SU (N) generator-spectrum

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Abstract

This paper provides an accurate mathematical method for determining $SU(N)$ symmetry group generators in a general N-dimensional quantum state space. We identify the generators as well-defined quantum state transition and eigenvalue operators occurring in an orderly pattern within a system of $(N-1)$ *focal state transition spaces* specified by *focal state vectors* $|m\rangle$, $m = 2, 3, ..., N$ which constitute an $SU(N)$ generator-spectrum. Each focal state transition space specified by a focal state vector $|m\rangle$ (denoted by FSTS-$|m\rangle$) contains $2m-1$ *traceless* non-diagonal and diagonal symmetric and antisymmetric generators plus 1 *non-traceless* diagonal symmetric generator. The full $SU(N)$ generator-spectrum composed of an orderly pattern of $(N-1)$ focal state transition spaces contains a total of N^2-1 traceless non-diagonal and diagonal symmetric and antisymmetric generators plus $(N-1)$ non-traceless diagonal symmetric generators. A well-defined weighted sum of the $(N-1)$ non-traceless diagonal symmetric generators constitutes a completeness relation within the N-dimensional quantum state space of the $SU(N)$ symmetry group. Noting that each focal state transition space FSTS-$|m\rangle$ contains $m-1$ two-state subspaces, we determine an orderly distribution of the $\frac{1}{2}N(N-1)$ two-state subspaces among the $N-1$ focal state transition spaces in an $SU(N)$ generator-spectrum. Realizing that the basic $SU(N)$ generator-spectrum for $N \geq 3$ is not algebraically closed, we introduce "Cartan" and "conjugate-Cartan" generators which provide the desired closed $SU(N)$ algebra.

1 Introduction

The work in this paper is a follow up to earlier work [1 , 2] by the present author where an accurate mathematical method for determining all the standard N^2-1 traceless non-diagonal and diagonal symmetric and antisymmetric generators plus $(N-1)$ non-traceless diagonal symmetric generators of an $SU(N)$ symmetry group was developed. We provide a precise clarification of the procedure, leading to identification of an orderly arrangement of the generators in an $SU(N)$ generator-spectrum.

Considering that $SU(N)$ symmetry group elements generate transformations governing the dynamical evolution of interacting physical systems, we introduce an $SU(N)$ symmetry group quantum state space defined as an N-dimensional (integer $N = 2, 3, 4, ...$) state space specified by N mutually orthonormal state vectors $|n\rangle$, $n = 1, 2, 3, ..., N$ defined as column matrices, i.e., $N \times 1$ matrices, with entries 0 in all rows except entry 1 in the n-th row according to

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \quad ; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \quad ; \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix} \quad ; \quad .... \quad ; \quad |N-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ 0 \end{pmatrix} \quad ; \quad |N\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} \qquad (1a)$$

satisfying orthonormalization relation

$$\langle n|m\rangle = \delta_{nm} \qquad (1b)$$

These unit state vectors may specify either the energy level or spin angular momentum spectrum of a system of interacting particles and fields. The general dynamics of the system of particles and gauge fields in an N-state quantum space is characterized by transitions among the quantum states. To gain a clear understanding of the state transition processes, the N-state quantum space is decomposed into $\frac{1}{2}N(N-1)$ two-state subspaces, specified by paired state numbers nm, within which transitions occur, such that each transition couples only two states $|n\rangle$, $|m\rangle$ at a time. The decomposition of the N-state quantum space into $\frac{1}{2}N(N-1)$ two-state subspaces is determined by a rich spectrum of state transition processes, which we classify in two types, namely *random state* and *focal state* transition processes, where a *random state transition process* is composed of a collection of scattered transitions coupling random pairs nm , $n \neq m = 1,2,3,...,N$ of quantum states, while a *focal state transition process* specified by a *focal state* $|m\rangle$ is composed of a collection of $(m-1)$ focussed transitions from $(m-1)$ various states $|n\rangle$, $n = 1,2,3,...,m-1$ into the focal state $|m\rangle$, $m = 2,3,4,...,\ N$. A focal state is defined as a quantum state into which transitions from a specified number of different quantum states converge. There are a total of $(N-1)$ focal states $|m\rangle$, $m = 2,3,4,...,\ N$ within the N-state quantum space.

In a physical interpretation, a focal state transition process composed of a stream of electromagnetic radiation propagating from $(m-1)$ different states $|n\rangle$, $n = 1,2,....,m-1$ into a focal state $|m\rangle$, $m = 2,3,....,N$ within an N-state quantum space is equivalent to a stream of light rays from various sources converging at a focal point in classical geometrical optics, such that a focal state in the quantum state space corresponds to a focal point in classical geometrical optics.

2 Focal state transition spaces and the $SU(N)$ generator-spectrum

Considering that $SU(N)$ symmetry group generators are associated with the focussed (non-chaotic) transitions within focal state transition processes, we introduce a *focal state transition space* defined as a quantum subspace within which a focal state transition process comprising $(m-1)$ transitions into a focal state $|m\rangle$ occurs. A focal state transition space specified by a focal state vector $|m\rangle$ is thus composed of $(m-1)$ two-state subspaces each coupling a state $|n\rangle$, $n = 1,2,3,...,\ m-1$ to the focal state $|m\rangle$. The physical property that there are $(N-1)$ focal states each specifying a focal state transition space means that there are $(N-1)$ focal state transition spaces in the general N-state quantum space. In a group theoretic interpretation, a focal state transition space corresponds to a Cartan subspace defined by a Cartan subalgebra and the number $(N-1)$ of the focal state transition spaces corresponds to the rank of the underlying $SU(N)$ symmetry group.

Within each of the $(m-1)$ two-state subspaces specified by $|n\rangle$, $|m\rangle$ in a focal state transition space, the basic $N \times N$ matrices I_{nm} , σ^z_{nm} , σ^x_{nm} , σ^y_{nm} , $n = 1,2,3,...,m-1$, obtained as diagonal or non-diagonal symmetric and antisymmetric tensor products of the unit state vectors $|n\rangle$, $|m\rangle$ constitute a set of $SU(N)$ symmetry group generators within the focal state transition space. The full set of $SU(N)$ symmetry group generators formed in all the $(N-1)$ focal state transition spaces, each specified by focal state vector $|m\rangle$, $m = 2,3,4,...,\ N$, constitutes the $SU(N)$ symmetry group generator-spectrum, which we determine explicitly below.

In a two-state subspace $|n\rangle$, $|m\rangle$ within a focal state transition space specified by a focal state $|m\rangle$, we identify I_{nm} , σ^z_{nm} as the basic diagonal symmetric and antisymmetric $N \times N$ matrices obtained in the respective unit state vector tensor product forms

$$I_{nm} = |n\rangle\langle n| + |m\rangle\langle m| \qquad ; \qquad \sigma^z_{nm} = |n\rangle\langle n| - |m\rangle\langle m| \tag{2a}$$

and σ^x_{nm} , σ^y_{nm} as the basic non-diagonal symmetric and antisymmetric $N \times N$ matrices obtained in the respective unit state vector tensor product forms

$$\sigma^x_{nm} = |n\rangle\langle m| + |m\rangle\langle n| \qquad ; \qquad \sigma^y_{nm} = -i(|n\rangle\langle m| - |m\rangle\langle n|) \tag{2b}$$

where in the full N-state quantum space containing $(N-1)$ focal states $|m\rangle$, n takes values $n = 1,2,....,m-1$ for each $m = 2,3,....,N$. The indices x,y,z denote components in the Cartesian coordinate axes as usual. The imaginary number factor $-i$ in the definition of the non-diagonal antisymmetric matrix σ^y_{nm} effects the algebraic property that the symmetry group matrices are interpreted as Hermitian quantum operators. We observe that the non-diagonal symmetric and antisymmetric matrices σ^x_{nm} , σ^y_{nm} and the diagonal antisymmetric matrix σ^z_{nm} are *traceless*, but the diagonal symmetric matrix I_{nm} is *non-traceless*.

