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Research Article

Weighted Composition Groups on the Little Bloch Space

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We determine both the semigroup and spectral properties of a group of weighted composition operators on the little Bloch space. It turns out that these are strongly continuous groups of invertible isometries on the Bloch space. We then obtain the norm and spectra of the infinitesimal generator as well as the resulting resolvents which are given as integral operators. As a consequence, we complete the analysis of the adjoint composition group on the predual of the nonreflexive Bergman space and a group of isometries associated with a specific automorphism of the upper half-plane.

Dedicated to Prof. Len Miller (PhD advisor to J. O. Bonyo) and Prof. Vivien Miller of Mississippi State University on their retirement

1. Introduction

The (open) unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ is defined as $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$, while the upper half-plane of $\mathbb{C}$, denoted by $\mathbb{U}$, is given by $\mathbb{U} = \{\omega \in \mathbb{C}: \mathfrak{I}(\omega) > 0\}$, where $\mathfrak{I}(\omega)$ stands for the imaginary part of ω. The Cayley transform $\psi(z) := i(1+z)/(1-z)$ maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ with inverse $\psi^{-1}(\omega) = (\omega - i)/(\omega + i)$. For every $\alpha > -1$, we define a positive Borel measure dm_α on $\mathbb{D}$ by $dm_\alpha(z) = (1-|z|^2)^\alpha dA(z)$, where dA denotes the area measure on $\mathbb{D}$.

For an open subset Ω of $\mathbb{C}$, let $\mathscr{H}(\Omega)$ denote the Fréchet space of analytic functions $f: \Omega \longrightarrow \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω. Let $\mathrm{Aut}(\Omega) \subset \mathscr{H}(\Omega)$ denote the group of biholomorphic maps $f: \Omega \longrightarrow \Omega$. For $1 \le p < \infty$, $\alpha > -1$, the weighted Bergman spaces of the unit disc $\mathbb{D}$, $L^p_a(\mathbb{D}, m_\alpha)$, are defined by

$$L^p_a(\mathbb{D}, m_\alpha) := \left\{ f \in \mathscr{H}(\mathbb{D}): \|f\|_{L^p_a(\mathbb{D},m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p dm_\alpha(z) \right)^{1/p} < \infty \right\}. \tag{1}$$

Clearly, $L^p_a(\mathbb{D}, m_\alpha) = L^p(\mathbb{D}, m_\alpha) \cap \mathscr{H}(\mathbb{D})$, where $L^p(\mathbb{D}, m_\alpha)$ denotes the classical Lebesgue spaces. For every $f \in L^p_a(\mathbb{D}, m_\alpha)$, the growth condition is given by

$$|f(z)| \le \frac{K\|f\|}{\left(1-|z|^2\right)^\gamma}, \tag{2}$$

where K is a constant and $\gamma = (\alpha+2)/p$, see, for example, [1], Theorem 4.14.

The Bloch space of the unit disc, denoted by $\mathscr{B}_\infty(\mathbb{D})$, is defined as the space of analytic functions $f \in \mathscr{H}(\mathbb{D})$ such that the seminorm

$$\|f\|_{\mathscr{B}_{\infty,1}(\mathbb{D})} := \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)\left|f'(z)\right| < \infty. \tag{3}$$

Following [1, 2], $\mathscr{B}_\infty(\mathbb{D})$ is a Banach space with respect to the norm $\|f\|_{\mathscr{B}_\infty(\mathbb{D})} := |f(0)| + \|f\|_{\mathscr{B}_{\infty,1}(\mathbb{D})}$. On the contrary, the little Bloch space of the disc, denoted by $\mathscr{B}_{\infty,0}(\mathbb{D})$, is defined to be the closed subspace of $\mathscr{B}_\infty(\mathbb{D})$ such that

$$\mathscr{B}_{\infty,0}(\mathbb{D}) := \mathrm{cl}_{\mathscr{B}_\infty} \mathbb{C}[z], \tag{4}$$

where $\mathrm{cl}_{\mathscr{B}_\infty}\mathbb{C}[z]$ denotes $\mathscr{B}_\infty$ closure of the set of analytic polynomials in z. Equivalently,

$$\mathscr{B}_{\infty,0}(\mathbb{D}) := \left\{ f \in \mathscr{H}(\mathbb{D}): \lim_{|z|\longrightarrow 1^-, z\in\mathbb{D}} \left(1-|z|^2\right)\left|f'(z)\right| = 0 \right\}, \tag{5}$$

and possesses the same norm as $\mathscr{B}_\infty(\mathbb{D})$. Since $\mathscr{B}_{\infty,0}(\mathbb{D})$ is a closed subspace of the Banach space $\mathscr{B}_\infty(\mathbb{D})$, it follows that

$\mathscr{B}_{\infty,0}(\mathbb{D})$ is a Banach space as well with respect to the norm $\|\cdot\|_{\mathscr{B}_\infty(\mathbb{D})}$. Note that every $f \in \mathscr{B}_\infty(\mathbb{D})$ (or $f \in \mathscr{B}_{\infty,0}(\mathbb{D})$) satisfies the growth condition:

$$|f(z)| \le \left(1 + \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)\right)\|f\|_{\mathscr{B}_\infty(\mathbb{D})}. \tag{6}$$

See, for instance, [3] for details. Let $1 < p < \infty$ and q be conjugate to p in the sense that $(1/p) + (1/q) = 1$. If $(L_a^p(\mathbb{D}, m_\alpha))^*$ is the dual space of $L_a^p(\mathbb{D}, m_\alpha)$, then

$$\left(L_a^p(\mathbb{D}, m_\alpha)\right)^* \approx L_a^q(\mathbb{D}, m_\alpha), \quad \alpha > -1, \tag{7}$$

under the integral pairing

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\mathrm{d}m_\alpha, \quad \left(f \in L_a^p(\mathbb{D}, m_\alpha),\ g \in L_a^q(\mathbb{D}, m_\alpha)\right). \tag{8}$$

It is well known that for $1 < p < \infty$, $L_a^p(\mathbb{D}, m_\alpha)$ is reflexive. The case $p = 1$ is the nonreflexive case and the duality relations have been determined as follows:

$$\begin{aligned}\left(L_a^1(\mathbb{D}, m_\alpha)\right)^* &\approx \mathscr{B}_\infty(\mathbb{D}),\\ \left(\mathscr{B}_{\infty,0}(\mathbb{D})\right)^* &\approx L_a^1(\mathbb{D}, m_\alpha),\end{aligned} \tag{9}$$

under the duality pairings given by, respectively:

$$\begin{aligned}\langle f, g\rangle &= \int_{\mathbb{D}} f(z)\overline{g(z)}\mathrm{d}m_\alpha, \quad \left(f \in L_a^1(\mathbb{D}, m_\alpha), g \in \mathscr{B}_\infty(\mathbb{D})\right),\\ \langle f, g\rangle &= \int_{\mathbb{D}} f(z)\overline{g(z)}\mathrm{d}m_\alpha, \quad \left(f \in \mathscr{B}_{\infty,0}(\mathbb{D}), g \in L_a^1(\mathbb{D}, m_\alpha)\right).\end{aligned} \tag{10}$$

In other words, the dual and predual spaces of the nonreflexive Bergman space $L_a^1(\mathbb{D}, m_\alpha)$ are the Bloch and little Bloch spaces, respectively. For a comprehensive account of the theory of Bloch and Bergman spaces, we refer to [1, 2, 4–6].

