ON THE THEORY OF RANDOM SEARCH

By

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other University.

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This thesis has been submitted for examination with my approval as University supervisor.

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CONTENTS

literature relevant to the problem under study is given, whereas Section 1.4 gives a concise statement of the problem together with a list of specific objectives of the study. The envisaged significance of the results of the study are mentioned in Section 1.5.

Chapter 2 deals with the properties of separating systems of a finite set S_n. Some useful properties of separating system proved by Renyi (1965) are given in Section 2.1, whereas binary minimal separating systems and non-binary separating systems are discussed in Sections 2.2 and 2.3 respectively.

In Chapter 3, random search models based on binary structures are examined. Properties of binary search models proved by Renyi (1965) are given in Section 3.1. Section 3.2 deals with random search models based on finite plane projective geometries while random search models based on finite plane Euclidean geometries are discussed in Section 3.3. Search models based on random 0-1 matrices are given in Section 3.4.

Chapter 4 is concerned with search models for detecting more than one unknown element from a finite set. Section 4.1 introduces two types of search designs namely; the 2-Complete search design and the partition search design. A detail study of the 2-complete search design is given in Sections 4.2 and

4.3. Section 4.4 considers construction and properties of the partition search designs. The problem of detecting more than two unknown elements from a finite set is discussed in Section 4.5.

In Chapter 5 duration of the search process for detecting two unknown elements is studied. Examples to illustrate the computation of the duration of the search process for detection of two unknown elements using the 2-complete search design and the partition search design are given in Section 5.2. In Section 5.3 some results concerning the duration of the search process for detection of two unknown elements are derived.

Search models for the detection of unknown element(s) in the presence of noise are studied in Chapter 6. The possibility of an observed function being in error is introduced in Section 6.1. Section 6.2 deals with separating systems which determine one unknown element in the presence of noise while Sections 6.3 and 6.4 deal with the problem of detecting two unknown elements in the presence of noise using a 2-complete search design and a partition search design.

Chapter 7, contains a brief summary of some concluding remarks together with a list of some problems that require further investigation.

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CHAPTER 1

INTRODUCTION

1.1 WHAT IS RANDOM SEARCH?

Consider a set $S_n = \{a_1, a_2, \ldots\ldots, a_n\}$ containing n elements and a system F of test-functions defined on S_n. Suppose that k of the elements, say $x_1, x_2, \ldots\ldots\ldots, x_k$ $(k < n)$ in S_n are not known. Then the problem of search is concerned with determining the identities of these unknown elements using the test-functions in F. It is assumed that it is not possible to observe these unknown elements directly but one can choose a sequence of functions $f_1, f_2, \ldots\ldots, f_N$ from the system F and observe the values of these functions at each of the elements $x_1, x_2, \ldots\ldots\ldots\ldots$ and x_k, until enough information is obtained to determine the identities of these unknown elements.

A method for the successive choice of the test-functions $f_1, f_2, \ldots\ldots\ldots, f_N$ from a system F of functions, which leads in the end to the determination of the unknown element(s) is called a *strategy of search.* A strategy can either be pure or mixed. It is called *pure* if it uniquely specifies the choice of the test-functions and it is called *mixed* if the choice of these test-functions depends on chance. In a mixed strategy, test-functions are chosen according to some probability distribution. A mixed strategy is therefore called *random search.* A pure strategy is said to be *predetermined* if the number N and the choice of each of the test-functions is determined before beginning the observation. It is called ***sequential*** if only the choice of f_1 is determined in advance and the choice of f_k $(k \geq 2)$ is made only after observing $f_1(x)$, $f_2(x), \ldots\ldots, f_{k-1}(x)$ and may depend on these observed values. When observed values may be in error due to noise the search process is called *noisy*. Otherwise the search process is called *noiseless*.

If the system F of functions contains a function which takes on different values for different elements of S_n, then a single observation of this function at the unknown element(s) will identify the element(s). In practice the number of different values taken by a test-function in F is much smaller than N. In the special case where each function can take only two values 0 and 1, the system of functions F is called a *binary search system.* A search strategy based on F is then described as *binary search strategy.*

1.2 BASIC CONCEPTS AND NOTATIONS.

The following are some basic concepts and notations [illegible] useful in our discussion of the search problem.

Types of search systems.

A system F of functions defined on the set S_n is called a *separating system* if for every pair of distinct elements $a_i, a_j \in S_n$ there exists a function f in F such that $f(a_i) \neq f(a_j)$.

A separating system F can also be defined as follows:

Let

$M = (f_i(a_j)),\quad i = 1,2,\ldots\ldots,m;\quad j = 1,2,\ldots\ldots,n$

denote an mxn matrix whose (i,j)-th entry is $f_i(a_j)$. Then F is a separating system if and only if all the columns of the matrix M are distinct. We shall call M the search matrix of the system F. A system F of functions is said to be a *minimal separating system* if no proper subset of F is a separating system on S_n.

We shall also need the notion of homogeneity of a separating system of functions in the situation where all the elements in S_n have the same chance of being the unknown element, that is, when we assume that,

$$Pr(x = a_i) = 1/n,\qquad i = 1,2,\ldots\ldots,n.$$

For any choice of $k(2 \leq k \leq n)$ distinct elements a_{i_1}, $a_{i_2},\ldots\ldots,a_{i_k}$ of S_n, let R_k denote the number of functions f in F such that $f(a_{i_1}) = f(a_{i_2}) = \ldots\ldots\ldots= f(a_{i_k})$. Then if R_k does not depend on the choice of the k elements, F is called a *weakly homogeneous system* of order k. The system F is called a *strongly homogeneous system* of order k, if for every k distinct elements $a_{i_1}, a_{i_2},\ldots,a_{i_k}$ of S_n and a sequence of k numbers $y_{i_1}, y_{i_2},\ldots\ldots\ldots\ldots,y_{i_k}$ where y_{i_j}.s are values taken on by members of F and they are not necessarily all different, the number

$R_k(y_{i_1}, y_{i_2}, \ldots, y_{i_k})$ of functions f in F for which $f(a_{i_1}) = y_{i_1}, f(a_{i_2}) = y_{i_2}, \ldots, f(a_{i_k}) = y_{i_k}$, does not depend on the choice of elements $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ but may depend on the values of $y_{i_1}, y_{i_2}, \ldots, y_{i_k}$.