Applying the orthonormalization relation in equation $(1b)$, we identify the basic non-diagonal symmetric and antisymmetric matrices σ^x_{nm} , σ^y_{nm} defined in equation $(2b)$ as state transition operators generating state transition algebraic operations

$$\sigma^x_{nm}|n\rangle = |m\rangle \quad ; \quad \sigma^x_{nm}|m\rangle = |n\rangle \quad ; \quad \sigma^y_{nm}|n\rangle = i|m\rangle \quad ; \quad \sigma^y_{nm}|m\rangle = -i|n\rangle \tag{2c}$$

and the basic diagonal symmetric and antisymmetric $SU(2)$ matrices I_{nm} , σ^z_{nm} defined in equation $(2a)$ as state identity and eigenvalue operators, respectively, generating state transition algebraic operations

$$I_{nm}|n\rangle = |n\rangle \quad ; \quad I_{nm}|m\rangle = |m\rangle \quad ; \quad \sigma^z_{nm}|n\rangle = |n\rangle \quad ; \quad \sigma^z_{nm}|m\rangle = -|m\rangle \tag{2d}$$

The algebraic property that state transformations generated by elements of $SU(N)$ symmetry groups govern the dynamical evolution of particle and field interactions, taken together with the group theoretic interpretation that the $(N-1)$ focal state transition spaces within the N-state quantum space correspond to the $(N-1)$ Cartan subspaces of an $SU(N)$ symmetry group of rank $(N-1)$, means that the generators of the underlying $SU(N)$ symmetry group are composed of the basic diagonal and non-diagonal symmetric and antisymmetric $N \times N$ matrices I_{nm} , σ^z_{nm} , σ^x_{nm} , σ^y_{nm} determined as tensor products of the unit state vectors $|n\rangle$, $|m\rangle$ specifying the $(m-1)$ two-state subspaces within each of the $(N-1)$ focal state transition spaces according to equations $(2a)$, $(2b)$.

Due to their algebraic property as state transition operators, *all* the $(m-1)$ traceless non-diagonal symmetric matrices σ^x_{nm} and *all* the $(m-1)$ traceless non-diagonal antisymmetric matrices σ^y_{nm} defined in equation $(2b)$ form a set of $2(m-1)$ *traceless* non-diagonal symmetric and antisymmetric $SU(N)$ symmetry group generators $\lambda_j = \sigma^x_{nm}$, σ^y_{nm} within a focal state transition space specified by a focal state vector $|m\rangle$.

The algebraic property that the basic diagonal symmetric and antisymmetric matrices I_{nm} , σ^z_{nm} are the respective state identity and eigenvalue operators within each of the $(m-1)$ two-state subspaces $|n\rangle$, $|m\rangle$, each leaving the state vectors unchanged or only changing the sign of a state vector according to equations $(2c)$, $(2d)$, means that the basic matrices I_{nm} , σ^z_{nm} do not separately constitute the expected effective diagonal symmetric and antisymmetric $SU(N)$ generators within a focal state transition space. The algebraic property that each focal state transition space specified by a focal state $|m\rangle$ is composed of $(m-1)$ two-state subspaces each specified by a state $|n\rangle$, $n = 1, 2, 3, ...,\ m-1$ coupled to the focal state $|m\rangle$ means that an *effective* traceless diagonal antisymmetric $SU(N)$ generator Λ_{m-1} is obtained as a normalized sum of the basic traceless diagonal antisymmetric generators σ^z_{nm} from each of the $(m-1)$ two-state subspaces within the focal state transition space according to the composition formula

$$\Lambda_{m-1} = \sqrt{\frac{2}{m(m-1)}} \sum_{n=1}^{m-1} \sigma^z_{nm} \qquad ; \qquad m = 2, 3, ..., N \tag{2e}$$

with a corresponding *effective* non-traceless diagonal symmetric $SU(N)$ generator $\overline{\Lambda}_{m-1}$ obtained as a normalized sum of the basic non-traceless diagonal symmetric generators I_{nm} according to the composition formula

$$\overline{\Lambda}_{m-1} = \sqrt{\frac{2}{m(m-1)}} \sum_{n=1}^{m-1} I_{nm} \qquad ; \qquad m = 2, 3, ..., N \tag{2f}$$

We characterize the non-traceless diagonal symmetric generators $\overline{\Lambda}_{m-1}$ determined through the formula in equation $(2f)$ as the symmetric counterparts of the standard traceless diagonal antisymmetric generators Λ_{m-1} determined through the formula in equation $(2e)$. As presented in equation $(2h)$ below, the non-traceless diagonal symmetric generators $\overline{\Lambda}_{m-1}$ provide the completeness relation in the N-state quantum space of the $SU(N)$ symmetry group.

In summary, there are $2(m-1)+1 = 2m-1$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(N)$ symmetry group generators $\lambda_j = \sigma^x_{nm}$, σ^y_{nm} and Λ_{m-1}, plus 1 non-traceless diagonal symmetric generator $\overline{\Lambda}_{m-1}$ in each of the $(N-1)$ focal state transition spaces specified by a focal state $|m\rangle$, $m = 2, 3, 4, ..., N$, adding up to a total of N^2-1 traceless non-diagonal and diagonal symmetric and antisymmetric generators plus $(N-1)$ non-traceless diagonal symmetric generators of the $SU(N)$ group. The distribution of these generators among the $(N-1)$ focal state transition spaces specified by focal state vectors $|m\rangle$, each containing $2m-1$ traceless non-diagonal and diagonal symmetric and antisymmetric generators plus 1 non-traceless diagonal symmetric generator, constitutes an $SU(N)$ *generator-spectrum.* In this interpretation, a focal state transition space in an $SU(N)$ generator-spectrum corresponds to an orbital shell in an atomic energy-spectrum.

In specifying *all* the $(N^2-1)+(N-1) = (N+2)(N-1)$ traceless and non-traceless generators in the full $SU(N)$ generator-spectrum, we consider it necessary to introduce a revised notation, denoting the $N(N-1)$

traceless non-diagonal symmetric and antisymmetric generators σ^x_{nm} , σ^y_{nm} obtained using equation $(2b)$ by the usual Gell-Mann symbols λ_j, $j = 1, 2, ..., N(N-1)$ in an ascending order through the $(N-1)$ focal state transition spaces, the $(N-1)$ traceless diagonal antisymmetric generators obtained as normalized sums of the basic traceless diagonal antisymmetric generators σ^z_{nm} using equations $(2a)$, $(2e)$ by the upper case symbols Λ_k , $k = 1, 2, ..., N-1$ and the $(N-1)$ non-traceless diagonal symmetric generators obtained as normalized sums of the basic non-traceless diagonal generators I_{nm} using equations $(2a)$, $(2f)$ by the upper case symbols $\overline{\Lambda}_k$, $k = 1, 2, ..., N-1$. We observe that only the $N(N-1) + (N-1) = N^2 - 1$ traceless non-diagonal and diagonal symmetric and antisymmetric generators λ_j , Λ_k , $j = 1, 2, ...$, $N(N-1)$, $k = 1, 2, ...$, $(N-1)$ are generally known to be the standard form of the full set of generators of an $SU(N)$ symmetry group [3-15]. The $(N-1)$ non-traceless diagonal symmetric generators $\overline{\Lambda}_k$, $k = 1, 2, ...$, $(N-1)$ emerged for the first time in [1] and have been elaborated in a recent article [2] as the symmetric counterparts of the standard traceless diagonal antisymmetric generators Λ_k , $k = 1, 2, ...$, $(N-1)$. It turns out that the non-traceless diagonal symmetric generators $\overline{\Lambda}_k$, $k = 1, 2, ...$, $(N-1)$ complete the specification of the $SU(N)$ generator-spectrum by providing the completeness relation in the N-state quantum space of the $SU(N)$ symmetry group.