In [7], all the self analytic maps $(\varphi_t)_{t\ge 0} \subseteq \mathrm{Aut}(\mathbb{U})$ of the upper half-plane $\mathbb{U}$ were identified and classified according to the location of their fixed points into three distinct classes, namely, scaling, translation, and rotation groups. For each self-analytic map φ_t, we define a corresponding group of weighted composition operator on $\mathscr{H}(\mathbb{U})$ by

$$S_{\varphi_t} f(z) = \left(\varphi_t'(z)\right)^\gamma f(\varphi_t(z)), \tag{11}$$

for some appropriate weight γ.

It is noted in [7] Section 5 that for the rotation group, we consider the corresponding group of weighted composition operators defined on the analytic spaces of the disc $\mathscr{H}(\mathbb{D})$ given by

$$T_t f(z) = e^{ict} f\left(e^{ikt} z\right), \quad \text{with } c, k \in \mathbb{R},\ k \ne 0. \tag{12}$$

The study of composition operators on spaces of analytic functions still remains an active area of research. For Bloch spaces, most studies have only focussed on the boundedness and compactness of these operators. See, for instance, [3, 8–11]. In [7, 12], both the semigroup and spectral properties of the group $(T_t)_{t\in\mathbb{R}}$ were studied in detail on the Hardy and Bergman spaces. The aim of this paper is to extend the analysis of the group $(T_t)_{t\in\mathbb{R}}$ from the Hardy and Bergman spaces to the setting of the little Bloch space. Specifically, we apply the theory of semigroups as well as spectral theory of linear operators on Banach spaces to study the properties of the group of weighted composition operators given by equation (12) on the little Bloch space of the disk. As a consequence, we shall complete the analysis of the adjoint group on the dual of the nonreflexive Bergman space $L_a^1(\mathbb{D}, m_\alpha)$. The analysis of the adjoint group on the reflexive Bergman space, that is, $L_a^p(\mathbb{D}, m_\alpha)$ for $1 < p < \infty$, was considered exhaustively in [12]. We shall also consider a specific automorphism of $\mathbb{U}$ and carry out an analysis of the corresponding composition operator.

If X is an arbitrary Banach space, let $\mathscr{L}(X)$ denote the algebra of bounded linear operators on X. For a linear operator T with domain $\mathscr{D}(T) \subset X$, denote the spectrum and point spectrum of T by $\sigma(T)$ and $\sigma_p(T)$, respectively. The resolvent set of T is $\rho(T) = \mathbb{C}\backslash\sigma(T)$, while $r(T)$ denotes its spectral radius. For a good account of the theory of spectra, see [13–15]. If X and Y are arbitrary Banach spaces and $U \in \mathscr{L}(X, Y)$ is an invertible operator, then clearly $(A_t)_{t\in\mathbb{R}} \subset \mathscr{L}(X)$ is a strongly continuous group if and only if $B_t := UA_tU^{-1}$, $t \in \mathbb{R}$, is a strongly continuous group in $\mathscr{L}(Y)$. In this case, if $(A_t)_{t\in\mathbb{R}}$ has generator Γ, then $(B_t)_{t\in\mathbb{R}}$ has generator $\Delta = U\Gamma U^{-1}$ with domain $\mathscr{D}(\Delta) = U\mathscr{D}(\Gamma) := \{y \in Y\colon Uy \in \mathscr{D}(\Gamma)\}$. Moreover, $\sigma_p(\Delta) = \sigma_p(\Gamma)$ and $\sigma(\Delta) = \sigma(\Gamma)$, since if λ is in the resolvent set $\rho(\Gamma) := \mathbb{C}\backslash\sigma(\Gamma)$, we have that $R(\lambda, \Delta) = UR(\lambda, \Gamma)U^{-1}$. See, for example, [16], Chapter II and [15], Chapter 3.

2. Groups of Composition Operators on the Little Bloch Space

We consider the group of weighted composition operators $(T_t)_{t\in\mathbb{R}}$ given by equation (12) and defined on the little Bloch space $\mathscr{B}_{\infty,0}(\mathbb{D})$ as $T_tf(z) = e^{ict}f(e^{ikt}z)$, where $c, k \in \mathbb{R}$, $k \ne 0$ and $\forall f \in \mathscr{B}_{\infty,0}(\mathbb{D})$. We denote the infinitesimal generator of the group $(T_t)_{t\in\mathbb{R}}$ by $\Gamma_{\{c,\ k\}}$ and give some of its properties in the following proposition.

Proposition 1

(1) $(T_t)_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $\mathscr{B}_{\infty,0}(\mathbb{D})$

(2) The infinitesimal generator $\Gamma_{c,k}$ of $(T_t)_{t\in\mathbb{R}}$ on $\mathscr{B}_{\infty,0}(\mathbb{D})$ is given by $\Gamma_{c,k} f(z) = i(cf(z) + kzf'(z))$. with domain $\mathscr{D}(\Gamma_{c,k}) = \{f \in \mathscr{B}_{\infty,0}(\mathbb{D})\colon zf' \in \mathscr{B}_{\infty,0}(\mathbb{D})\}$.

Proof. To prove isometry, we have

$$\begin{aligned}\|T_t f\|_{\mathscr{B}_\infty(\mathbb{D})} &= |T_t f(0)| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|(T_t f)'(z)\right|\\ &= \left|e^{ict} f(0)\right| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|e^{ict}e^{ikt} f'\left(e^{ikt}z\right)\right|\\ &= |f(0)| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|f'\left(e^{ikt}z\right)\right|.\end{aligned} \tag{13}$$

By change of variables, let $\omega = e^{ikt}z$. Then,

$$\begin{aligned}\|T_t f\|_{\mathscr{B}_\infty(\mathbb{D})} &= |f(0)| + \sup_{\omega\in\mathbb{D}}\left(1-|\omega|^2\right)|f'(\omega)| \\ &= \|f\|_{\mathscr{B}_\infty(\mathbb{D})}, \quad \text{as desired.}\end{aligned} \tag{14}$$

To prove strong continuity, we shall use the density of polynomials in $\mathscr{B}_{\infty,0}(\mathbb{D})$. Therefore, it suffices to show that, for $(z^n)_{n\geq 0}$,

$$\lim_{t\to 0^+} \|T_t z^n - z^n\|_{\mathscr{B}_{\infty,0}(\mathbb{D})} = 0. \tag{15}$$

Now, $T_t z^n - z^n = e^{ict}(e^{ikt}z)^n - z^n = (e^{i(c+kn)t}-1)z^n$. Therefore,

$$\begin{aligned}\lim_{t\to 0^+}\|T_t z^n - z^n\|_{\mathscr{B}_{\infty,0}(\mathbb{D})} &= \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|(T_t z^n - z^n)'\right|\right) \\ &= \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|n\left(e^{i(k+kn)t}-1\right)z^{n-1}\right|\right) \\ &= 0, \quad \text{as claimed.}\end{aligned} \tag{16}$$

Now, for the infinitesimal generator $\Gamma_{c,k}$, let $f\in\mathscr{D}(\Gamma_{c,k})$ in $\mathscr{B}_{\infty,0}(\mathbb{D})$, then the growth condition (6) implies that

$$\begin{aligned}\Gamma_{c,k}f(z) &= \lim_{t\to 0^+}\frac{e^{ict}f(e^{ikt}z)-f(z)}{t} = \frac{\partial}{\partial t}\left(e^{ict}f\left(e^{ikt}z\right)\right)\Big|_{t=0} \\ &= i\left(cf(z) - izf'(z)\right).\end{aligned} \tag{17}$$