Efficiency of a separating system

Let the range of the function $f_1, f_2, \ldots, f_m$ in F be a finite set $Y = \{y_1, y_2, \ldots, y_q\}$ and let $k_{j\ell}$ be the number of points in S_n such that

$$f_j(a) = y_\ell, \quad j = 1,2,\ldots,m$$

$$\ell = 1,2,\ldots,q, \; a \in S_n$$

and

$$\sum_{\ell=1}^{q} k_{j\ell} = n.$$

If the element $a_i \in S_n$ is assigned probability p and the function $f_j \in F$ is assigned probability p′, then the entropy of $a \in S_n$ is given by

$$H(a) = -\sum_{i=1}^{n} P_r(a = a_i)\log P_r(a = a_i)$$

$$= -\sum_{i=1}^{n} p\log p \qquad (1.1)$$

and the entropy of $f \in F$ is

$$H(f) = -\sum_{j=1}^{m} P_r(f = f_j)\log P_r(f = f_j)$$

$$= -\sum_{j=1}^{m} p'\log p'. \qquad (1.2)$$

Thus H(a) and H(f) give the average uncertainty associated with the selection of a_i from S_n and f_i from F respectively.

The joint entropy of a and f is given by:

$$H(a,f) = H(a) + H(f) \qquad (1.3)$$

since the choice of any element $a_i \in S_n$ is stochastically independent of the choice of any function $f_j \in F$.

Now, the probability distribution of f(x) conditional on $f = f_j$ is

$$Pr(f(x) = y_\ell | f = f_j) = Pr(f_j(x) = y_\ell) = p_{j\ell}$$

and so the conditional entropy of f(x) given $f = f_j$ is given by

$$H(f|f_j) = -\sum_{\ell=1}^{q} p_{j\ell} \log p_{j\ell}, \qquad \text{for } p_{j\ell} \neq 0$$

and

$$H(f|f_j) = 0, \qquad \text{for } p_{j\ell} = 0 \qquad (1.4)$$

Renyi (1965) has proved the inequality

$$\sum_{j=1}^{m} H(f|f_j) \geq \log_2 n \qquad (1.5)$$

for any separating system F on S_n. The ratio

$$\log_2 n / \sum_{j=1}^{m} H(f|f_j) \qquad (1.6)$$

is used as a measure of the *efficiency* of the separating system F. The closer this ratio is to one, the more efficient the system $\{f_1, f_2, \ldots\ldots\ldots, f_m\}$ is in separating the elements of S_n.

The Duration of a Search Process.

We shall first consider the duration of the search process for detecting one unknown element. Let F be a system of m functions defined on the set $S_n = \{a_1, a_2, \ldots\ldots\ldots, a_n\}$ which separates the elements of S_n. Let x be an unknown element in S_n and let us suppose that we search for x in the following way: we choose first a function f from F at random so that each function of F has the same probability 1/m to be chosen. We observe $f_1(x)$, the value of f_1 at x and after this we choose again a function f_2 from F so that the choice of f_2 is independent of the choice of f_1 and each element f of F has the same probability 1/m to be chosen as f_2. We observe $f_2(x)$ and continue with the process until f_N is selected and its value at x observed.

We shall denote the probability that the sequence $f_1(x), f_2(x), \ldots\ldots, f_N(x)$ determines the unknown element x by $P_1(N,x)$ and the probability that the process of detecting x terminates exactly at the Nth step by $p_1(N,x)$. The expected duration of the search process for detecting the unknown element x is then given by:

$$E_1(x) = \sum_{N=0}^{\infty} N\ p_1(N,x). \qquad (1.7)$$

Next, we consider the duration of the search process for detecting two unknown elements. Let F be a system of m functions defined on the set $S_n = \{a_1, a_2, \ldots\ldots\ldots, a_n\}$ which separates any pair of elements of S_n. Let (u,v) be the unknown pair of elements and let us suppose that we search for the pair of elements in the following way: we choose first a function f_1 from F at random so that each function of F has the same probability 1/m to be chosen. We observe f_1 at u and v. Each observation specifies a subset of S_n, say A_{1u} and A_{1v}, where

$$A_{1u} = f_1^{-1}(f_1(u))$$

and

$$A_{1v} = f_1^{-1}(f_1(v)). \qquad (1.8)$$

Next, we choose f_2 at random such that the choice of f_2 is independent of the choice of f_1 and each function f in F has the same probability 1/m of being chosen as f_2. Again by observing f_2 at u and v we obtain subsets A_{2u} and A_{2v}. The process is continued until we are able to determine the unknown pair {u,v} uniquely; that is, until

$$\left(\bigcap_{i=1}^{N} A_{iu}\right) \cup \left(\bigcap_{i=1}^{N} A_{iv}\right) = \{u,v\}. \qquad (1.9)$$

If this happens then we require the sequence $f_1, f_2, \ldots\ldots\ldots, f_N$ of functions in order to detect the pair {u,v}.

We shall denote the probability that the sequence

$f_1, f_2, \ldots\ldots\ldots, f_N$ determines the pair (u,v) of the unknown elements by $P_1(N,u,v)$ and the probability that the process of detecting {u,v} terminates exactly at the Nth step by $p_1(N,u,v)$. The expected duration of the search process for detecting a pair of unknown elements is then given by:

$$E_1(u,v) = \sum_{N=0}^{\infty} N\ p_1(N,u,v). \qquad (1.10)$$

The concepts and notations of the duration of the search process stated here will be useful in Chapters 3 and 5, in the computation of the duration of the search process for detecting the unknown element(s).

Finite Plane Projective Geometries: PG(2,s).

In plane projective geometry, a point is defined by an ordered set of three elements (x_0, x_1, x_2) not all zeros belonging to GF(s), where s is prime or power of prime and a line is defined by the equation

$$a_0x_0 + a_1x_1 + a_2x_2 = 0, \qquad a_0, a_1, a_2 \in GF(s).$$

This geometry is denoted by PG(2,s). In plane projective geometry the following basic properties hold.

(i) Two different points are incident with one line, that is, given two points there exists only one line through them.