All the $N(N-1) + N - 1 = N^2 - 1$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(N)$ generators λ_1 , $\lambda_2, ..., \lambda_{N(N-1)}$, Λ_1 , $\Lambda_2, ..., \Lambda_{N-1}$ obtained using equations $(2a)$, $(2b)$, $(2e)$ satisfy the standard $SU(N)$ generator normalization conditions $(i, j = 1, 2, ..., N(N-1)$; $k, l = 1, 2, ..., (N-1))$

$$\mathrm{Tr}\lambda_j = 0 \quad ; \quad \mathrm{Tr}\Lambda_k = 0 \quad ; \quad \mathrm{Tr}\lambda_i\lambda_j = 2\delta_{ij} \quad ; \quad \mathrm{Tr}\Lambda_k\Lambda_l = 2\delta_{kl} \quad ; \quad \mathrm{Tr}\lambda_j\Lambda_k = 0 \qquad (2g)$$

while the $(N-1)$ non-traceless diagonal symmetric generators $\overline{\Lambda}_k$, $k = 1, 2, ...$, $(N-1)$ in the $SU(N)$ generator-spectrum provide the completeness relation in the N-state quantum space of the $SU(N)$ symmetry group obtained in the form

$$\sum_{m=2}^{N} \sqrt{\frac{1}{2}m(m-1)}\ \overline{\Lambda}_{m-1} = (N-1)\ I \qquad ; \qquad I = N \times N \text{ identity matrix} \qquad (2h)$$

The set of equations $(1a)$, $(1b)$ and $(2a)$-$(2h)$ provide a complete specification and normalization properties of an $SU(N)$ generator-spectrum.

For completeness, we clarify the notation for the generators in an $SU(N)$ generator spectrum. We start by noting that a two-state subspace $\{\ |n\rangle\ ,\ |m\rangle\ \}$ is specified by a set of four basic matrices $I_{nm}, \sigma^x_{nm}, \sigma^y_{nm}, \sigma^z_{nm}$ satisfying an $SU(2)$ algebra, represented in the form

$$\{\ |n\rangle\ ,\ |m\rangle\ \} = \{\ I_{nm}, \sigma^x_{nm}, \sigma^y_{nm}, \sigma^z_{nm}\ \} \qquad ; \qquad \mathrm{SU(2)} \qquad (3a)$$

For brevity, we introduce an abbreviation FSTS-$|m\rangle$ to represent a focal state transition space specified by a focal state vector $|m\rangle$. The $(m-1)$ two-state subspaces $\{\ |n\rangle\ ,\ |m\rangle\ \}$, $n = 1, 2, ..., m-1$, $m = 2, 3, ..., N$ in an FSTS-$|m\rangle$ are specified as

$$FSTS - |m\rangle :\ \{\ |1\rangle\ ,\ |m\rangle\ \} \quad , \quad \{\ |2\rangle\ ,\ |m\rangle\ \} \quad , \quad .... \quad , \quad \{\ |m-1\rangle\ ,\ |m\rangle\ \} \qquad (3b)$$

In an $SU(N)$ generator-spectrum, the $2m-1$ traceless non-diagonal and diagonal symmetric and antisymmetric generators, plus the 1 non-traceless diagonal symmetric generator determined from the $(m-1)$ two-state subspaces $\{\ |n\rangle\ ,\ |m\rangle\ \}$, $n = 1, 2, ..., m-1$, $m = 2, 3, ..., N$ in a focal state transition space specified by a focal state vector $|m\rangle$ are determined and denoted according to the definitions

$$FSTS - |m\rangle$$

$$\{\ |1\rangle\ ,\ |m\rangle\ \} = (\ I_{1m}, \sigma^x_{1m}, \sigma^y_{1m}, \sigma^z_{1m}\ ) \quad ; \quad \lambda_{(m-1)(m-2)+1} = \sigma^x_{1m}\ ,\ \lambda_{(m-1)(m-2)+2} = \sigma^y_{1m}$$

$$\{\ |2\rangle\ ,\ |m\rangle\ \} = (\ I_{2m}, \sigma^x_{2m}, \sigma^y_{2m}, \sigma^z_{2m}\ ) \quad ; \quad \lambda_{(m-1)(m-2)+3} = \sigma^x_{2m}\ ,\ \lambda_{(m-1)(m-2)+4} = \sigma^y_{2m}$$

$$.....$$

$$.....$$

$$\{\ |m-1\rangle\ ,\ |m\rangle\ \} = (\ I_{(m-1)m}, \sigma^x_{(m-1)m}, \sigma^y_{(m-1)m}, \sigma^z_{(m-1)m}\ )$$

$$\lambda_{m(m-1)-1} = \sigma^x_{(m-1)m}\ ,\ \lambda_{m(m-1)} = \sigma^y_{(m-1)m}$$

$$\Lambda_{m-1} = \sqrt{\frac{2}{m(m-1)}} \; \left(\sigma^z_{1m} + \sigma^z_{2m} + ... + \sigma^z_{(m-1)m} \right)$$

$$\overline{\Lambda}_{m-1} = \sqrt{\frac{2}{m(m-1)}} \; \left(I_{1m} + I_{2m} + ... + I_{(m-1)m} \right) \qquad (3c)$$

2.1 General $SU(N)$ generator-spectrum

The N-dimensional quantum state space of a general $SU(N)$ symmetry group generator-spectrum is specified by

$$N \geq 2 \;:\quad n = 1, 2, 3, ..., m-1 \quad ; \quad m = 2, 3, 4, ..., N$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \;;\; |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \;;\; |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix} \;;\; .... \;;\; |N-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ 0 \end{pmatrix} \;;\; |N\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} \qquad (1a')$$

An $SU(N)$ generator-spectrum specified by $N-1$ focal state vectors $|m\rangle$, $m = 2, 3, ..., N$ is composed of $N-1$ focal state transition spaces FSTS-$|m\rangle$, the first specified by $|2\rangle$ denoted as FSTS-$|2\rangle$, the second specified by $|3\rangle$ denoted as FSTS-$|3\rangle$, so on up to the last focal state transition space specified by $|N\rangle$ denoted as FSTS-$|N\rangle$.

We apply the state vector tensor product relations in equations $(2a)$, $(2b)$ and the formulae in equations $(2e)$, $(2f)$ using the unit state vectors defined in equation $(1a')$ to determine the complete generator-spectrum for a general $SU(N)$ $(N \geq 2)$ symmetry group. Following the definitions elaborated in equation $(3c)$ above, the generators are arranged within the $N-1$ focal state transition spaces which constitute the full $SU(N)$ generator-spectrum. We use the abbreviation FSTS-$|m\rangle$ defined above for a focal state transition space specified by focal state vector $|m\rangle$ in the generator-spectrum.

The N^2-1 traceless non-diagonal and diagonal symmetric and antisymmetric generators $\lambda_1, ..., \lambda_{N(N-1)}, \Lambda_1, ..., \Lambda_{N-1}$ in a general $SU(N)$ generator-spectrum are obtained as

$$m = 2 \;:\; FSTS - |2\rangle \quad \begin{cases} \lambda_1 = \sigma^x_{12} \quad ; \quad \lambda_2 = \sigma^y_{12} \\ \qquad \Lambda_1 = \sigma^z_{12} \end{cases}$$

$$m = 3 \;:\; FSTS - |3\rangle \quad \begin{cases} \lambda_3 = \sigma^x_{13} \quad ; \quad \lambda_4 = \sigma^y_{13} \\ \lambda_5 = \sigma^x_{23} \quad ; \quad \lambda_6 = \sigma^y_{23} \\ \qquad \Lambda_2 = \frac{1}{\sqrt{3}} \left(\sigma^z_{13} + \sigma^z_{23} \right) \end{cases}$$

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$$FSTS - |m\rangle \quad \begin{cases} \lambda_{(m-1)(m-2)+1} = \sigma^x_{1m} \quad ; \quad \lambda_{(m-1)(m-2)+2} = \sigma^y_{1m} \\ \lambda_{(m-1)(m-2)+3} = \sigma^x_{2m} \quad ; \quad \lambda_{(m-1)(m-2)+4} = \sigma^y_{2m} \\ ..... \\ ..... \\ \lambda_{m(m-1)-1} = \sigma^x_{(m-1)m} \quad ; \quad \lambda_{m(m-1)} = \sigma^y_{(m-1)m} \\ \qquad \Lambda_{m-1} = \sqrt{\frac{2}{m(m-1)}} \; \sum_{n=1}^{m-1} \sigma^z_{nm} \end{cases}$$