Therefore, $\mathscr{D}(\Gamma_{c,k})\subseteq\left\{f\in\mathscr{B}_{\infty,0}(\mathbb{D}): zf'\in\mathscr{B}_{\infty,0}(\mathbb{D})\right\}$. Conversely, if $f\in\mathscr{B}_{\infty,0}(\mathbb{D})$ is such that $zf'\in\mathscr{B}_{\infty,0}(\mathbb{D})$, then $F(z) = i(cf(z)+kzf'(z))\in\mathscr{B}_{\infty,0}(\mathbb{D})$ and for all $t>0$,

$$\begin{aligned}\frac{T_t f(z) - f(z)}{t} &= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}(T_s f(z))\,\mathrm{d}s \\ &= \frac{1}{t}\int_0^t e^{ics}\left(i\left(cf\left(e^{iks}z\right) + k\left(e^{iks}z\right)f'\left(e^{iks}z\right)\right)\right)\mathrm{d}s \\ &= \frac{1}{t}\int_0^t T_s F(z)\,\mathrm{d}s.\end{aligned} \tag{18}$$

Strong continuity of $(T_s)_{s\geq 0}$ implies that

$$\left\|\frac{1}{t}\int_0^t T_s F\,\mathrm{d}s - F\right\| \leq \frac{1}{t}\int_0^t \|T_s F - F\|\,\mathrm{d}s \longrightarrow 0, \quad \text{as } t\longrightarrow 0^+. \tag{19}$$

Thus, $\mathscr{D}(\Gamma_{c,k})\supseteq\left\{f\in\mathscr{B}_{\infty,0}(\mathbb{D}): zf'\in\mathscr{B}_{\infty,0}(\mathbb{D})\right\}$. □

Define M_z, Q on $\mathscr{H}(\mathbb{D})$ by $M_z f(z) = zf(z)$ and $Qf(z) = (f(z)-f(0))/z$, $(Qf(0) = f'(0))$. More generally, $Q^m f(z) = \sum_{k=m}^{\infty}((f^{(k)}(0))/k!)z^{k-m}$, $Q^m f(0) = ((f^m(0))/m!)$. Then, $M_z^m Q^m f = \sum_m^\infty ((f^{(k)}(0))/k!)z^k$ and $Q^m M_z^m f = f$. We now give the following proposition.

Proposition 2

(1) M_z: $\mathscr{B}_\infty(\mathbb{D})\longrightarrow\mathscr{B}_\infty(\mathbb{D})$ *is bounded.*
(2) $M_z\mathscr{B}_{\infty,0}(\mathbb{D})\subseteq\mathscr{B}_{\infty,0}(\mathbb{D})$.
(3) Q: $\mathscr{B}_{\infty,0}(\mathbb{D})\longrightarrow\mathscr{B}_{\infty,0}(\mathbb{D})$ *is bounded.*
(4) For $m\geq 1$, $M_z^m\mathscr{B}_{\infty,0}(\mathbb{D}) = \left\{f\in\mathscr{B}_{\infty,0}(\mathbb{D}): f^k(0) = 0\,\forall k<m\right\}$. *In particular,* $M_z\mathscr{B}_{\infty,0}(\mathbb{D})$ *is closed in* $\mathscr{B}_{\infty,0}(\mathbb{D})$.

Proof. If $f\in\mathscr{B}_\infty(\mathbb{D})$, then for all $z\in\mathbb{D}$,

$$\begin{aligned}\left(1-|z|^2\right)\left|(zf)'\right| &= \left(1-|z|^2\right)\left|zf'(z)+f(z)\right| \\ &\leq \left(1-|z|^2\right)\left|f'(z)\right| + \left(1-|z|^2\right)|f(z)| \\ &\leq \left(1-|z|^2\right)\left|f'(z)\right| + \left(1-|z|^2\right) \\ &\quad\cdot\left(1+\frac{1}{2}\log((1+|z|)/(1-|z|))\right)\|f\|_{\mathscr{B}_\infty(\mathbb{D})}.\end{aligned} \tag{20}$$

Therefore, assertions (1) and (2) follow. For (3), if $f\in\mathscr{B}_{\infty,0}(\mathbb{D})$, then for $|z|<1$,

$$\begin{aligned}&\left(1-|z|^2\right)\left|(Qf)'(z)\right| \\ &= \left(1-|z|^2\right)\left|\frac{zf'(z)-f(z)+f(0)}{z^2}\right| \\ &\leq \frac{\left(1-|z|^2\left|f'(z)\right|\right)}{|z|} \\ &\quad+ \frac{\left(1-|z|^2\right)(1+(1/2)\log((1+|z|)/(1-|z|)))\|f\|_{\mathscr{B}_\infty(\mathbb{D})}}{|z|^2} \\ &\quad+ \frac{\left(1-|z|^2\right)\|f\|_{\mathscr{B}_\infty(\mathbb{D})}}{|z|^2} \longrightarrow 0 \text{ as } |z|\longrightarrow 1.\end{aligned} \tag{21}$$

Thus, $Qf\in\mathscr{B}_{\infty,0}(\mathbb{D})$. To prove (4), let $f\in\mathscr{B}_{\infty,0}(\mathbb{D})$ and $f(0)=0$. Then, $f = M_z Qf\in M_z\mathscr{B}_{\infty,0}(\mathbb{D})$. The reverse inclusion is obvious. Therefore, the one-to-one and onto mapping M_z: $\mathscr{B}_{\infty,0}(\mathbb{D})\longrightarrow\left\{f\in\mathscr{B}_{\infty,0}(\mathbb{D}): f(0)=0\right\}$ is bounded. So, the open mapping theorem implies that the inverse is bounded. It therefore follows that Q: span $(1)\oplus M_z\mathscr{B}_{\infty,0}(\mathbb{D})\longrightarrow\mathscr{B}_{\infty,0}(\mathbb{D})$ is bounded. □

Proposition 3. *Let* $\Gamma_{c,k}$ *be the infinitesimal generator of the group* $(T)_{t\in\mathbb{R}}$ *given by (12) on* $\mathscr{B}_{\infty,0}(\mathbb{D})$*, then*

(1) $\Gamma_{c,k} = ic + k\Gamma_{0,1}$ *with domain* $\mathscr{D}(\Gamma_{c,k}) = \mathscr{D}(\Gamma_{0,1}) = \left\{f\in\mathscr{B}_{\infty,0}(\mathbb{D}): zf'\in\mathscr{B}_{\infty,0}(\mathbb{D})\right\}$.
(2) $\sigma(\Gamma_{c,k}) = \left\{ic+k\sigma(\Gamma_{0,1})\right\}$ *and* $\sigma_p(\Gamma_{c,k}) = \left\{ic+k\sigma_p(\Gamma_{0,1})\right\}$.

In fact, $\lambda\in\rho(\Gamma_{0,1})$ if and only if $ic+k\lambda\in\rho(\Gamma_{c,k})$, and

$$R\left(ic+k\lambda,\Gamma_{c,k}\right) = \frac{1}{k}R\left(\lambda,\Gamma_{0,1}\right). \tag{22}$$

Proof. See [12], Lemma 4.3. □

As a result of Proposition 3 above and without loss of generality, we restrict our attention to the generator $\Gamma_{0,1}$instead of $\Gamma_{c,k}$ as the cases $c\neq 0$ and $k\neq 1$ where $k\neq 0$ can be easily obtained from $\Gamma_{0,1}$. Indeed, $\Gamma_{0,1}f(z) = izf'(z)$ with domain $\mathscr{D}(\Gamma_{0,1}) = \{f \in \mathscr{B}_{\infty,0}(\mathbb{D})\colon zf' \in \mathscr{B}_{\infty,0}(\mathbb{D})\}$ is the infinitesimal generator of the group $T_t = f(e^{itz})$ which is exactly the case when $c = 0$ and $k = 1$ in equation (12). We now give the spectral properties of the generator $\Gamma_{0,1}$ as well as the resulting resolvents in the following theorem.

