-\sqrt{\beta})} = -\frac{(\beta+1)(\beta-1)^2}{2\beta\sqrt{\beta}(\beta-\alpha)} \tag{19}$$

If ΣR denotes sum of residues of $f(z)$ in c, then from (17), (18) and (19),

$$\Sigma R = -\frac{2\sqrt{\alpha}}{\alpha^2\beta}$$

and from Cauchy integral theorem

$$\int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta}{t^2\cos^2\theta - x^2} \right) \sin^2\theta d\theta = -\frac{1}{2it} 2\pi i (-\frac{2\sqrt{\alpha}}{\alpha^2\beta})$$

or

$$\int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta}{t^2\cos^2\theta - x^2} \right) \sin^2\theta d\theta = \frac{2\pi\sqrt{\alpha}}{\alpha^2\beta t}$$

Now our original integral occupied only $\frac{1}{4}$ of c, hence we have

$$\int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta}{t^2\cos^2\theta - x^2}\right)\sin^2\theta d\theta = \frac{2\pi\sqrt{\alpha}}{\alpha^2\beta t}.\frac{1}{4}$$

or

$$\int_0^{\frac{\pi}{2}} \left(\frac{t\cos\theta}{t^2\cos^2\theta - x^2}\right)\sin^2\theta d\theta = \frac{\pi}{2\beta t\alpha\sqrt{\alpha}} \tag{20}$$

Using (20) in (16) we find

$$Q(x,t) = \frac{1}{2^{\frac{5}{2}} i\alpha\beta t\sqrt{\alpha}\sqrt{\pi}} \tag{21}$$

or

$$Q(x,t) = \frac{1}{2^{\frac{5}{2}} i\sqrt{\pi}\sqrt{\{(2x^2 - t^2) + 2x\sqrt{x^2 - t^2}\}}} \tag{22}$$

Definition 4. For $x \in \mathbb{R}$, let $f(x)$ and $g(x)$ be two given functions. The convolution of $f(x)$ and $g(x)$ denoted $f(x) * g(x)$ is defined to be the function

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

Theorem 5 (Convolution theorem). *Let* $\mathfrak{F}^{-1}[F_1(\xi)] = f(x)$ *and* $\mathfrak{F}^{-1}[F_2(\xi)] = g(x)$, *then*

$$\mathfrak{F}^{-1}[F_1(\xi) * F_2(\xi)] = f(x) * g(x)$$

In (10), set $n = 4$. Taking the inverse Fourier transform of this equation and using (22), (10) becomes

$$u(x,t) = 4\pi f(x) * Q(x,t) \tag{23}$$

Using the convolution theorem in (23), we find

$$u(x,t) = \frac{1}{i}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}\left(\frac{f(y)dy}{\sqrt{\{(2(x-y)^2 - t^2) + 2(x-y)\sqrt{(x-y)^2 - t^2}\}}}\right) \tag{24}$$

2.2 Weak solution for n=1, the case of 1-D wave equation

Consider (10) when $n = 1$, i.e.

$$\mathfrak{F}[u](x,t) = \sqrt{\frac{\pi}{2}} F(|\xi|)\,(|\xi|t)^{\frac{1}{2}} J_{-\frac{1}{2}}(|\xi|t) \tag{25}$$

and (11) becomes

$$(|\xi|t)^{\frac{1}{2}} J_{-\frac{1}{2}}(|\xi|t) = \sqrt{\frac{2}{\pi}} \cos(|\xi|t) \tag{26}$$

according to [9]. Using this in (25) we find

$$\mathfrak{F}[u](x,t) = F(|\xi|) \cos(|\xi|t)$$

or

$$u(x,t) = \frac{1}{i2\pi} f(x) * \left(\frac{t}{t^2 - x^2} \right) \tag{27}$$

on use of convolution theorem. Hence

$$u(x,t) = \frac{t}{i2\pi} \int_{-\infty}^{\infty} \frac{f(y)dy}{[t^2 - (x-y)^2]} \tag{28}$$

being the weak solution of the $1 - D$ wave equation.

3 Conclusion

Equations (24) and (28) obtained represent the weak solutions of the EPD for $n = 4$ and $n = 1$ the $1 - D$ wave equations respectively.

References

[1] Euler, L.: *Foundations of Integral Calculus*, vol. 3, Academia Scientiarum Imperialis Petropolitanae, 1770.

[2] Martin, M.(1951): 'Riemann's method and the Cauchy Problem'*Bull. Amer. Math. Soc.*, vol. 57, pp238-249.

[3] Diaz, J. and Weinberger, H.(1953): 'A Solution of the singular INtial Value Problem for the EPD Equation'*Proc. Amer. Math. Soc.*, vol. 4, pp703-718.

[4] Blum, E.(1954): 'The Euler-poisson-darboux equation in exceptional Cases' *Pro. Amer. Math. Soc.*, vol. 5, pp511-520.

[5] Weinstein, A.(1954): 'On the Wave Equation and the Euler-Poisson-Darboux Equation' *Pro. Fifth Symp. App. Math; Amer. Math. Soc.*, pp137-147.

[6] Fusaro, B.(1966): 'A Solution of a singular Mixed Problem for the Equation of Euler-Poisson-Darboux' *Amer. Math. Monthly*, vol. 8, pp197-219.

[7] Dernek, N.(2000): 'On the Singular Cauchy Problem of Euler-Poisson-Darboux Equation' *Istanbul Univ. Fen. Fak. Mat. Der.*,vol. 59, pp17-27.

[8] Uchikoshi, K.(2007): 'Singular Cauchy Problems for Perfect incompressible Fluids' *Jour. Math. Univ. Tokyo.*, vol. 14 pp157-176.

[9] Carroll, R. W. and Showalter, R.: *Singular and Degenerate Cauchy Problems*, vol. 1, New York, Academic Press, 1976.

Received: September, 2012