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$$FSTS-|N\rangle \quad \begin{cases} \lambda_{(N-1)(N-2)+1} = \sigma^x_{1N} \quad ; \quad \lambda_{(N-1)(N-2)+2} = \sigma^y_{1N} \\ \lambda_{(N-1)(N-2)+3} = \sigma^x_{2N} \quad ; \quad \lambda_{(N-1)(N-2)+4} = \sigma^y_{2N} \\ ..... \\ ..... \\ \lambda_{N(N-1)-1} = \sigma^x_{(N-1)N} \quad ; \quad \lambda_{N(N-1)} = \sigma^y_{(N-1)N} \\ \qquad \Lambda_{N-1} = \sqrt{\frac{2}{N(N-1)}} \ \sum_{n=1}^{N-1} \sigma^z_{nN} \end{cases} \tag{4a}$$

The $N-1$ non-traceless diagonal symmetric generators $\overline{\Lambda}_1, ..., \overline{\Lambda}_{N-1}$ are obtained as

$$m = 2,3,4,...,N \ : \quad \overline{\Lambda}_{m-1} = \sqrt{\frac{2}{m(m-1)}} \sum_{n=1}^{m-1} I_{nm}$$

$$\sum_{m=2}^{N} \sqrt{\frac{1}{2}m(m-1)} \ \overline{\Lambda}_{m-1} = (N-1)I \qquad ; \qquad I = N \times N \text{ identity matrix} \tag{4b}$$

As an illustration, the $SU(2)$ $(N=2: \ n=1 \ , m=2)$ generator-spectrum consists of $2-1=1$ focal state transition space specified by $|2\rangle$ containing a total of $(2.2-1)=3$ traceless generators ($\lambda_1 = \sigma^x_{12}, \lambda_2 = \sigma^y_{12}$) , $\Lambda_1 = \sigma^z_{12}$, plus 1 non-traceless generator $\overline{\Lambda}_1 = I_{12}$, while the $SU(3)$ $(N=3: \ n=1,2 \ , \ m=2,3)$ generator-spectrum consists of $3-1=2$ focal state transition spaces, the first specified by $|2\rangle$ contains $(2.2-1)=3$ traceless generators ($\lambda_1 = \sigma^x_{12}, \lambda_2 = \sigma^y_{12}$) , $\Lambda_1 = \sigma^z_{12}$, plus 1 non-traceless generator $\overline{\Lambda}_1 = I_{12}$, the second specified by $|3\rangle$ contains $(2.3-1)=5$ traceless generators ($\lambda_3 = \sigma^x_{13}, \lambda_4 = \sigma^y_{13}$) , ($\lambda_5 = \sigma^x_{23}, \lambda_6 = \sigma^y_{23}$) , $\Lambda_2 = \sqrt{2/m(m-1)}(\ \sigma^z_{13} + \sigma^z_{23} \ )$, plus 1 non-traceless generator $\overline{\Lambda}_2 = \sqrt{2/m(m-1)}(\ I_{13} + I_{23} \ )$, giving a total of 8 traceless generators and 2 non-traceless generators. We note that, as determined explicitly in the examples given below, the $I_{12}, \sigma^x_{12}, \sigma^y_{12}, \sigma^z_{12}$ in the $SU(2)$ generator-spectrum are 2×2 matrices, while the $I_{12}, \sigma^x_{12}, \sigma^y_{12}, \sigma^z_{12}$ in the $SU(3)$ generator-spectrum are 3×3 matrices.

We now set $N = 2,3,4,5,6,7$ in equations $(4a)$, $(4b)$ and substitute the respective unit state vectors into the tensor product definitions in equations $(2a)$, $(2b)$ to determine the explicit $N \times N$ matrix forms of the generator-spectra of the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$, $SU(6)$, $SU(7)$ symmetry groups which have commonly been used as algebraic frameworks for formulating models of gauge theories, including grand unified theories, of elementary particle interactions [3-15] in quantum field theory. According to the general form in equation $(4a)$, the generators are arranged within the $N-1$ focal state transition spaces FSTS-$|m\rangle$ which constitute the complete generator-spectrum.

2.1.1 $SU(2)$ generator-spectrum

$$N=2 \ : \quad n=1 \quad ; \quad m=2 \ : \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad ; \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The $2^2-1=3$ traceless non-diagonal and diagonal symmetric and antisymmetric generators $\lambda_1, \lambda_2, \Lambda_1$ in the $SU(2)$ generator-spectrum are obtained as

$$m=2 \ : \ FSTS-|2\rangle \quad \begin{cases} \lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \lambda_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \qquad \Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases} \tag{5a}$$

The $2-1=1$ non-traceless diagonal symmetric generator $\overline{\Lambda}_1$ is obtained as

$$\overline{\Lambda}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad ; \qquad \overline{\Lambda}_1 = I \tag{5b}$$

2.1.2 $SU(3)$ generator-spectrum

$$N=3\ :\quad n=1\ ,\ 2\quad ;\quad m=2\ ,\ 3\ :\qquad |1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad ;\quad |2\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad ;\quad |3\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

The $3^2-1=8$ traceless non-diagonal and diagonal symmetric and antisymmetric generators $\lambda_1,...,\lambda_6,\Lambda_1,\Lambda_2$ in the $SU(3)$ generator-spectrum are obtained as

$$m=2\ :\ FSTS-|2\rangle\quad\begin{cases}\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\quad ;\quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\\ \Lambda_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\end{cases}$$

$$m=3\ :\ FSTS-|3\rangle\quad\begin{cases}\lambda_3=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\quad ;\quad \lambda_4=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\\ \lambda_5=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad ;\quad \lambda_6=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\\ \Lambda_2=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\end{cases}$$

$$(6a)$$

The $3-1=2$ non-traceless diagonal symmetric generators $\overline{\Lambda}_1,\overline{\Lambda}_2$ are obtained as

$$\overline{\Lambda}_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\quad ;\quad \overline{\Lambda}_2=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\quad ;\quad \overline{\Lambda}_1+\sqrt{3}\,\overline{\Lambda}_2=2I \qquad (6b)$$

We observe that all the 8 generators in the $SU(3)$ generator-spectrum determined here in equation (6a) are exactly equal to the corresponding $SU(3)$ generators in the original Gell-Man's work, standard textbooks [3-6] and the general literature on the quark model in quantum field theory.

2.1.3 $SU(4)$ generator-spectrum

$$N=4\ :\quad n=1\ ,\ 2\ ,\ 3\quad ;\quad m=2\ ,\ 3\ ,\ 4$$

$$|1\rangle=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad ;\quad |2\rangle=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad ;\quad |3\rangle=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\quad ;\quad |4\rangle=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The $4^2-1=15$ traceless non-diagonal and diagonal symmetric and antisymmetric generators $\lambda_1,...,\lambda_{12},\Lambda_1,...,\Lambda_3$ in the $SU(4)$ generator-spectrum are obtained as

$$m=2\ :\ FSTS-|2\rangle\quad\begin{cases}\lambda_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad ;\quad \lambda_2=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\\ \Lambda_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{cases}$$

$$m=3\ :\ FSTS-|3\rangle \quad \begin{cases} \lambda_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \lambda_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \Lambda_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$m=4\ :\ FSTS-|4\rangle \quad \begin{cases} \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_8 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\ \lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \\ \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad ; \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\ \Lambda_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \end{cases}$$

$$(7a)$$

The $4-1=3$ non-traceless diagonal symmetric generators $\overline{\Lambda}_1, ..., \overline{\Lambda}_3$ are obtained as

$$\overline{\Lambda}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\overline{\Lambda}_1 + \sqrt{3}\ \overline{\Lambda}_2 + \sqrt{6}\ \overline{\Lambda}_3 = 3I \qquad (7b)$$

We observe that all the 15 generators in the $SU(4)$ generator-spectrum determined here in equation $(7a)$ are exactly equal to the corresponding $SU(4)$ generators obtained in [6 , 10 , 16] using an ad-hoc pattern building procedure explained in those works.