(ii) Two lines are incident with one point, that is, they intersect.

(iii) Not all points are incident with the same line.

(iv) There are at least three different points in the same line.

(v) The number of points incident with at least one line is finite.

The following results are derived from the above properties of PG(2,s).

(i) The total number of points is s^2+s+1.

(ii) The total number of lines is s^2+s+1.

(iii) Each line is incident with (s+1) points.

(iv) Each point is incident with (s+1) lines.

Let $n = s^2+s+1$, then with PG(2,s) we can associate an nxn matrix $M = ((a_{ij}))$ where $a_{ij} = 0$ or 1 depending on whether the ith point is incident with the jth line or not ($i = 1,2,\ldots,n$, $j = 1,2,\ldots\ldots n$). This matrix M is the incidence matrix of PG(2,s).

Finite Plane Euclidean Geometries EG(2,s).

In plane Euclidean geometry, a point is defined by an ordered set of two elements (x_1,x_2) belonging to GF(s), where s is prime or power of prime and a line is defined by the equation

$$a_0 + a_1x_1 + a_2x_2 = 0, \qquad a_0,a_1,a_2 \in GF(s).$$

This geometry is denoted by EG(2,s). In plane Euclidean geometry the following basic properties hold.

(i) Two distinct points are incident with one and only one common line.

(ii) Through every point not incident with a given line there passes one and only one line which has no common point with the given line. This line is said to be parallel to the given line. All other lines through the point have one common point with the given line.

(iii) Not all points are incident with the same line.

(iv) There are at least two distinct points on the same line.

The following results are derived from the above properties of EG(2,s).

(i) The total number of points is s^2.

(ii) The total number of lines is s^2+s.

(iii) Each line is incident with exactly s points.

(iv) Each point is incident with exactly (s+1) lines.

Let $m = s^2+s$ and $n = s^2$ then with EG(2,s) we can associate an mxn matrix $M = ((a_{ij}))$, where $a_{ij} = 0$ or 1 depending on whether the ith point is incident with the jth line or not ($i = 1,2,..,n$, $j = 1,2,..,r$). This matrix M is the incidence matrix of EG(2,s).

We shall make use of the incidence matrices of EG(2,s) and PG(2,s) in the study of the homogeneity of separating systems in Chapter 3.

A t-Complete search design.

A system $\{S_1,S_2,....,S_\tau\}$ where S_i ($i = 1,2,..,\tau$) is a subset of the set S_n, is said to be a t-Complete search design if for every t distinct elements $a_{i_1}, a_{i_2}, \ldots, a_{i_t} \in S_n$, we can select subsets $\{S_j,\ j \in T\}$, where $T = \{j |\ a_{i_k} \notin S_j$, for $k = 1,2,\ldots,t\}$, such that

$$\bigcup_{j \in T} S_j = S_n - \{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}.$$

This definition was given by Bush and Federer (1984). A t-complete search design can also be defined in terms of the intersection of the subset $\{S_j,\ j \in T\}$. We shall use this approach to define a 2-Complete search design in Chapter 4.

A Balanced Incomplete Block design

A balanced incomplete block (BIB) design is an arrangement of v objects into b subsets (blocks) such that each block contains k distinct objects, each object occurring in r different blocks, and each pair of distinct objects occurring together on λ different blocks. For construction of these designs see Hall (1967) and Bose (1969).

An arrangement of v objects in b blocks such that each block contains either $k_1, k_2, \ldots$ or k_m objects and every pair of objects occurs in exactly λ blocks is called a *pairwise balanced* design of index λ. It is denoted by PWB $(v;\ k_1, k_2, \ldots, k_m;\ b_1, b_2, \ldots, b_m;\ \lambda)$, where b_i denotes the number of blocks of size k_i. All the blocks of size k_i form the equiblock component D_i of the PWB design D. PWB designs in which all the objects have the same number of replications are called *equireplicated* PWB designs.

The balanced incomplete block designs and the related designs will be useful in the construction of 2-Complete search designs in Chapter 4.

A t-(v,k,λ_t) design.

An arrangement of v objects into b subsets (blocks) such that each block consists of k distinct objects is called a *t-(v,k,λ_t) design*. A balanced incomplete block design is a special case of t-(v,k,λ_t) design with $t = 2$.

A t-$(v,k\ \lambda_t)$ design will also be useful in the construction of 2-Complete search designs in Chapter 4.

Some concepts from Coding Theory.

The basic concepts and properties of codes mentioned here will be useful in discussing error-correcting search systems in Chapter 6.

Consider the set $\{a_1, a_2, \ldots\ldots, a_p\}$ of p symbols. In coding theory, these symbols are referred to as *code characters*. A finite sequence of code characters called a *code word* and the number of code characters in a code word is the *length* of the code word. For example the code word 1101100 has length seven. The collection of all code words is called a *code;* and the collection of all code words of the same length is called a *block code*.

The *Hamming distance* between two code words $\underline{v}_1$ and

$\underline{v}_2$ is the number of places in which they differ. The *Hamming weight* is the number of non-zero co-ordinates in a code word. For example, the code word $\underline{v}$ = 1101100 has a Hamming weight of four. The minimum distance d of a block code $\mathcal{C}$ is defined by:

$$d = \underset{\substack{u,v \in \mathcal{C} \\ \underline{u} \neq \underline{v}}}{\text{minimum}} \; d(u,v).$$

The following two properties of block codes will be useful in discussing error-correcting search systems.

(i) A block code with distance d is capable of correcting all patterns of t or fewer errors and detecting all patterns of $t+j, 0 < j < s$ errors if $2t+s < d$, $s > 0$.

(ii) The minimum distance of a block code is the weight of the minimum weight code word.

For a more complete discussion of these results see for example Blake and Mullin (1975).

1.3 BRIEF LITERATURE REVIEW.

The problem of search was in the early stages concerned with developing models for solving specific problems. For example Bose and Nelson (1962) constructed a network for sorting n elements. They gave an upper bound for minimum number of comparators needed in an n-element sorting network and conjectured that this upper bound is the exact minimum number of comparators needed in such a network. Subsequent construction by Floyd and Knuth (1967) showed that this upper bound given by Bose and Nelson can be improved for all n>8. In a later paper Floyd (1972) proved that the Bose-Nelson conjecture was correct for n≤8.