Theorem 1

(1) *$\sigma(\Gamma_{0,1}) = \sigma_p(\Gamma_{0,1}) = \{in\colon n \in \mathbb{Z}_+\}$, and for each $n \geq 0$, $\ker(in - \Gamma_{0,1}) = \operatorname{span}(z^n)$.*

(2) *If $\lambda \in \rho(\Gamma_{0,1})$, then $M_z\mathscr{B}_{\infty,0}(\mathbb{D})$ is $R(\lambda, \Gamma_{0,1})$, invariant $\forall m \in \mathbb{Z}_+$, $m > \mathfrak{I}(\lambda)$. Moreover, if $h \in M_z^m\mathscr{B}_{\infty,0}(\mathbb{D})$, then*

$$R(\lambda, \Gamma_{0,1})h(z) = iz^{-\lambda t}\int_0^z \omega^{i\lambda-1}h(\omega)\mathrm{d}\omega = iz^m \cdot \int_0^1 t^{m+i\lambda-1}(Q^m h)(tz)\mathrm{d}t. \tag{23}$$

(3) *For $\lambda \in \rho(\Gamma_{0,1})$, the resolvent operator $R(\lambda, \Gamma_{0,1})$ is compact.*

(4) *$\sigma(R(\lambda,\Gamma_{0,1})) = \sigma_p(R(\lambda,\Gamma_{0,1})) = \{w \in \mathbb{C}\colon |w - (1/(2\mathfrak{R}(\lambda)))| = (1/2\mathfrak{R}(\lambda))\}$. Moreover,*

$$r(R(\lambda,\Gamma_{0,1})) = \|R(\lambda,\Gamma_{0,1})\| = \frac{1}{|\mathfrak{R}(\lambda)|}. \tag{24}$$

Proof. Since each T_t is an invertible isometry, its spectrum satisfies $\sigma(T_t) \subseteq \partial\mathbb{D}$, and the spectral mapping theorem for strongly continuous groups (see, for example, [16], Theorem V.2.5 or [17]) implies that $e^{t\sigma(\Gamma_{0,1})} \subseteq \sigma(T_t)$. Thus, $e^{t\sigma(\Gamma_{0,1})} \subseteq \partial\mathbb{D} \Longrightarrow |e^{t\sigma(\Gamma_{0,1})}| = 1 \Longrightarrow e^{t\mathfrak{R}(\omega)} = 1 \Longrightarrow \mathfrak{R}(\omega) = 0$ for $\omega \in \sigma(\Gamma_{0,1})$. It immediately follows that $\sigma(\Gamma_{0,1}) \subseteq i\mathbb{R}$.

We now solve the resolvent equation: If $\lambda \in \mathbb{C}$ and $h \in \mathscr{H}(\mathbb{D})$, $(\lambda - \Gamma)f = h$. This is equivalent to

$$f'(z) + \frac{i\lambda}{z}f(z) = \frac{i}{z}h(z), \quad (z \neq 0), \qquad z^{i\lambda}f(z) = iz^{i\lambda-1}h(z), \quad (z \in \mathbb{D}\backslash(-1,0]). \tag{25}$$

In particular, $(\lambda - \Gamma)f = 0$ if and only $f(z) = Kz^{-i\lambda}$, where K is a constant. Since $z^{-i\lambda} \in \mathscr{H}(\mathbb{D})$ if and only if $-i\lambda \in \mathbb{Z}_+$, it follows that

$$\sigma_p(\Gamma_{0,1}) = \{in\colon n \in \mathbb{Z}_+\}, \tag{26}$$

with $\ker(in - \Gamma_{0,1}) = \operatorname{span}(z^n)$. Moreover, if $n \in \mathbb{Z}_+$ and $\lambda \in \sigma_p(\Gamma_{0,1})$, then

$$(\lambda - \Gamma)f = z^n, \tag{27}$$

has a unique solution

$$f(z) = \frac{1}{\lambda - in}z^n. \tag{28}$$

Notice that, for $\lambda \notin \sigma_p(\Gamma_{0,1})$ and $f \in \mathscr{D}(\Gamma_{0,1})$, $(\lambda - \Gamma)f(0) = \lambda f(0)$. More generally, if $f(z) = z^n g(z)$ with $g(0) \neq 0$, then

$$(\lambda - \Gamma)f = \lambda f - z(z^m g)' = z^m(\lambda g - mz^m g - z^{m+1}g'). \tag{29}$$

Note that the functions $(\lambda - \Gamma)f$ and f have the same order of zero at 0. Thus, $M_z^m\mathscr{B}_{\infty,0}(\mathbb{D})$ is invariant under $\lambda - \Gamma_{0,1}$.

Fix $\lambda \in \mathbb{C}\backslash\sigma_p(\Gamma_{0,1})$ and let $m > \mathfrak{I}(\lambda)$. If $h = z^m g$ with $g \in \mathscr{B}_{\infty,0}(\mathbb{D})$, then

$$i\int_0^z \omega^{i\lambda-1}h(\omega)\mathrm{d}\omega = iz^{m+i\lambda}\int_0^1 t^{m+i\lambda-1}g(tz)\mathrm{d}t. \tag{30}$$

Thus, $(\lambda - \Gamma)h$ has a unique solution:

$$f(z) = iz^m\int_0^1 t^{m+i\lambda-1}(Q^m h)(tz)\mathrm{d}t. \tag{31}$$

If $u \in \mathscr{B}_\infty(\mathbb{D})$ and $0 \leq t < 1$, then

$$\begin{aligned}\|u(tz)\|_{\mathscr{B}_\infty(\mathbb{D})} &= \sup_{|z|<1}(1-|z|^2)t|u'(tz)| \\ &\leq \sup_{|z|<1}(1-t^2|z|^2)|u'(tz)| \\ &\leq \|u\|_{\mathscr{B}_\infty(\mathbb{D})}.\end{aligned} \tag{32}$$

Thus, $\|f\| \leq (1/(m - \mathfrak{I}(\lambda)))\|M_z^m\|\|Q^m\|\|h\|$. Now, $\forall m \geq 1$,

$$\mathscr{B}_{\infty,0}(\mathbb{D}) = \operatorname{span}(z^n) \oplus M_z^m\mathscr{B}_{\infty,0}(\mathbb{D}),$$

$$R(\lambda,\Gamma_{0,1})\big|_{\operatorname{span}(z^n)_{0\leq n<m}} = \begin{pmatrix} \frac{1}{\lambda} & & & \\ & \frac{1}{\lambda - i} & 0 & \\ & & \ddots & \\ & 0 & & \frac{1}{\lambda - (m-1)i} \end{pmatrix}. \tag{33}$$

Thus, $\lambda \notin \sigma_p(\Gamma_{0,1})$ implying that $R(\lambda, \Gamma_{0,1})$ is bounded on $\mathscr{B}_{\infty,0}(\mathbb{D})$. Therefore, $\sigma(\Gamma_{0,1}) = \sigma_p(\Gamma_{0,1})$. This proves (1) and (2).

To prove the compactness of the resolvent operator, we argue as in [7], Theorem 5.2. Fix $\lambda \in \rho(\Gamma_{0,1})$ and let $m \in \mathbb{Z}_+$ be such that $\mathfrak{I}(\lambda) < m$. Then, by equation (33), it suffices to show that $R_m(\lambda,\Gamma_{0,1}) = R(\lambda,\Gamma_{0,1})|_{M_z^m\mathscr{B}_{\infty,0}(\mathbb{D})}$ is compact.