2.1.4 $SU(5)$ generator-spectrum

$$N=5\ :\quad n=1\ ,\ 2\ ,\ 3\ ,\ 4\quad ;\quad m=2\ ,\ 3\ ,\ 4\ ,\ 5$$

$$|1\rangle=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\quad ;\quad |2\rangle=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\quad ;\quad |3\rangle=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}\quad ;\quad |4\rangle=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\quad ;\quad |5\rangle=\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

The $5^2-1=24$ traceless non-diagonal and diagonal symmetric and antisymmetric generators $\lambda_1,...,\lambda_{20},\Lambda_1,...,\Lambda_4$ in the $SU(5)$ generator-spectrum are obtained as

$$m=2\ :\ FSTS-|2\rangle\quad \begin{cases}\lambda_1=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\quad ;\quad \lambda_2=\begin{pmatrix}0&-i&0&0&0\\i&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\\ \Lambda_1=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\end{cases}$$

$$m=3\ :\ FSTS-|3\rangle\quad \begin{cases}\lambda_3=\begin{pmatrix}0&0&1&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\quad ;\quad \lambda_4=\begin{pmatrix}0&0&-i&0&0\\0&0&0&0&0\\i&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\\ \lambda_5=\begin{pmatrix}0&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\quad ;\quad \lambda_6=\begin{pmatrix}0&0&0&0&0\\0&0&-i&0&0\\0&i&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\\ \Lambda_2=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&-2&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\end{cases}$$

$$
m=4\ :\ FSTS-|4\rangle \quad \left\{ \begin{array}{l}
\lambda_7 = \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_8 = \begin{pmatrix} 0&0&0&-i&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ i&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \\
\lambda_9 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{10} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&-i&0\\ 0&0&0&0&0\\ 0&i&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \\
\lambda_{11} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{12} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-i&0\\ 0&0&i&0&0\\ 0&0&0&0&0 \end{pmatrix} \\
\Lambda_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&-3&0\\ 0&0&0&0&0 \end{pmatrix}
\end{array} \right.
$$

$$
m=5\ :\ FSTS-|5\rangle \quad \left\{ \begin{array}{l}
\lambda_{13} = \begin{pmatrix} 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{14} = \begin{pmatrix} 0&0&0&0&-i\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ i&0&0&0&0 \end{pmatrix} \\
\lambda_{15} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{16} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&-i\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&i&0&0&0 \end{pmatrix} \\
\lambda_{17} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix} \quad ; \quad \lambda_{18} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-i\\ 0&0&0&0&0\\ 0&0&i&0&0 \end{pmatrix} \\
\lambda_{19} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix} \quad ; \quad \lambda_{20} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-i\\ 0&0&0&i&0 \end{pmatrix} \\
\Lambda_4 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&-4 \end{pmatrix}
\end{array} \right. \qquad (8a)
$$

The $5-1=4$ non-traceless diagonal symmetric generators $\overline{\Lambda}_1, ..., \overline{\Lambda}_4$ are obtained as

$$
\overline{\Lambda}_1 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&2&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}
$$

$$\overline{\Lambda}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_4 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\overline{\Lambda}_1 + \sqrt{3}\,\overline{\Lambda}_2 + \sqrt{6}\,\overline{\Lambda}_3 + \sqrt{10}\,\overline{\Lambda}_4 = 4I \tag{8b}$$

It emerges here that the incorrect forms of the two traceless diagonal antisymmetric generators L^{11} , L^{12} determined and used in [6-9 , 11 , 12] and other related work in formulating the $SU(5)$ Grand Unified Theory have to be replaced with the correct traceless diagonal antisymmetric $SU(5)$ generators Λ_3 , Λ_4, particularly noting that in the 5-representation of fermions as defined in [6-9 , 11 , 12], the charge operator Q determined there as a linear combination of the generators L^{11} and L^{12} in the form $Q = \frac{1}{2}\left(L^{12} + \sqrt{\frac{5}{3}}L^{12}\right)$ is completely specified by the correct traceless diagonal antisymmetric generator Λ_3 obtained here in equation $(8a)$ in the form

$$\Lambda_3 = \sqrt{\frac{3}{2}} \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad Q = -\sqrt{\frac{2}{3}}\,\Lambda_3 \tag{8c}$$

In general, the determination of the correct generator-spectrum achieved in the present and earlier work [2], means that the identification of various types of elementary particle states, fermions or bosons, with the $SU(5)$ generators, together with the value of the weak-interaction angle (sometimes called Weinberg angle) parameter $\sin^2\theta_W$ predicted within the $SU(5)$ grand unified theory, may change radically.

2.1.5 $SU(6)$ generator-spectrum

$$N = 6 \;:\quad n = 1,2,3,4,5 \quad ; \quad m = 2,3,4,5,6$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \; |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \; |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \; |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \; |5\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \; |6\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The $6^2 - 1 = 35$ traceless non-diagonal and diagonal symmetric and antisymmetric generators $\lambda_1, ..., \lambda_{30}, \Lambda_1, ..., \Lambda_5$ in the $SU(6)$ generator-spectrum are obtained as

$$m = 2 \;:\; FSTS - |2\rangle \quad \begin{cases} \lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \Lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$
m=3\ :\ FSTS-|3\rangle \quad
\begin{cases}
\lambda_3 = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad ; \quad
\lambda_4 = \begin{pmatrix} 0&0&-i&0&0&0\\ 0&0&0&0&0&0\\ i&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \\
\lambda_5 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad ; \quad
\lambda_6 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&-i&0&0&0\\ 0&i&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \\
\Lambda_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&-2&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}
\end{cases}
$$

$$
m=4\ :\ FSTS-|4\rangle \quad
\begin{cases}
\lambda_7 = \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad ; \quad
\lambda_8 = \begin{pmatrix} 0&0&0&-i&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ i&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \\
\lambda_9 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad ; \quad
\lambda_{10} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&-i&0&0\\ 0&0&0&0&0&0\\ 0&i&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \\
\lambda_{11} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad ; \quad
\lambda_{12} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&-i&0&0\\ 0&0&i&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \\
\Lambda_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&-3&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}
\end{cases}
$$

$$m=5\ :\ FSTS-|5\rangle \quad \begin{cases} \lambda_{13}=\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix} \quad ; \quad \lambda_{14}=\begin{pmatrix}0&0&0&0&-i&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\i&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix} \\ \lambda_{15}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix} \quad ; \quad \lambda_{16}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&-i&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&i&0&0&0&0\\0&0&0&0&0&0\end{pmatrix} \\ \lambda_{17}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\end{pmatrix} \quad ; \quad \lambda_{18}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-i&0\\0&0&0&0&0&0\\0&0&i&0&0&0\\0&0&0&0&0&0\end{pmatrix} \\ \lambda_{19}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix} \quad ; \quad \lambda_{20}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-i&0\\0&0&0&i&0&0\\0&0&0&0&0&0\end{pmatrix} \\ \Lambda_4=\frac{1}{\sqrt{10}}\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-4&0\\0&0&0&0&0&0\end{pmatrix} \end{cases}$$

$$m=6\ :\ FSTS-|5\rangle \quad \left\{ \begin{array}{l}
\lambda_{21}=\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{22}=\begin{pmatrix} 0&0&0&0&0&-i\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ i&0&0&0&0&0 \end{pmatrix} \\
\lambda_{23}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{24}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&-i\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&i&0&0&0&0 \end{pmatrix} \\
\lambda_{25}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{26}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-i\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&i&0&0&0 \end{pmatrix} \\
\lambda_{27}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix} \quad ; \quad \lambda_{28}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-i\\ 0&0&0&0&0&0\\ 0&0&0&i&0&0 \end{pmatrix} \\
\lambda_{29}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix} \quad ; \quad \lambda_{30}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-i\\ 0&0&0&0&i&0 \end{pmatrix} \\
\Lambda_5=\frac{1}{\sqrt{15}}\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&-5 \end{pmatrix}
\end{array} \right. \tag{9a}$$

The $6-1=5$ non-traceless diagonal symmetric generators $\overline{\Lambda}_1,...,\overline{\Lambda}_5$ are obtained as

$$\overline{\Lambda}_1=\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_2=\frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&2&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$$

$$\overline{\Lambda}_3=\frac{1}{\sqrt{6}}\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&3&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_4=\frac{1}{\sqrt{10}}\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&4&0\\ 0&0&0&0&0&0 \end{pmatrix}$$

$$\overline{\Lambda}_5 = \frac{1}{\sqrt{15}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \quad ; \quad \overline{\Lambda}_1 + \sqrt{3}\,\overline{\Lambda}_2 + \sqrt{6}\,\overline{\Lambda}_3 + \sqrt{10}\,\overline{\Lambda}_4 + \sqrt{15}\,\overline{\Lambda}_5 = 5I \qquad (9b)$$

Here again, we observe that the $SU(6)$ grand unified theory formulated in [13] and related works must be reviewed to take account of the correct $SU(6)$ generator-spectrum determined here in equation $(9a)$.