Other authors who developed models to solve specific problems include: Bose and Koch (1969) who studied combinatorial information retrieval systems for files with multiple-valued attributes. They developed a model for filing systems which is capable of handling large volumes of data and permitting efficient information retrieval. Koch (1969) extended this work by studying a class of covers for finite projective geometries which are related to the design of combinatorial filing systems. He gave a method for selecting a certain subset of m-flats from a finite projective geometry PG(N,q) which cover all (t-1) flats. His results have application in the problem of designing efficient information retrieval systems.

In an attempt to unify the various models that had been proposed before to solve specific problems, Renyi (1965,1969, 1970) developed a mathematical model for a general search problem. He examined in detail the use of a rooted directed tree of degree q with n vertices as search system with a sequential strategy for noiseless search. He also defined separating systems of functions and introduced different notions of homogeneity of separating systems.

Katona (1966) also studied the separating systems of functions. He gave lower and upper bounds for the number of functions required to form a separating system under some specified conditions. Dickson (1969) later extended the concept of separating system when he defined a completely separating system. He considered the problem of finding the cardinality of a minimal completely separating system and showed that this cardinality is asymptotic to the cardinality of a minimal separating system. The cardinalities of minimal binary separating systems and non-binary separating systems under various conditions is studied in chapter 2 of this thesis.

After developing a model which solves a general search problem and introducing the concepts of separating systems and different homogeneities of separating systems, the next problem was the application of this model to solve specific problems and the construction of these designs. Chakravarti and Manglik (1972) considered the problem of applying the random search model developed by Renyi (1965). They used binary search systems derived from incidence matrices of PG(2,2) and PG(2,3) to determine the identity of one unknown element in a finite set S_n. However, their study did not cover other known geometrical structures like Euclidean geometries, random 0-1 matrices or general projective geometries.

Manglik (1972) on the other hand studied the construction of different homogeneities of separating systems. He related strongly homogeneous systems of order 2 to incidence matrices of equireplicated pairwise balanced designs. He also studied and gave properties of weakly homogeneous binary systems of order 2. Strongly homogeneous and weakly homogeneous systems of higher orders were not considered in his paper.

An extension of the work done by Chakravarti and Manglik (1972) and later by Manglik (1972) is given in chapter 3. The chapter mainly concentrates on areas not covered by the two papers, namely: the use of binary search systems derived from incidence matrices of Euclidean geometries, random 0-1 matrices and general projective geometries, to identify one unknown element in a finite set S_n and the *relation of* strongly homogeneous and weakly homogeneous systems of higher orders to incidence matrices of equireplicated pairwise balanced designs.

In applying Renyi's model to detect one unknown element in a finite set S_n, Chakravarti and Manglik (1972) assumed a noiseless search model. A noisy search was later studied by Chakravarti (1976). He constructed search systems and strategies which are separating in the presence of noise. He also gave a statistical decision rule for identifying an unknown element which maximizes the probability of correct identification in the presence of noise. A combinatorial approach of solving this identification problem in the presence of noise is discussed in chapter 6 of this thesis.

After applying Renyi's model of search to detect one unknown element in a finite set S_n, the attention was directed at detecting two or more unknown elements in the set S_n. Tosic (1980) considered the problem of detecting two unknown elements in S_n. He developed an optimal search procedure which identifies two unknown elements in S_n by testing some subsets of S_n which may contain all the two unknown elements or just one of the unknown elements. The same problem of detecting two unknown elements using subsets of S_n was later studied by Bush and Federer (1984). They examined the case where each subset of S_n contains the two unknown elements and called such a design a 2-Complete search design. They also discussed properties of these designs. Construction of 2-Complete search designs which was not considered by Bush and Federer (1984) is given in Chapter 4 of this thesis.

A more general design for detecting more than one unknown element was given by Sebo (1988). He considered the problem of detecting an unknown subset of cardinality k (k = 1,2......) of the finite set S_n and developed a probabilistic strategy of detecting the unknown subsets using minimum number of subsets. Although Sebo (1988) gave a method of detecting the unknown subset of S_n with a small error probability, explicit detection of two or more unknown elements in the presence of noise was not given. A model for detecting two or more unknown elements in the presence of noise is given in Chapter 6 of this thesis.

In this study we take up the problem of developing search strategies for identifying one, two and three unknown elements in a finite set. The search models will be based on logical extensions and generalizations of geometrical structures like projective and Euclidean geometries and 2-Complete search designs.

1.4 STATEMENT OF THE PROBLEM.

The present study investigates some properties of binary and non-binary separating systems studied by Renyi (1965) and Katona (1966). The relationship between separating systems and incidence matrices of projective geometries, Euclidean geometries and random 0-1 matrices are also investigated along the line of Chakravarti and Manglik (1972). Duration of the search process for detecting one unknown element in a finite set S_n using these incidence matrices as separating systems is also discussed.

The problem of detecting two unknown elements was studied by Tosic (1980).and later extended by Bush and Federer (1984) and Sebo (1988). The present study attempts to develop search models for detecting more than one unknown element. In particular, the study gives a method of constructing 2-Complete search designs introduced by Bush and Federer (1984) and develops new designs which are capable of detecting two unknown elements. The duration of the search process for detecting two unknown elements using a 2-Complete search design and the newly developed designs are also calculated.

Lastly, the study examines the problem of detecting one unknown element and two unknown elements in the presence of noise.

SPECIFIC OBJECTIVES OF THE STUDY.

The specific objectives of the present study may be summarized as follows:

(i) To obtain some useful properties of separating systems.

(ii) To use the existing geometrical structures like projective and Euclidean geometries to construct search systems for detecting one unknown element in a finite set.

(iii) To compute duration of the search process for detecting one unknown element.

(iv) To develop models for detecting two unknown elements.

(v) To compute duration of the search process for detecting two unknown elements.

(vi) To investigate detection of one unknown element and two unknown elements in the presence of noise.

1.5 SIGNIFICANCE OF THE STUDY.

The results of the present study are expected to provide useful search models for detecting one or more unknown elements in a set under investigation.