Let $\mathscr{A}(r\mathbb{D})$, $r > 0$, be the disc algebra $\mathscr{A}(r\mathbb{D}) = C(r\overline{\mathbb{D}}) \cap \mathscr{H}(r\mathbb{D})$, equipped with the supremum norm, and for each t, $0 \leq t < 1$, and $f \in \mathscr{H}(\mathbb{D})$, let $H_t f(z) = f_t(z) = f(tz)$. Then, by equation (32), for every $t \in [0, 1)$, H_t is a contraction on $\mathscr{B}_{\infty,0}(\mathbb{D})$.

Now, by equation (23), $R_m(\lambda,\Gamma_{0,1}) = iM_z^m\int_0^1 t^{m+i\lambda-1} H_t Q^m\mathrm{d}t$ with convergence in norm. Define $C_r = iM_z^m\int_0^r t^{m+i\lambda-1}H_t Q^m\mathrm{d}t$ on $M_z^m\mathscr{B}_{\infty,0}(\mathbb{D})$, for $0 < r < 1$. Then,

$$\begin{aligned}\left\|R_m - C_r\right\| &\leq \int_r^1 t^{m-\mathfrak{I}(\lambda)-1}\|Q\|^m \mathrm{d}t \\ &= \frac{\|Q\|^m}{m-\mathfrak{I}(\lambda)}\left(1 - r^{m-\mathfrak{I}(\lambda)}\right) \longrightarrow 0,\end{aligned} \tag{34}$$

as $r \longrightarrow 1^-$. Choosing s so that $1 < s < r^{-1}$, we have that C_r: $M_z^m\mathscr{B}_{\infty,0}(\mathbb{D}) \longrightarrow M_z^m\mathscr{B}_{\infty,0}(\mathbb{D})$ factors through $\mathscr{A}(s\mathbb{D})$. If $\mathbb{B}$ denotes the closed unit ball of $M_z^m\mathscr{B}_{\infty,0}(\mathbb{D})$, let $h = Q^m f$ ($f \in M_z^m\mathscr{B}_{\infty,0}(\mathbb{D})$). Then, $\forall$t, $0 \leq t \leq r$, the growth condition (6) implies that, for $|z| \leq s$,

$$\begin{aligned}|h(tz)| &\leq \left(1 + \frac{1}{2}\log\left(\frac{1+rs}{1-rs}\right)\right)\|h\|_{\mathscr{B}_{\infty,0}(\mathbb{D})}, \\ \left|\frac{\mathrm{d}}{\mathrm{d}t}h(tz)\right| &\leq \frac{\|h\|_{\mathscr{B}_{\infty,0}(\mathbb{D})}}{1-rs}.\end{aligned} \tag{35}$$

Let $K = (1 + (1/2)\log((1+rs)/(1-rs)))\|h\|_{\mathscr{B}_{\infty,0}(\mathbb{D})}$. Thus, for $|z| \leq s$,

$$\begin{aligned}\left|C_r f(z)\right| &\leq K\frac{s^m r^{m-\mathfrak{I}(\lambda)}}{m-\mathfrak{I}(\lambda)}, \\ \left|\frac{\mathrm{d}}{\mathrm{d}z}C_r f(z)\right| &\leq K\frac{m s^{m-1} r^{m-\mathfrak{I}(\lambda)}}{m-\mathfrak{I}(\lambda)} + \frac{s^m r^{m-\mathfrak{I}(\lambda)}}{m-\mathfrak{I}(\lambda)}\frac{\|h\|_{\mathscr{B}_{\infty,0}(\mathbb{D})}}{1-rs}.\end{aligned} \tag{36}$$

Thus, by Arzela-Ascoli, $C_r\mathbb{B}$ is precompact in $\mathscr{A}(s\mathbb{D})$ which further implies that $C_r\mathbb{B}$ is precompact in $\mathscr{B}_{\infty,0}(\mathbb{D})$ by the continuous embeddedness of $\mathscr{A}(s\mathbb{D})$ in $\mathscr{B}_{\infty,0}(\mathbb{D})$. Therefore, each C_r is compact in $\mathscr{L}(M_z^m\mathscr{B}_{\infty,0}(\mathbb{D}))$ and as a result, $R_m(\lambda, \Gamma_{0,1}) = (\text{norm})\lim_{r\longrightarrow 1^-} C_r$ is compact as well.

The spectral mapping theorem for resolvents as well as assertion (1) above implies that

$$\begin{aligned}\sigma\left(R\left(\lambda,\Gamma_{0,1}\right)\right) = \sigma_p\left(R\left(\lambda,\Gamma_{0,1}\right)\right) &= \left\{\frac{1}{\lambda - im}: m \in \mathbb{Z}_+\right\} \cup \{0\} \\ &= \left\{\omega \in \mathbb{C}: \left|\omega - \frac{1}{2\mathfrak{R}(\lambda)}\right| = \frac{1}{2|\mathfrak{R}(\lambda)|}\right\}.\end{aligned} \tag{37}$$

Clearly, the spectral radius $r(R(\lambda,\Gamma_{0,1})) = (1/|\mathfrak{R}(\lambda)|)$ and therefore by the Hille–Yosida theorem, it follows that $(1/|\mathfrak{R}(\lambda)|) = r(R(\lambda,\Gamma_{0,1})) \leq \|R(\lambda,\Gamma_{0,1})\| \leq (1/|\mathfrak{R}(\lambda)|)$, as desired. □

As a consequence, the properties of the general group T_t given by equation (12) is as follows.

Corollary 1

(1) $\sigma(\Gamma_{c,k}) = \sigma_p(\Gamma_{c,k}) = \{i(c+kn): n \in \mathbb{Z}_+\}$, and for each $n \geq 0$, $\ker(i(c+kn) - \Gamma_{c,k}) = \operatorname{span}(z^n)$.

(2) If $\mu \in \rho(\Gamma_{c,k})$, then $M_z\mathscr{B}_{\infty,0}(\mathbb{D})$ is $R(\mu,\Gamma_{c,k})$-invariant $\forall m \in \mathbb{Z}_+$, $m > \mathfrak{I}((\mu - ic)/k)$. Moreover, if $h \in M_z^m\mathscr{B}_{\infty,0}(\mathbb{D})$, then

$$\begin{aligned}R\left(\mu,\Gamma_{c,k}\right)h(z) &= \frac{i}{k}z^{-((\mu-ic)/k)t}\int_0^z \omega^{i((\mu-ic)/k)-1}h(\omega)\mathrm{d}\omega \\ &= \frac{i}{k}z^m\int_0^1 t^{m+i(\mu-ic/k)-1}\left(Q^m h\right)(tz)\mathrm{d}t.\end{aligned} \tag{38}$$

(3) For $\mu \in \rho(\Gamma_{c,k})$, the resolvent $R(\mu,\Gamma_{c,k})$ is compact.

(4) $\sigma(R(\mu,\Gamma_{c,k})) = \sigma_p(R(\mu,\Gamma_{c,k})) = \{w \in \mathbb{C}: |w - (1/2\mathfrak{R}(\mu))| = (1/2\mathfrak{R}(\mu))\}$.

(5) $r(R(\mu,\Gamma_{c,k})) = \|R(\mu,\Gamma_{c,k})\| = (1/(2|\mathfrak{R}(\mu)|))$.