2.1.6 $SU(7)$ generator-spectrum

$$N = 7 \; : \quad n = 1\,,\; 2\,,\; 3\,,\; 4\,,\; 5\,,\; 6 \quad ; \quad m = 2\,,\; 3\,,\; 4\,,\; 5\,,\; 6\,,\; 7$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; |5\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; ; \; |6\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; |7\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The $7^2 - 1 = 48$ traceless non-diagonal and diagonal symmetric and antisymmetric generators $\lambda_1, ..., \lambda_{42}, \Lambda_1, ..., \Lambda_6$ in the $SU(7)$ generator-spectrum are obtained as

$$m = 2 \; : \; FSTS - |2\rangle \left\{ \begin{array}{l} \lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \Lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \right.$$

$$m = 3 \ : \ FSTS - |3\rangle \quad \left\{ \begin{array}{l}
\lambda_3 = \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_4 = \begin{pmatrix} 0&0&-i&0&0&0&0\\ 0&0&0&0&0&0&0\\ i&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \\
\lambda_5 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_6 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&-i&0&0&0&0\\ 0&i&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \\
\Lambda_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&-2&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}
\end{array} \right.$$

$$m = 4 \ : \ FSTS - |4\rangle \quad \left\{ \begin{array}{l}
\lambda_7 = \begin{pmatrix} 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_8 = \begin{pmatrix} 0&0&0&-i&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ i&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \\
\lambda_9 = \begin{pmatrix} 0&0&0&0&0&0&\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{10} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&-i&0&0&0\\ 0&0&0&0&0&0&0\\ 0&i&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \\
\lambda_{11} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{12} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-i&0&0&0\\ 0&0&i&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \\
\Lambda_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&-3&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}
\end{array} \right.$$

$$
m=5\ :\ FSTS-|5\rangle \quad \begin{cases}
\lambda_{13}=\begin{pmatrix}0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix} \quad ; \quad \lambda_{14}=\begin{pmatrix}0&0&0&0&-i&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\i&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\\
\lambda_{15}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix} \quad ; \quad \lambda_{16}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&-i&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&i&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\\
\lambda_{17}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix} \quad ; \quad \lambda_{18}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-i&0&0\\0&0&0&0&0&0&0\\0&0&i&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\\
\lambda_{19}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix} \quad ; \quad \lambda_{20}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-i&0&0\\0&0&0&i&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\\
\Lambda_4=\frac{1}{\sqrt{10}}\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&-4&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}
\end{cases}
$$

$$
m=6\ :\ FSTS-|6\rangle \quad \left\{
\begin{array}{ccc}
\lambda_{21}=\begin{pmatrix}0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix} & ; & \lambda_{22}=\begin{pmatrix}0&0&0&0&0&-i&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\i&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\\
\lambda_{23}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix} & ; & \lambda_{24}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&-i&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&i&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\\
\lambda_{25}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix} & ; & \lambda_{26}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-i&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&i&0&0&0&0\\0&0&i&0&0&0&0\end{pmatrix}\\
\lambda_{27}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&0\end{pmatrix} & ; & \lambda_{28}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-i&0\\0&0&0&0&0&0&0\\0&0&0&i&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\\
\lambda_{29}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\end{pmatrix} & ; & \lambda_{30}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-i&0\\0&0&0&0&i&0&0\\0&0&0&0&0&0&0\end{pmatrix}\\
\multicolumn{3}{c}{\Lambda_5=\frac{1}{\sqrt{15}}\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&-5&0\\0&0&0&0&0&0&0\end{pmatrix}}
\end{array}\right.
$$

$$
m=7\ :\ FSTS-|7\rangle \quad \left\{
\begin{array}{lcl}
\lambda_{31}=\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\end{pmatrix} & ; & \lambda_{32}=\begin{pmatrix}0&0&0&0&0&0&-i\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\i&0&0&0&0&0&0\end{pmatrix}\\
\lambda_{33}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\end{pmatrix} & ; & \lambda_{34}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&-i\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&i&0&0&0&0&0\end{pmatrix}\\
\lambda_{35}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\end{pmatrix} & ; & \lambda_{36}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&-i\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&i&0&0&0&0\end{pmatrix}\\
\lambda_{37}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&1&0&0&0\end{pmatrix} & ; & \lambda_{38}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&-i\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-i&0&0&0\end{pmatrix}\\
\lambda_{39}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\end{pmatrix} & ; & \lambda_{40}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&-i\\0&0&0&0&0&0&0\\0&0&0&0&i&0&0\end{pmatrix}\\
\lambda_{41}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0\end{pmatrix} & ; & \lambda_{42}=\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-i&0\\0&0&0&0&0&i&0\end{pmatrix}\\
\multicolumn{3}{c}{\Lambda_6=\frac{1}{\sqrt{21}}\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&-6\end{pmatrix}}
\end{array}\right.
\tag{10a}
$$

The $7-1=6$ non-traceless diagonal symmetric generators $\overline{\Lambda}_1,...,\overline{\Lambda}_6$ are obtained as

$$\overline{\Lambda}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_4 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_5 = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_6 = \frac{1}{\sqrt{21}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$\overline{\Lambda}_1 + \sqrt{3}\,\overline{\Lambda}_2 + \sqrt{6}\,\overline{\Lambda}_3 + \sqrt{10}\,\overline{\Lambda}_4 + \sqrt{15}\,\overline{\Lambda}_5 + \sqrt{21}\,\overline{\Lambda}_6 = 6I \tag{10b}$$

The determination of $SU(N)$ symmetry group generators as symmetric and antisymmetric tensor products of orthonormal state vectors spanning two-state subspaces defined within focal state transition spaces specified by focal state vectors in a general N-state quantum space gives both algebraic and physical structure to the full set of generators taking a well defined form of group generator-spectrum. The focal state transition spaces which contain specified numbers of group generators correspond to orbital shells in an atomic energy-spectrum. Defining the rank r of an $SU(N)$ symmetry group as the number of focal state transition spaces ($r = N-1$) containing a specified number of generators in the group generator-spectrum, it is evident in equations ($4a$)-($9b$) that the generator-spectrum of a higher rank $SU(N)$ group contains the generator-spectrum of a lower rank group. The number of focal state transition spaces in group generator-spectrum increases in unit steps as symmetry group advances progressively from $SU(2)$ to $SU(N>2)$. It emerges that all $(N-1)^2-1$ traceless non-diagonal and diagonal symmetric and antisymmetric generators, plus the $N-2$ non-traceless diagonal symmetric generators in the lower rank ($r = N-2$) $SU(N-1)$ generator-spectrum are contained in the corresponding set of N^2-1 traceless non-diagonal and diagonal symmetric and antisymmetric generators plus the $N-1$ non-traceless diagonal symmetric generators in the higher rank ($r = N-1$) $SU(N)$ generator-spectrum, noting that in this case, each $(N-1)\times(N-1)$ generator in the $SU(N-1)$ generator-spectrum is extended to a corresponding $N \times N$ generator in the $SU(N)$ generator-spectrum by simply adding a column of entries 0 to the right and a row of entries 0 at the bottom as clearly evident in the set of equations ($5a$)-($10b$). In addition to the $N-2$ focal state transition spaces containing a total of $(N-1)^2-1$ generators in the lower rank $SU(N-1)$ sector, the $SU(N)$ generator-spectrum is completed by an additional focal state transition space FSTS-$|N\rangle$ containing $2N-1$ distinct traceless non-diagonal and diagonal symmetric and antisymmetric generators, making a total of $(N-1)^2-1+(2N-1) = N^2-1$ as expected.