Also, the results demonstrate further use of projective and Euclidean geometries as separating systems.

The search models derived in the study presume both noiseless and noisy conditions, thus widening the scope of practical applications of the results of the study.

Examples of practical problems in which the search models proposed in the study are expected to be usefully applicable include: identification of an unmarked chemical in a laboratory, searching for a mistake in a computer program, decoding a received message, searching for failure in a complicated mechanism, diagnosis of a disease by clinical tests, forensic identification and so on.

CHAPTER 2

ON SEPARATING SYSTEMS OF A FINITE SET.

2.1 INTRODUCTION.

We recall here the two definitions of separating systems given in Chapter 1 as follows:

(i) A system F of functions $f_1, f_2, \ldots, f_m$ defined on a finite set S_n is a separating system if for every pair of distinct elements $a_i, a_j \in S_n$ there exists in F a function f such that $f(a_i) \neq f(a_j)$.

(ii) A system F of functions $f_1, f_2, \ldots, f_m$ defined on S_n is a separating system if an mxn matrix whose (i,j)-th entry is $f_i(a_j)$ has distinct columns.

An example of a separating system is given below.

Example 2.1: Consider a system $F = \{f_1, f_2, f_3\}$ defined on the set $S_3 = \{a_1, a_2, a_3\}$ as follows;

$$f_i\{a_j\} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j, \ i = 1,2,3; \ j = 1,2,3, \end{cases}$$

For any pair of distinct elements $a_i, a_j \in S_3$ there exists a function f_i in F such that $f_i(a_i) = 0$ and $f_i(a_j) = 1$, that is $f_i(a_i) \neq f_i(a_j)$. Thus, the system $F = \{f_1, f_2, f_3\}$ is a separating system.

The search matrix of this system is;

$$M = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{ccc} a_1 & a_2 & a_3 \\ \end{array}$$

$$M = \begin{matrix} & a_1 & a_2 & a_3 \\ f_1 & 0 & 1 & 1 \\ f_2 & 1 & 0 & 1 \\ f_3 & 1 & 1 & 0 \end{matrix} .$$

The columns of the matrix M are distinct as expected, since the system $\{f_1, f_2, f_3\}$ is a separating system.

Some properties of separating systems.

The following are some useful properties of separating systems; see Renyi (1965).

(i) Let F be a minimal separating system of functions separating the elements of the finite set S_n, having n elements. If m denotes the number of functions in F then $m \leq n - 1$.

(ii) The minimum number of functions m which separates n elements of the set S_n is $\{\log_2 n\}$, where $\{x\}$ denotes the least integer greater than or equal to x.

2.2 BINARY MINIMAL SEPARATING SYSTEMS.

We call a system F of functions defined on a finite set S_n a *binary minimal separating system* if the system consists of the minimum number of functions which separates any two elements of the set S_n and each function takes only two values 0 and 1. It has been proved by Renyi (1965) that the minimal binary separating system of a set of n elements has exactly $\{\log_2 n\}$ functions, (Where $\{x\}$ denotes the least integer $\geq x$).

Example 2.2: The minimal separating system of a set consisting of, 8 elements has $\log_2 8 = 3$ functions and one possible search matrix of the functions which separates the 8 elements is

$$M = \begin{matrix} \\ f_1 \\ f_2 \\ f_3 \end{matrix} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ \end{matrix}\left[\begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{matrix}\right].$$

A minimal separating system in which

$$\sum_{i=1}^{m} H(f_i(x)) = \log_2 n \qquad (2.1)$$

where $H(f)$ denotes the entropy of $f \in F$, is called an *optimal separating system.*

Lemma 2.1: Every optimal separating system is a minimal separating system. However, a minimal separating system is an optimal separating system if and only if the random functions $f_1, f_2, .., f_m$ are independent.

Proof

Suppose the functions $f_1, f_2, \ldots\ldots, f_m$ form a minimal separating system, that is the vector $(f_1(x), f_2(x), \ldots, f_m(x))'$ which is a column of the search matrix M of the functions $f_1, f_2, \ldots, f_m$ takes on different values for different values of $x \in S_n$. Assuming that the vector $(f_1(x), f_2(x), \ldots, f_m(x))'$ is equally likely to be any of the m columns of the search matrix M, the probability that the vector $(f_1(x), f_2(x), \ldots, f_m(x))'$ is the ith column $(i = 1, \ldots, n)$ of the matrix M is $\frac{1}{n}$ and the entropy of $(f_1(x), f_2(x), \ldots, f_m(x))$ is

$$H(f_1(x), f_2(x), \ldots, f_m(x)) = \sum_{i=1}^{n} \frac{1}{n} \log_2 n = \log_2 n.$$

But

$$\sum_{i=1}^{m} H(f_i(x)) \geq H(f_1(x), f_2(x), \ldots, f_m(x)) = \log_2 n$$

with equality if and only if $f_1(x), f_2(x), \ldots, f_m(x)$ are independent. Thus an optimal separating system corresponds to a minimal separating system with $f_1(x), f_2(x), \ldots, f_m(x)$ independent.

Remark: An optimal separating system F can be characterized by saying that the partial bits of information obtained by observing different functions f belonging to F do not overlap. Thus an optimal separating system corresponds to a most economic strategy.

Lemma 2.2: Suppose $F = \{f_1, f_2, \ldots, f_m\}$ is a separating system for the set $S_n = \{a_1, a_2, \ldots, a_n\}$

with $m = \log_2 n$, then F is an optimal separating system.

Proof

Since $m = \log_2 n$, the number of columns of the search matrix $M_{m,n}$ of the functions $f_1, f_2, \ldots, f_m$ consists of all possible combinations of i ones (zeros) and $(m-i)$ zeros (ones), $i = 1,2,\ldots\ldots,m$. To determine the number of ones in a row, say the jth row of $M_{m,n}$ we form a matrix $M'_{m,n'}$ whose columns are all the columns of the matrix $M_{m,n}$ with entry 1 in the j-th row. That is, the j-th row of the matrix $M'_{m,n'}$ consists of all ones, with n' giving the number of ones in the j-th row of $M_{m,n}$.