Proof. Following Proposition 3, $\mu \in \rho(\Gamma_{c,k})$ if and only if $(\mu - ic)/k \in \rho(\Gamma_{0,1})$. The proof now follows at once from Theorem 1. We omit the details. □

3. Adjoint of the Composition Group on the Predual of Nonreflexive Bergman Space $L_a^1(\mathbb{D}, m_\alpha)$

In studying the adjoint properties of the rotation group isometries given by equation (12) on Bergman spaces $L_a^p(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$; the second author in [12] considered the reflexive case, that is, when $1 < p < \infty$. This was an extension of the investigation of adjoint properties of the Cesáro operator in [18] on Hardy spaces, and later generalized to Bergman spaces in [7]. For the nonreflexive Bergman space $L_a^1(\mathbb{D}, m_\alpha)$ (that is, $p = 1$), the analysis of the adjoint of the rotation group isometries remains open and forms the basis of this section. Specifically, we complete the analysis of the adjoint group of the group of isometries $T_t f(z) = e^{ict}f(e^{ikt}z)$, where $c, k \in \mathbb{R}$ with $k \neq 0$ and $\forall f \in L_a^1(\mathbb{D}, m_\alpha)$.

Recall from Section 1, the duality relation $(\mathscr{B}_{\infty,0}(\mathbb{D}))^* \approx L_a^1(\mathbb{D}, m_\alpha)$ under the integral pairing $\langle g, f\rangle = \int_{\mathbb{D}} g(z)\overline{f(z)}\mathrm{d}m_\alpha$ ($g \in \mathscr{B}_{\infty,0}(\mathbb{D})$, $g \in L_a^1(\mathbb{D}, m_\alpha)$). In particular, the predual of $L_a^1(\mathbb{D}, m_\alpha)$ is the little Bloch space $\mathscr{B}_{\infty,0}(\mathbb{D})$. Thus, using this duality pairing, for every $g \in \mathscr{B}_{\infty,0}(\mathbb{D})$, we have

$$\begin{aligned}\langle g, T_t f\rangle &= \int_{\mathbb{D}} g(z)\overline{e^{ict}f(e^{ikt}z)}\mathrm{d}m_\alpha(z) \\ &= \int_{\mathbb{D}} e^{-ict}g(z)\overline{f(e^{ikt}z)}\left(1 - |z|^2\right)^\alpha \mathrm{d}A(z).\end{aligned} \tag{39}$$

By a change of variables argument: Let $\omega = e^{ikt}z$ so that $z = e^{-ikt}\omega$ and

$$\begin{aligned}\langle g, T_t f\rangle &= \int_{\mathbb{D}} e^{-ict}g\left(e^{-ikt}\omega\right)\overline{f(\omega)}\left(1 - \left|e^{-ikt}\omega\right|^2\right)^\alpha \mathrm{d}A(\omega) \\ &= \int_{\mathbb{D}} e^{-ict}g\left(e^{-ikt}\omega\right)\overline{f(\omega)}\mathrm{d}m_\alpha(\omega) \\ &= \int_{\mathbb{D}} T_{-t}g(\omega)\overline{f(\omega)}\mathrm{d}m_\alpha(\omega) = \langle T_{-t}g, f\rangle,\end{aligned} \tag{40}$$

where $T_{-t}g(\omega) = e^{-ict}g(e^{-ikt}\omega)$ for all $g \in \mathscr{B}_{\infty,0}(\mathbb{D})$. Thus, the adjoint group T_t^* of T_t for $t \in \mathbb{R}$ is therefore given by

$$T_t^* g(\omega) := T_{-t} g(\omega) = e^{-ict} g\left(e^{-ikt}\omega\right), \quad \text{for all } g \in \mathscr{B}_{\infty,0}(\mathbb{D}). \tag{41}$$

Let Γ denote the infinitesimal generator of the adjoint group T_t^*. Using the results of Section 2, we easily obtain the properties of the group $(T_t^*)_{t\in\mathbb{R}}$ as we give in the following theorem.

Theorem 2. *Let $(T_t^*)_{t\in\mathbb{R}} \subseteq \mathscr{L}(\mathscr{B}_{\infty,0}(\mathbb{D}))$ be the adjoint group of the group of weighted composition operators $(T_t)_{t\in\mathbb{R}} \subseteq \mathscr{L}(L_a^1(\mathbb{D}, m_\alpha))$ given by (41). Then, the following hold:*

(1) $(T_t^*)_{t\in\mathbb{R}}$ is strongly continuous group of isometries on $\mathscr{B}_{\infty,0}(\mathbb{D})$.

(2) The infinitesimal generator Γ of $(T_t^*)_{t\geq 0}$ is given by $\Gamma g(\omega) = -i(cg(\omega) + k\omega g'(\omega))$ with domain $\mathscr{D}(\Gamma) = \{g \in \mathscr{B}_{\infty,0}(\mathbb{D}) : \omega g' \in \mathscr{B}_{\infty,0}(\mathbb{D})\}$.

(3) $\sigma(\Gamma) = \sigma_p(\Gamma) = \{-i(c+kn) : n \in \mathbb{Z}_+\}$, and for each $n \geq 0$, $\ker(-i(c+kn) - \Gamma) = \operatorname{span}(\omega^n)$

(4) If $\mu \in \rho(\Gamma)$, then $M_\omega \mathscr{B}_{\infty,0}(\mathbb{D})$ is $R(\mu, \Gamma)$-invariant $\forall m \in \mathbb{Z}_+$, $m > \mathfrak{I}((-\mu - ic)/k)$. Moreover, if $h \in M_\omega^m \mathscr{B}_{\infty,0}(\mathbb{D})$, then

$$\begin{aligned} R(\mu,\Gamma)h(w) &= -\frac{i}{k}\omega^{((\mu+ic)/k)t} \int_0^\omega z^{-i((\mu+ic)/k)-1} h(z)\,dz \\ &= -\frac{i}{k}\omega^m \int_0^1 t^{m-i((\mu+ic)/k)-1} (Q^m h)(t\omega)\,dt. \end{aligned} \tag{42}$$

(5) $\sigma(R(\mu,\Gamma)) = \sigma_p(R(\mu,\Gamma)) = \{w \in \mathbb{C} : |w - (1/2\mathfrak{R}(\mu))| = (1/2\mathfrak{R}(\mu))\}$.

(6) $r(R(\mu,\Gamma)) = \|R(\mu,\Gamma)\| = (1/|\mathfrak{R}(\mu)|)$.

Proof. The proof follows immediately by replacing c and k with $-c$ and $-k$, respectively, in Proposition 3 and Corollary 1. We omit the details. □

4. Specific Automorphism of the Half-Plane

In this section, we consider a specific automorphism group $(\varphi_t)_{t\in\mathbb{R}} \subset \operatorname{Aut}(\mathbb{U})$ corresponding to the rotation group given by

$$\varphi_t(z) = \frac{z\cos t - \sin t}{z\sin t + \cos t}. \tag{43}$$

It can be easily verified that $\varphi_t(z) = \psi \circ u_t \circ \psi^{-1}(z)$, where $u_t(z) = e^{-2it}z$. The associated group of weighted composition operators on $\mathscr{H}(\mathbb{U})$ is given by S_{φ_t}, and by the chain rule, it follows that $S_{\varphi_t} = S_{\psi^{-1}} S_{u_t} S_\psi$, where $S_{\psi^{-1}} = S_\psi^{-1}$.