In general, it turns out that generators of an $SU(N)$ symmetry group are well defined quantum operators falling into a beautiful pattern composed of a system of $N-1$ focal state transition spaces FSTS-$|m\rangle$, $m = 2,3,...,N$, each containing a specified number $(2m-1)$ of generators. The generators are formed within $m-1$ two-state subspaces contained in each focal state transition space FSTS-$|m\rangle$ as explained earlier in accordance with equations ($3a$)-($3c$). The orderly arrangement of $SU(N)$ generators in a pattern similar to an orbital shell system in an atomic energy-spectrum then means that the two-state subspaces also form an orderly pattern within the N-dimensional quantum state space of the $SU(N)$ symmetry group, which we present below.

2.2 Distribution of two-state subspaces in $SU(N)$ generator-spectrum

The N-dimensional $SU(N)$ symmetry group quantum state space contains $\frac{1}{2}N(N-1)$ two-state subspaces distributed among the $N-1$ focal state transition spaces FSTS-$|m\rangle$ each containing $m-1$ subspaces according to the general $SU(N)$ subspace-spectrum

$$m=2 \;:\; FSTS-|2\rangle \;;\; 1 \text{ subspace}: \quad \{\,\{\ |1\rangle\ ,\ |2\rangle\ \} = (\ I_{12}, \sigma_{12}^x, \sigma_{12}^y, \sigma_{12}^z\ )$$

$$m=3 \;:\; FSTS-|3\rangle \;;\; 2 \text{ subspaces}: \quad \begin{cases} \{\ |1\rangle\ ,\ |3\rangle\ \} = (\ I_{13}, \sigma_{13}^x, \sigma_{13}^y, \sigma_{13}^z\ ) \\ \{\ |2\rangle\ ,\ |3\rangle\ \} = (\ I_{23}, \sigma_{23}^x, \sigma_{23}^y, \sigma_{23}^z\ ) \end{cases}$$

$$m=4 \;:\; FSTS-|4\rangle \;;\; 3 \text{ subspaces}: \quad \begin{cases} \{\ |1\rangle\ ,\ |4\rangle\ \} = (\ I_{14}, \sigma_{14}^x, \sigma_{14}^y, \sigma_{14}^z\ ) \\ \{\ |2\rangle\ ,\ |4\rangle\ \} = (\ I_{24}, \sigma_{24}^x, \sigma_{24}^y, \sigma_{24}^z\ ) \\ \{\ |3\rangle\ ,\ |4\rangle\ \} = (\ I_{34}, \sigma_{34}^x, \sigma_{34}^y, \sigma_{34}^z\ ) \end{cases}$$

$$m=5 \;:\; FSTS-|5\rangle \;;\; 4 \text{ subspaces}: \quad \begin{cases} \{\ |1\rangle\ ,\ |5\rangle\ \} = (\ I_{15}, \sigma_{15}^x, \sigma_{15}^y, \sigma_{15}^z\ ) \\ \{\ |2\rangle\ ,\ |5\rangle\ \} = (\ I_{25}, \sigma_{25}^x, \sigma_{25}^y, \sigma_{25}^z\ ) \\ \{\ |3\rangle\ ,\ |5\rangle\ \} = (\ I_{35}, \sigma_{35}^x, \sigma_{35}^y, \sigma_{35}^z\ ) \\ \{\ |4\rangle\ ,\ |5\rangle\ \} = (\ I_{45}, \sigma_{45}^x, \sigma_{45}^y, \sigma_{45}^z\ ) \end{cases}$$

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$$FSTS-|N\rangle \;;\; N-1 \text{ subspaces}: \quad \begin{cases} \{\ |1\rangle\ ,\ |N\rangle\ \} = (\ I_{1N}, \sigma_{1N}^x, \sigma_{1N}^y, \sigma_{1N}^z\ ) \\ \{\ |2\rangle\ ,\ |N\rangle\ \} = (\ I_{2N}, \sigma_{2N}^x, \sigma_{2N}^y, \sigma_{2N}^z\ ) \\ \{\ |3\rangle\ ,\ |N\rangle\ \} = (\ I_{3N}, \sigma_{3N}^x, \sigma_{3N}^y, \sigma_{3N}^z\ ) \\ ..... \\ ..... \\ \{\ |N-1\rangle\ ,\ |N\rangle\ \} = (\ I_{(N-1)N}, \sigma_{(N-1)N}^x, \sigma_{(N-1)N}^y, \sigma_{(N-1)N}^z\ ) \end{cases} \tag{11}$$

This orderly distribution of the two-state subspaces among the $N-1$ focal state transition spaces provides better clarity on the specification of the quantum operators $\sigma_{nm}^j\ ,\ j=x,y,z$, which constitute the general $SU(N)$ generator-spectrum in equation $(4a)$. From each two-state subspace, the non-diagonal operators $\sigma_{nm}^x\ ,\ \sigma_{nm}^y$ are separately identified as the traceless non-diagonal symmetric and antisymmetric generators, while the diagonal operator σ_{nm}^z is identified as a component of the effective traceless diagonal antisymmetric generator Λ_{m-1} determined using the composition formula in equation $(2e)$.

3 Algebraically closed $SU(N)$ generator-spectrum : "Cartan" and "conjugate-Cartan" generators

Having determined the $SU(N)$ generator-spectrum in the general form in equations $(4a)$, $(4b)$ and in explicit forms in the examples evaluated in the set of equations $(5a)$-$(10b)$, we now consider the algebraic properties of the generators. We note that all the N^2-1 traceless non-diagonal and diagonal symmetric and antisymmetric generators $\lambda_1\ ,\ \lambda_2, ..., \lambda_{N(N-1)}\ ,\ \Lambda_1\ ,\ \Lambda_2, ..., \Lambda_{N-1}$ and the $N-1$ non-traceless diagonal symmetric generators obtained in the $SU(N)$ generator-spectra for $N \geq 2$ satisfy the standard $SU(N)$ generator normalization conditions and completeness relation in equations $(2g)$ and $(2h)$, respectively.

The $N-1$ traceless diagonal antisymmetric generators $\Lambda_1\ ,\ \Lambda_2, ..., \Lambda_{N-1}$ characterizing the $N-1$ focal state transition spaces in an $SU(N)$ generator-spectrum mutually commute, satisfying commutation relations

$$[\ \Lambda_j\ ,\ \Lambda_k\ ] = 0 \tag{12a}$$

which we identify as a Cartan subalgebra of the $SU(N)$ symmetry group in a group theoretic interpretation where we identify each focal state transition space as a Cartan subspace. In this respect, we call the commuting traceless diagonal antisymmetric generators $\Lambda_k\ ,\ k=1,2,...,N-1$ in an $SU(N)$ generator-spectrum the z-component "Cartan" generators, which we denote by $C_k^z = \Lambda_k$ for reasons clarified below.