With the jth row of the matrix $M'_{m,n'}$ consisting of all ones, the remaining rows which consist of ones and zeros is $(m-1)$ and the number of columns of the matrix $M_{m,n'}$, n' is given by the number of all possible combinations of i ones (zeros) and $(m-1-i)$ zeros (ones). Thus, the number of ones in the jth row of the matrix $M_{m,n}$ is:

$$\binom{m-1}{0} + \binom{m-1}{1} + \binom{m-1}{2} + \ldots\ldots\ldots + \binom{m-1}{m-1}$$

$$= \sum_{i=0}^{m-1} \binom{m-1}{i} = 2^{m-1}$$

and the number of zeros in the jth row of $M_{m,n}$ is;

$$2^m - 2^{m-1} = 2^{m-1}.$$

Using the relative frequency interpretation of probability we have:

$$Pr(f_i(x) = 0) = Pr(f_i(x) = 1) = 2^{m-1} / 2^m = \frac{1}{2}.$$

The entropy of f_i in F is thus;

$$H(f_i) = \frac{1}{2} \log_2 n + \frac{1}{2} \log_2 n = \log_2 n$$

and

$$\sum_{i=1}^{m} H(f_i) = m = \log_2 n$$

which is the required condition for the separating system F to be optimal. Thus F is an optimal separating system.

Next, we consider the problem of determining the lower bound of the integer m for which there exists a binary search matrix $M_{m,n}$, in which each row contains k ones and no two columns are identical. We shall denote this integer by m(n,k).

Theorem 2.1: The integer m(n,k) described above satisfies the inequality:

$$m(n,k) \geq \frac{\log_2 n}{\frac{k}{n} \log_2 \frac{n}{k} + \frac{n-k}{n} \log_2 \frac{n}{n-k}}.$$

Proof

Let $F = \{f_1, f_2, \ldots, f_m\}$ be a system of functions defined on the set $S_n = \{a_1, a_2, \ldots, a_n\}$

whose search matrix is the matrix $M_{m,n}$ and the element $a_j \in S_n$ corresponds to the jth column of $M_{m,n}$. Assuming that the function f_i takes the values 0 or 1 with equal probabilities we have:

$$Pr(f_i(x) = 1) = \frac{k}{n}$$

and

$$Pr(f_i(x) = 0) = 1 - \frac{k}{n}$$

since each row of $M_{m,n}$ consists of k ones and (n-k) zeros. The entropy of $f_i \in F$ is then given by;

$$H(f_i) = \frac{k}{n} \log_2 \frac{n}{k} + \frac{n-k}{n} \log_2 \frac{n}{n-k}.$$

But

$$\sum_{i=1}^{m(n,k)} H(f_i) = m(n,k)\left(\frac{k}{n}\log_2 \frac{n}{k} + \frac{n-k}{n} \log_2 \frac{n}{n-k}\right),$$

$$\geq \log_2 n, \quad \text{(see Renyi (1965)).}$$

Therefore,

$$m(n,k) \geq \log_2 n \Big/ \left(\frac{k}{n} \log_2 \frac{n}{k} + \frac{n-k}{n} \log_2 \frac{n}{n-k}\right) \qquad (2.2)$$

Which is the required result.

Corollary 2.2: For k close to but less than or equal to $\frac{n}{2}$ -1, the integer m(n,k) satisfies the inequality:

$$m(n,k) \geq \log_2 n \Big/ \left(\frac{1}{2}\log_2\left(\frac{n^2}{k(n-k)}\right) + \frac{1}{n}\log_2\left(\frac{k}{n-k}\right)\right). \quad (2.3)$$

Proof.

From 2.3

$$m(n,k) \geq \log_2 n \Big/ \left(\frac{k}{n}\log_2 \frac{n}{k} + \frac{n-k}{n}\log_2 \frac{n}{n-k}\right)$$

But

$$\frac{k}{n} \log_2 \frac{n}{k} + \frac{n-k}{n}\log_2 \frac{n}{n-k} = \frac{1}{n}\Big[k\log_2 n - k\log_2 k + (n-k)\log_2 n - (n-k)\log_2(n-k)\Big]$$

$$= \frac{1}{n}\Big[n\log_2 n - k\log_2 k - n\log_2(n-k) + k \log_2(n-k)\Big]$$

$$= \frac{1}{n}\Big[n\log_2 \frac{n}{n-k} + k\log_2 \frac{n-k}{k}\Big]$$

$$\leq \frac{1}{n}\Big[n\log_2 \frac{n}{n-k} + \frac{n-2}{2}\log_2 \frac{n-k}{k}\Big]$$

for k close to but less than or equal to $\frac{n}{2} - 1$,

we have,

$$\frac{k}{n} \log_2 \frac{n}{k} + \frac{n-k}{n}\log_2 \frac{n}{n-k} = \frac{1}{2n}\Big[2n\log_2 n - 2n \log_2(n-k) + n \log_2(n-k) - 2\log_2(n-k) - n\log_2 k + [illegible]\Big]$$

$$= \frac{1}{2n}\Big[2n\log_2 n - n\log_2(n-k) + 2\log_2 \frac{k}{n-k} - n\log_2 k\Big]$$

$$= \frac{1}{2n}\Big[n\log_2 \frac{n^2}{k(n-k)} + 2\log_2 \frac{k}{n-k}\Big]$$

$$= \frac{1}{2}\log_2 \frac{n^2}{k(n-k)} + \frac{1}{n}\log_2 \frac{k}{n-k}$$

Thus

$$m(n,k) \geq \log_2 n \Big/ \left(\frac{1}{2}\log_2\left(\frac{n^2}{k(n-k)}\right) + \frac{1}{n}\log_2\left(\frac{k}{n-k}\right)\right)$$

Hence the proof of Corollary (2.2).

Remark: Corollary 2.2 gives a weaker but easier to compute estimate of the integer $m(n,k)$.

Example 2.3: Let $n = 11$ and $k = 4$, then $k = 4$ is close to $(n-2)/2 = (11-2)/2 = 4.5$. Thus corollary 2.2 could be applied to obtain an estimate for $m(11,4)$. This estimate is:

$$\frac{\log_2 11}{\frac{1}{2}\log_2(121/4x7) + \frac{1}{11}\log_2 4/7} = 3.52.$$

That is, $m(11,4) \geq 4$, since $m(n,k)$ must be an integer.