Now, for $f \in \mathscr{B}_{\infty,0}(\mathbb{D})$,

$$\begin{aligned} S_{u_t} f(z) &= (u_t'(z))^\gamma f(u_t(z)) \\ &= e^{-2i\gamma t} f\left(e^{-2it} z\right). \end{aligned} \tag{44}$$

Apparently, S_{u_t} can be obtained as a special case of the group $(T_t)_{t\geq 0}$ given by equation (12) when $c = -2\gamma$ and $k = -2$. Let $\Gamma = \Gamma_{-2\gamma,-2}$ be the infinitesimal generator of the group S_{u_t}, then the properties of Γ can be summarized by the following proposition.

Proposition 4. *Let Γ be the infinitesimal generator of the group of isometries S_{u_t} on $\mathscr{B}_{\infty,0}(\mathbb{D})$. Then,*

(1) $\Gamma f(z) = i(-2\gamma f(z) - 2zf'(z))$ for every $f \in \mathscr{B}_{\infty,0}(\mathbb{D})$, with domain

$$\mathscr{D}(\Gamma) = \left\{f \in \mathscr{B}_{\infty,0}(\mathbb{D}) : f' \in \mathscr{B}_{\infty,0}(\mathbb{D})\right\}. \tag{45}$$

(2) $\sigma(\Gamma) = \sigma_p(\Gamma) = \{-2(\gamma+n)i : n \in \mathbb{Z}_+\}$, and for each $n \geq 0$,

$$\ker(-2(\gamma+n)i - \Gamma) = \operatorname{span}(z^n). \tag{46}$$

(3) If $\mu \in \rho(\Gamma)$, then $\mathscr{R}(M_z^m)$ is $R(\mu, \Gamma)$-invariant for every $m \in \mathbb{Z}_+$, $m > \mathfrak{I}(-(\mu + 2\gamma i)/2)$. Moreover, if $h \in \mathscr{R}(M_z^m)$, then

$$\begin{aligned} R(\mu,\Gamma)h(z) &= -\frac{i}{2} z^{((\mu+2i\gamma)/2)i} \int_0^z \omega^{-((\mu+2i\gamma)/2)i-1} h(\omega)\,d\omega \\ &:= R_\mu h(z). \end{aligned} \tag{47}$$

Proof. Take $c = -2\gamma$ and $k = -2$ in Proposition 1 and Corollary 1. The proof follows immediately.

Now, using the similarity theory of semigroups, we detail the properties of the group of weighted composition operators associated with the automorphism group $(\varphi_t)_{t\geq 0}$ given by (43) in the following theorem. □

Theorem 3. *Let $\varphi_t \in Aut(\mathbb{U})$ be given by $\varphi_t(z) = (z\cos t - \sin t)/(z\sin t + \cos t)$, for all $t \in \mathbb{R}, z \in \mathbb{U}$, and let $S_{\varphi_t} f(z) := (\varphi_t')^\gamma f(\varphi_t(z))$ be the corresponding group of isometries on $\mathscr{B}_{\infty,0}(\mathbb{U})$. Then,*

(1) The infinitesimal generator Δ of the group S_{φ_t} on $\mathscr{B}_{\infty,0}(\mathbb{U})$ is given by

$$\Delta(h(z)) = -2\gamma z h(z) - \left(1 + z^2\right) h'(z), \tag{48}$$

with domain $\mathscr{D}(\Delta) = \{h \in \mathscr{B}_{\infty,0}(\mathbb{U}) : 2\gamma(\omega + i)h + (\omega + i)^2 h' \in \mathscr{B}_{\infty,0}(\mathbb{D})\}$.

(2) $\sigma_p(\Delta) = \sigma(\Delta) = \{-2(\gamma+n)i : n \in \mathbb{Z}_+\}$, and for each $n \geq 0$, $\ker(-2(\gamma+n)i - \Delta) = \operatorname{span}(S_\psi^{-1} z^n)$.

(3) If $\mu \in \rho(\Delta)$ and if $m \in \mathbb{Z}_+$ is such that $m > \mathfrak{I}((-\mu)/(2 - i\gamma))$. Then, if $h \in \mathscr{R}(M_z^m)$, we have

$$\begin{aligned} R(\mu,\Delta)h(z) = (z-i)^{((\mu+2i\gamma)/2)i} &\cdot (z+i)^{-(((\mu+2i\gamma)/2)i+2\gamma)} \\ &\cdot \int_0^z (\omega - i)^{-((\mu+2i\gamma)/2)i-1} \\ &\cdot (\omega+i)^{((\mu+2i\gamma)/2)i+2\gamma-1} h(\omega)\,d\omega. \end{aligned} \tag{49}$$

(4) $R(\mu, \Delta)$ is compact on $\mathscr{B}_{\infty,0}(\mathbb{D})$.

(5) $\sigma(R(\mu,\Delta)) = \sigma_p(R(\mu,\Delta)) = \{w \in \mathbb{C}: |w - (1/2\Re(\mu))| = (1/|\Re(\mu)|)\}$. *Moreover*

$$r(R(\mu,\Delta)) = \|R(\mu,\Delta)\| = \frac{1}{\Re(\mu)}. \quad (50)$$

Proof. Let $g(z) = \psi^{-1}(z) = ((z-i)/(z+i))$ and $g^{-1}(z) = \psi(z) = ((i(1+z))/(1-z))$. Since $\varphi_t(z) = g^{-1} \circ u_t \circ g(z)$, it follows that $S_{\varphi_t} = S_g S_{u_t} S_{g^{-1}} = S_g S_{u_t} S_g^{-1}$, where S_g is invertible. Let Δ be the generator of S_{φ_t} and $\Gamma := \Gamma_{-2\gamma,-2}$ be the generator of S_{u_t}, then $\Delta = S_g \Gamma S_g^{-1}$ with domain $D(\Delta) = S_g D(\Gamma)$.

Let $f' \in B_{\infty,0}(\mathbb{D})$, then $f \in D(\Gamma)$ and define $h := S_g f$ belongs to $D(\Delta)$ with $f = S_g^{-1}(h)$. Then,

$$\begin{aligned}\Delta(h(z)) &= S_g \Gamma S_g^{-1} h(z) = S_g \Gamma f(z) \\ &= S_g(-2i\gamma f(z) - 2iz f'(z)) \\ &= (g'(z))^{\gamma}(-2i\gamma f(g(z)) - i2g(z) f'(g(z))).\end{aligned} \quad (51)$$

As stated earlier, $g(z) = ((z-i)/(z+i))$, implying that $g'(z) = (2i/(z+i)^2)$, and thus

$$\Delta(h(z)) = \frac{(2i)^{\gamma}}{(z+i)^{2\gamma}}(-2i\gamma f(g(z)) - i2g(z) f'(g(z))). \quad (52)$$

Since $g^{-1}(z) = ((i(1+z))/(1-z))$ and $(g^{-1}(z))' = (2i/(1-z)^2)$, then we have $f(z) = S_g^{-1} h(z) = S_{g^{-1}} h(z) = ((2i)^{\gamma}/(1-z)^{2\gamma}) h(g^{-1}(z))$ implying that $f(g(z)) = ((z+i)^{2\gamma}/(2i)^{\gamma}) h(z)$. Moreover, $f'(z) = ((2i)^{\gamma}/(1-z)^{2\gamma+2})(2\gamma(1-z)h(g^{-1}(z)) + 2ih'(g^{-1}(z)))$ implying that

$$f'(g(z)) = \frac{(z+i)^{2\gamma+1}}{(2i)^{\gamma+1}}(2\gamma h(z) + (z+i)h'(z)). \quad (53)$$