By evaluating commutation brackets of various pairs of the generators $\lambda_1\ ,\ \lambda_2, ..., \lambda_{N(N-1)}\ ,\ \Lambda_1\ ,\ \Lambda_2, ..., \Lambda_{N-1}$ in an $SU(N)$ generator-spectrum, we establish that all the $N-1$ focal state transition spaces are algebraically connected based on the property that commutation brackets of some pairs of generators in the higher level spaces produce generators in the lower level spaces, where in the $SU(N)$ generator-spectrum we consider FSTS-$|m\rangle$ to be a higher level space compared to FSTS-$|m-1\rangle$. For example, in the $SU(3)$ generator-spectrum in equation (6a), the algebraic connectedness of the higher level space FSTS-$|3\rangle$ to the lower level space FSTS-$|2\rangle$ is characterized by the property that, among other cases, the commutation bracket of two generators λ_3 , λ_5 in the higher level space FSTS-$|3\rangle$ produces the generator λ_2 in the lower level space FSTS-$|2\rangle$ according to $[\lambda_3\ ,\ \lambda_5] = i\lambda_2$, with the three generators λ_3 , λ_5 , λ_2 satisfying a closed algebra, noting $[\lambda_2\ ,\ \lambda_3] = i\lambda_5$, $[\lambda_5\ ,\ \lambda_2] = i\lambda_3$. Algebraic connectedness of the $N-1$ focal state transition spaces is a general property of an $SU(N)$ generator-spectrum.

A major problem arises on the general algebraic properties of $SU(N)$ symmetry group generators. A straightforward evaluation of the commutation brackets of the $SU(3)$ generators obtained here in equation (6a) and in the standard textbooks [3-6] does not provide a closed $SU(3)$ algebra, since there is no commutation bracket of any pair of the generators λ_j , $j = 1-6$ in equation (6a) (or λ_j , $j = 1-7$ in [3-6]) produces the diagonal generator Λ_2 in equation (6a) (or λ_8 in [3-6]). Contrary to the commonly stated results in [3-6] and related literature, the commutation relations of the generator pairs $\{\lambda_4\ ,\ \lambda_5\}$, $\{\lambda_6\ ,\ \lambda_7\}$ (standard Gell-Mann notation used in [3-6]) separately yield

$$[\lambda_4\ ,\ \lambda_5] = 2i\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \neq (...)\lambda_8 \qquad ; \qquad [\lambda_6\ ,\ \lambda_7] = 2i\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \neq (...)\lambda_8$$

$$\lambda_8 \equiv \Lambda_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix} \tag{12b}$$

Only the sum of the two commutation brackets in equation (12b) gives the results stated in [3-6] and related literature according to

$$[\lambda_4\ ,\ \lambda_5] + [\lambda_6\ ,\ \lambda_7] = 2i\sqrt{3}\lambda_8 \tag{12c}$$

The results in equations (12b) , (12c) show that the corresponding commutation brackets presented in [3-6] and related literature are not valid, thus revealing that the basic $SU(3)$ generators given in [3-6] and obtained here in equation (6a) do not provide a closed $SU(3)$ algebra. In general, an $SU(N)$ generator-spectrum for $N \geq 3$ is not algebraically closed, since no commutation brackets of any pair of the basic generators λ_1 , $\lambda_2, ..., \lambda_{N(N-1)}$, Λ_1 , $\Lambda_2, ..., \Lambda_{N-1}$ can produce any of the higher z-component Cartan generators $C_k^z = \Lambda_k$, $k = 2, 3, ..., N-1$. Only the $SU(2)$ generator-spectrum obtained here in equation (5a) is algebraically closed.

We address the problem of determining algebraically closed $SU(N)$ generator-spectrum by applying the group theoretic interpretation to identify the $N-1$ focal state transition spaces in the $SU(N)$ generator-spectrum as Cartan subspaces similarly specified by the $N-1$ focal state vectors $|m\rangle$, $m = 2, 3, ..., N$. We introduce a simple notation C-$|m\rangle$ for a Cartan subspace specified by a focal state vector $|m\rangle$ in an $SU(N)$ generator-spectrum. Each of the $N-1$ Cartan subspaces C-$|m\rangle$ is characterized by "Cartan" generators C_{m-1}^j and their associated "conjugate-Cartan" generators $\overline{C}_{m-1}^j$ obtained as normalized sums of the basic $SU(N)$ generators σ_{nm}^j for $j = x, y, z$ according to the composition formula in equation (2e) which we now generalize to determine the complete set of Cartan and "conjugate-Cartan" generators in the form

$$C-|m\rangle\ :\quad C_{m-1}^j = \sqrt{\frac{2}{m(m-1)}}\sum_{n=1}^{m-1}\sigma_{nm}^j \qquad ; \qquad \overline{C}_{m-1}^j = -\sqrt{\frac{2}{m(m-1)}}\sum_{n=1}^{m-1}(-1)^n\sigma_{nm}^j$$

$$m = 2, 3, ..., N \quad ; \quad j = x, y, z \tag{12d}$$

We note that setting $j = z$ in equation (12d) provides the composition formula $C_{m-1}^z = \Lambda_{m-1}$ in equation (2e), which we identify here as the z-component Cartan generator. We complete the characterization of the Cartan subspace by introducing the "Cartan" identity generator I_{m-1} and the associated "conjugate-Cartan" identity generator $\overline{I}_{m-1}$ obtained as normalized sums of the basic identity generators I_{nm} according to the composition formulae

$$C-|m\rangle\ :\quad I_{m-1} = \sqrt{\frac{2}{m(m-1)}}\sum_{n=1}^{m-1}I_{nm} \qquad ; \qquad \overline{I}_{m-1} = -\sqrt{\frac{2}{m(m-1)}}\sum_{n=1}^{m-1}(-1)^n I_{nm} \tag{12e}$$

Here again, we note that I_{m-1} in equation $(12e)$ equals $\overline{\Lambda}_{m-1}$ in equation $(2f)$.

Preliminary calculations confirm that the Cartan generators and their conjugates determined from equation $(12d)$ using the $SU(3)$ generator-spectrum in equation $(6a)$ provide the desired $SU(3)$ closed algebra. Details of calculations of closed algebra of $SU(N)$ $(N \geq 3)$ symmetry groups based on the respective Cartan and conjugate-Cartan generators will be presented in subsequent work.

4 Conclusion

We have identified $SU(N)$ symmetry group generators as state transition and eigenvalue quantum operators occurring in an orderly pattern within $N-1$ focal state transition spaces specified by focal state vectors $|m\rangle$ (FSTS-$|m\rangle$), $m = 2, 3, ..., N$, each containing $2m-1$ generators, forming an $SU(N)$ generator-spectrum in an N-dimensional quantum state space similar to the orderly arrangement of orbital shells in an atomic energy-spectrum. The $SU(N)$ generator-spectrum is effectively derived from an orderly distribution of $\frac{1}{2}N(N-1)$ two-state subspaces among the $N-1$ focal state transition spaces FSTS-$|m\rangle$ each containing $m-1$ two-state subspaces. While the $N-1$ focal state transition spaces in the full $SU(N)$ generator-spectrum are algebraically connected, the generator-spectrum for any $N \geq 3$ $SU(N)$ symmetry group is not algebraically closed. Applying a group theoretic interpretation of focal state transition spaces in an $SU(N)$ generator-spectrum as Cartan subspaces, we have introduced Cartan and conjugate-Cartan generators to determine the desired algebraically closed $SU(N)$ generator-spectrum.

The orderly arrangement of $SU(N)$ symmetry group generators within algebraically connected focal state transition spaces constituting an $SU(N)$ generator-spectrum in an algebraically closed structure similar to the orderly arrangement of orbital shells in an atomic energy-spectrum may provide greater insights into various models of $SU(N)$ gauge theories, including grand unified theories, of elementary particle interactions [3-9 , 11-15]. In particular, the determination of the correct sets of $SU(N)$ generators provides an algebraic platform for a thorough review or reformulation current models of $SU(N)$ Grand Unified Theories of elementary particle interactions, including the latest model of $SU(5)$ Grand Unified Theory Without Proton Decay [7 , 8].

As stated in the earlier article [2], we emphasize that the phenomenon of a focal state transition process composed of a collection of $(m-1)$ transitions, equivalent to a stream of (single mode) electromagnetic radiation from $(m-1)$ different sources, propagating into a common focal state $|m\rangle$ is an important physical property which brings a focal state in an N-state quantum space into direct correspondence with a focal point into which a stream of light rays from various sources converge in classical geometrical optics. The focal state transition process, characterized by $N-1$ focal states in an N-state quantum space, may find important practical applications in optics.

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