2.3 NON-BINARY SEPARATING SYSTEMS.

A system F of functions $f_1, f_2, \ldots, f_m$ defined on a finite set S_n is a *non-binary separating system* if for every pair of distinct elements $a_i, a_j \in S_n$, there exists a function $f \in F$ such that $f(a_i) \neq f(a_j)$ and each function in F takes p values $\emptyset, 1, 2, \ldots, p-1$ $(p \geq 3)$.

Example 2.4: Consider a system of two functions f_1 and f_2 defined on the set $S_n = \{a_1, a_2, \ldots, a_p\}$ as follows:

$f_1(a_1) = f_1(a_2) = f_1(a_3) = f_2(a_1) = f_2(a_4) = f_2(a_7) = \emptyset$

$f_1(a_4) = f_1(a_5) = f_1(a_6) = f_2(a_2) = f_2(a_5) = f_2(a_8) = 1$

$f_1(a_7) = f_1(a_8) = f_1(a_p) = f_2(a_3) = f_2(a_6) = f_2(a_p) = 2$.

Then the system $\{f_1, f_2\}$ is a non-binary separating system. This can be seen easily from its search matrix given as follows;

$$M = \begin{matrix} \\ f_1 \\ f_2 \end{matrix} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_p \\ \end{matrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}.$$

All the columns of the search matrix M are distinct: thus the system $\{f_1, f_2\}$ is a separating system.

Theorem 2.2: Suppose $F = \{f_1, f_2, \ldots, f_m\}$ is a separating system on the set S_n and each function $f \in F$ takes the value i ($i = 0, 1, .., p-1$) at k points in S_n, that is $n = kp$. Then m the number of functions in F satisfies the inequality;

$$m \geq \log_p n.$$

Proof

Let $M = ((f_i(a_j))$ be an mxn search matrix of the functions $f_1, f_2, \ldots, f_m$. Then the columns of the matrix M are distinct since $f_1, f_2, \ldots, f_m$ is a separating system. The joint entropy of $\{f_1, f_2, \ldots, f_m\}$ is;

$$H(f_1, f_2, \ldots, f_m) = \sum_{i=1}^{n} \frac{1}{n} \log_2 n$$

$$= \log_2 n .$$

And the entropy of $f \in F$ is

$$\sum_{i=0}^{p-1} Pr(f(x) = i)\log_2 \frac{1}{Pr(f(x) = i)}$$

$$= p\frac{k}{n} \log_2 \frac{n}{k}$$

$$= \log_2 \frac{n}{k}, \qquad \text{since } pk = n.$$

But

$$H(f_1) + H(f_2) + \ldots + H(f_m) \geq H(f_1, f_2, \ldots, f_m).$$

That is,

$$m \log_2 \frac{n}{k} \geq \log_2 n.$$

Changing from base 2 to base p, we have

$$\frac{m \log_p \frac{n}{k}}{\log_p 2} \geq \frac{\log_p n}{\log_p 2}$$

$$m \log_p \frac{n}{k} \geq \log_p n$$

$$m \log_p p \geq \log_p n$$

$$m \geq \log_p n$$

which is the required result.

Theorem 2.3: Suppose $F = \{f_1, f_2, \ldots, f_m\}$ is a separating system on the set S_n and each function $f \in F$ takes the value i $(i = 0, 1, \ldots, p-1)$ at $k_1, k_2, \ldots, k_p$ points in S_n where $k_1 \leq k_2 \leq \ldots \leq k_p$ and $\sum k_i = n$. Then m, the number of functions in F

satisfies the inequality:

$$m \geq \log_p n \Big/ \log_p \frac{n}{k_1}.$$

Proof

Let $M = ((f_i(a_j))$ be an mxn search matrix of the functions $f_1, f_2, \ldots, f_m$. Then the columns of the matrix M are distinct since $f_1, f_2, \ldots\ldots, f_m$ is a separating system. The joint entropy of $(f_1, f_2, \ldots, f_m)$ is;

$$H(f_1, f_2, \ldots\ldots, f_m) = \log_2 n.$$

And the entropy of $f \in F$ is;

$$H(f) = \sum_{i=0}^{p} \frac{k_i}{n} \log \frac{n}{k_i}.$$

But

$$H(f_1) + H(f_2) + \ldots + H(f_m) \geq H(f_1, f_2, \ldots, f_m)$$

$$= \log_2 n.$$

That is,

$$m\left[\frac{k_1}{n} \log_2 \frac{n}{k_1} + \frac{k_2}{n} \log_2 \frac{n}{k_2} + \ldots + \frac{k_p}{n} \log_2 \frac{n}{k_p}\right] \geq \log_2 n$$

i.e

$$\frac{m}{n}\Big[(k_1 \log_2 n + k_2 \log_2 n + \ldots + k_p \log_2 n)$$

$$- (k_1 \log_2 k_1 + k_2 \log_2 k_2 + \ldots + k_p \log_2 k_p)\Big] \geq \log_2 n.$$

Now, since $k_1 \leq k_2 \leq \ldots \leq k_p$

$$\frac{m}{n}\left[n \log_2 n - n \log_2 k_1\right] \geq \frac{m}{n}\Big[(k_1 \log_2 n + k_2 \log_2 n + \ldots$$

$$\ldots + k_p \log_2 n) - (k_1 \log_2 k_1$$

$$+ k_2 \log k_2 + \cdots\cdots$$

$$\cdots\cdots + k_p \log_2 k_p)\Big]$$

$$\geq \log_2 n .$$

Therefore,

$$\frac{m}{n}\left[n \log_2 \frac{n}{k_1}\right] \geq \log_2 n .$$

Changing from base 2 to base p, we have

$$\frac{m \log_p \frac{n}{k_1}}{\log_p 2} \geq \frac{\log_p n}{\log_p 2}$$

and

$$m \geq \log_p n \Big/ \log_p \frac{n}{k_1}$$

Hence the proof.