Therefore,

$$\begin{aligned}\Delta(h(z)) &= \frac{(2i)^{\gamma}}{(z+i)^{2\gamma}}\left(-i\gamma \frac{(z+i)^{2\gamma}}{(2i)^{\gamma}} h(z) - i\frac{z-i}{z+i}\frac{(z+i)^{2\gamma+1}}{(2i)^{\gamma+1}} \right. \\ &\quad \left. \cdot (2\gamma h(z) + (z+i)h'(z))\right) \\ &= -2i\gamma h(z) - 2\gamma(z-i)h(z) - (z-i)(z+i)h'(z) \\ &= -2\gamma z h(z) - (1+z^2)h'(z)).\end{aligned} \quad (54)$$

As given earlier, the domain of Δ, $D(\Delta)$ is given by $D(\Delta) = S_g D(\Gamma) = \{S_g f : \in D(\Delta)\}$. Now $h \in D(\Delta)$ implies that $S_g^{-1} h \in D(\Gamma)$ which implies that $(S_{g^{-1}} h)' \in B_{\infty,0}(\mathbb{D})$. But

$$\begin{aligned}(S_{g^{-1}} h)' &= \left(\frac{(2i)^{\gamma}}{(1-z)^{2\gamma}} h(g^{-1}(z))\right)' \\ &= 2\gamma(2i)^{\gamma}(1-z)^{-2\gamma-1} h(g^{-1}(z)) \\ &\quad + \frac{(2i)^{\gamma}}{(1-z)^{2\gamma}}\frac{2i}{(1-z)^2} h'(g^{-1}(z)) \\ &= \frac{(2i)^{\gamma}}{(1-z)^{2\gamma}}\left(2\gamma(1-z)^{-1} h(g^{-1}(z)) \right. \\ &\quad \left. + \frac{2i}{(1-z)^2} h'(g^{-1}(z))\right).\end{aligned} \quad (55)$$

Then, we have

$$\begin{aligned}(S_{g^{-1}} h)' &= \frac{(2i)^{\gamma}}{(1-z)^{2\gamma}}\left(\frac{2\gamma}{1-g\circ g^{-1}(z)} h(g^{-1}(z)) \right. \\ &\quad \left. + \frac{2i}{(1-g\circ g^{-1}(z))^2} h'(g^{-1}(z))\right).\end{aligned} \quad (56)$$

By change of variables, let $\omega = g^{-1}(z)$ which implies $g(\omega) = z = g \circ g^{-1}(z)$ and $(S_{g^{-1}} h)' = S_{g^{-1}}((2\gamma/(1-g(\omega))) h(\omega) + (2i/(1-g(\omega))^2) h'(\omega))$. Therefore,

$$\begin{aligned} h \in D(\Delta) &\Leftrightarrow S_{g^{-1}}\left(\frac{2\gamma}{1-g(\omega)} h(\omega) + \frac{2i}{(1-g(\omega))^2} h'(\omega)\right) \in B_{\infty,0}(\mathbb{D}) \\ &\Leftrightarrow \left(\frac{2\gamma}{1-g(\omega)} h(\omega) + \frac{2i}{(1-g(\omega))^2} h'(\omega)\right) \in B_{\infty,0}(\mathbb{D}) \\ &\Leftrightarrow \frac{\omega+i}{2i}(2\gamma h(\omega) + (\omega+i)h'(\omega)) \in B_{\infty,0}(\mathbb{D}),\end{aligned} \quad (57)$$

which implies that $U(\Delta) = \{h \in B_{\infty,0}(\mathbb{D}): 2\gamma h(\omega) + (\omega+i)h'(\omega) \in B_{\infty,0}(\mathbb{D})\}$.

From Section 1, the spectrum and point spectrum of Δ are given as $\sigma_p(\Delta) = \sigma_p(\Gamma) = \sigma(\Gamma) = \sigma(\Delta) = \{-i(\gamma+n): n \in \mathbb{Z}_+\}$.

For the resolvents, if $\mu \in \rho(\Delta) = \rho(\Gamma)$, then for $m \in \mathbb{Z}_+$, $m > \Im(-(\mu + i\gamma))$ and if $h \in R(M_z^m)$, we have $R(\mu,\Delta) = S_f R(\mu,\Gamma) S_f^{-1}$ and so

$$\begin{aligned}R(\mu,\Delta)h(z) &= S_g\left(-\frac{i}{2}z^{((\mu+2i\gamma)/2)i}\int_0^z \omega^{-i((\mu+2i\gamma)/2)-1}S_{g^{-1}}h(\omega)\mathrm{d}\omega\right)\\ &= S_g\left(-\frac{i}{2}z^{((\mu+2i\gamma)/2)i}\int_0^z \omega^{-((\mu+2i\gamma)/2)i-1}\frac{(2i)^\gamma}{(1-\omega)^{2\gamma}}h\left(g^{-1}(\omega)\right)\mathrm{d}\omega\right)\\ &= -\frac{i}{2}\frac{(2i)^\gamma}{(z+i)^{2\gamma}}(g(z))^{((\mu+2i\gamma)/2)i}\int_0^z (g(\omega))^{((\mu+2i\gamma)/2)i-1}\frac{(2i)^\gamma}{(1-g(\omega))^{2\gamma}}h(\omega)\frac{\mathrm{d}g}{\mathrm{d}\omega}\mathrm{d}\omega\\ &= \left(\frac{z-i}{(z+i)^{2\gamma}}\right)^{((\mu+2i\gamma)/2)i}\int_0^z (\omega-i)^{((\mu+2i\gamma)/2)i-1}(\omega+i)^{((\mu+2i\gamma)/2)i+2\gamma-1}h(\omega)\mathrm{d}\omega.\end{aligned} \tag{58}$$

Finally, from spectral mapping theorems it follows that, for all $\mu \in \rho(\Delta)$, the spectrum of $R(\mu, \Delta)$ is given by

$$\begin{aligned}\sigma(R(\mu,\Delta)) &= \left\{\frac{1}{\mu-z}: z\in\sigma(\Delta)\right\}\cup\{0\}\\ &= \left\{\frac{1}{\mu+i(\gamma+n)}: n\in\mathbb{Z}_+\right\}\cup\{0\}\\ &= \left\{\omega\in\mathbb{C}: \left|\omega-\frac{1}{2\Re(\mu)}\right| = \frac{1}{2\Re(\mu)}\right\}.\end{aligned} \tag{59}$$

Similarly, the point spectrum is given by

$$\begin{aligned}\sigma_p(R(\mu,\Delta)) &= \left\{\frac{1}{\mu-z}: z\in\sigma_p(\Delta)\right\}\cup\{0\}\\ &= \left\{\omega\in\mathbb{C}: \left|\omega-\frac{1}{2\Re(\mu)}\right| = \frac{1}{2\Re(\mu)}\right\}.\end{aligned} \tag{60}$$

Therefore, $\sigma(R(\mu,\Delta)) = \sigma_p(R(\mu,\Delta)) = \{\omega \in \mathbb{C}: |\omega - (1/(2\Re(\mu)))| = (1/(2\Re(\mu)))\}$. Finally, we conclude this section by proving the spectral radius $r(R(\mu,\Delta)) = (1/(|\Re(\mu)|))$ and $\|R(\mu,\Delta)\| = (1/(|\Re(\mu)|))$.

It is clear that the spectrum of the resolvent is $r(R(\mu,\Delta)) = (1/(|\Re(\mu)|))$. Hille–Yosida theorem yields $r(R(\mu,\Delta)) = (1/(|\Re(\mu)|)) \le \|R(\mu,\Delta)\| \le (1/(|\Re(\mu)|))$. □

Data Availability

No data used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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