Example 2.5: Let $F = \{f_1, f_2, \ldots, f_m\}$ be a separating system on the set $S_{64} = \{a_1, a_2, \ldots, a_{64}\}$ and each function $f \in F$ takes the value 0,1,2 and 3 at 4,12,20 and 28 points in S_{64} respectively. Then the minimum number of functions, m satisfies the inequality:

$$m \geq \log_4 64 \Big/ \log_4 64/4$$

$$= 1.5 .$$

That is, to separate the elements of the set S_{64} a minimum of two functions would be required.

CHAPTER 3

RANDOM SEARCH MODELS BASED ON BINARY STRUCTURES

3.1 INTRODUCTION.

The search models we are going to study in this Chapter consist of a system F of functions which identifies any unknown element x of the set $S_n = \{a_1, a_2, \ldots, a_n\}$, and each function in F takes only two values 0 and 1. Each function divides the set S_n into two subsets. The intersection of subsets in which the unknown element x belongs gives the identity of x. These search models were described in Chapter 1 as binary search models.

Renyi (1965) obtained the following properties concerning binary search models.

(i) A system F of binary functions which is weakly homogeneous of order 2 is also weakly homogeneous of orders 3.

(ii) If R_1 denotes the number of functions in F and R_2 denotes the number of functions for which $f(a_i) = f(a_j)$, $i \neq j$, $f \in F$ then

$$R_2/R_1 \geq (n-2)/2(n-1). \tag{3.1}$$

(iii) If the system F of functions defined on the set S_n is weakly homogeneous of order 2, then for all $x \in S_n$

$$P_1(N,x) \geq 1 - (n-1)\left(R_2/R_1\right)^N \tag{3.2}$$

where $P_1(N,x)$ denotes the probability that the sequence of functions $f_1, f_2, \ldots, f_N$ determines the unknown element x and R_1 and R_2 are as defined above.

(iv) If the system F of binary functions defined on the set S_n is weakly homogeneous of order 2 and thus weakly homogeneous of order 3, then for all x in S_n

$$P_1(N,x) \leq 1 - (n-1)\left[\frac{R_2}{R_1}\right]^N + \binom{n-1}{2}\left[\frac{R_3}{R_1}\right]^N \qquad (3.3)$$

where R_3 denotes the number of functions for which $f(a_i) = f(a_j) = f(a_k), i \neq j \neq k, f \in F$.

We also recall that the expected duration of the search process for detecting the unknown element x is given by;

$$E_1(x) = \sum_{N=0}^{\infty} Np_1(N,x) \qquad \text{c.f.}(1.7)$$

with $p_1(N,x)$ denoting the probability that the process for detecting x terminates exactly at the N th step.

The following is an example illustrating the computation of the duration of the search process for detecting one unknown element.

Example 3.1: Consider a set S_3 consisting of three elements a_1, a_2 and a_3 and suppose that we wish to determine one of these elements. Let

$F = \{f_1, f_2, f_3\}$ be a set of three functions defined as follows:

$$f_i(a_j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j, \quad i = 1,2,3, \; j = 1,2,3. \end{cases}$$

Then, the search matrix of the system F is;

$$M = \begin{matrix} & a_1 & a_2 & a_3 \\ f_1 \\ f_2 \\ f_3 \end{matrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We notice that the columns of the matrix M are distinct, therefore, the system F of functions f_1, f_2, f_3 is a separating system. For any choice of two distinct elements a_{i_1}, a_{i_2} in S_3 there is only one function in F such that $f(a_{i_1}) = f(a_{i_2})$. So F is a weakly homogeneous system of order 2 with $R_2 = 1$. Further, F is a strongly homogeneous system of order 3 with $R_2(1,1) = R_2(1,0) = R_2(0,1) = 1$, $R_2(0,0) = 0$, $R_3(0,1,1) = R_3(1,1,0) = R_3(1,0,1) = 1$, where R_1, R_2 and $R_k(y_{i_1}, y_{i_2}, \ldots\ldots, y_{i_k})$ are as defined in Section 1.2 of Chapter 1.

Now, let us compute $P_1(N,x)$ if the unknown element x is a_1. The following two sequences [illegible] not detect x;

$$f_2, f_2, \ldots\ldots\ldots \text{ and } f_3, f_3, \ldots\ldots\ldots$$

Thus, the probability of not detecting x within N steps is;

$$\left(\frac{1}{3}\right)^N + \left(\frac{1}{3}\right)^N = 2\left(\frac{1}{3}\right)^N. \tag{3.4}$$

Therefore, the probability of detecting x within N steps is;

$$P_1(N,x) = 1 - 2\left(\frac{1}{3}\right)^N. \qquad (3.5)$$

Hence the probability of detecting x in exactly N steps is;

$$p_1(N,x) = \left[1-2\left(\frac{1}{3}\right)^N\right]-\left[1-2\left(\frac{1}{3}\right)^{N-1}\right]$$

$$= \frac{4}{3}\left(\frac{1}{3}\right)^{N-1},\ N \geq 2$$

and

$$p_1(1,x) = \frac{1}{3},\quad \text{for } N = 1$$

since the function f_1 identifies the unknown element x. Using Equation (1.7) in Chapter 1, the expected duration of the search process is;

$$E_1(x) = \sum_{N=1}^{\infty} N \cdot p_1(N,x)$$

$$= \frac{1}{3} + \sum_{N=2}^{\infty} N . p_1(N,x)$$

$$= \frac{1}{3} + \frac{4}{3}\left[2\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 + \ldots\right]$$

$$= \frac{1}{3} + \frac{4}{3}\left[\frac{1}{(1-1/3)^2} - 1\right]$$

$$= \frac{1}{3} + \frac{5}{3} = 2.$$

That is, to determine x an average of two test-functions would be required.

3.2 RANDOM SEARCH MODELS BASED ON FINITE PLANE PROJECTIVE GEOMETRIES: PG(2, s).

We recall that the incidence matrix of PG(2,s) is an n x n matrix $M = ((a_{ij}))$, where $n = s^2 + s + 1$ and $a_{ij} = 0$ or 1 depending on whether the ith point is incident with the jth line or not ($i = 1,2,\ldots\ldots,n$; $j = 1,2,\ldots\ldots,n$).

Identifying the points of PG(2,s) with the elements of the set S_n and the lines with functions of F, the incidence matrix M of PG(2,s) forms a search matrix.

Lemma 3.1: The system of functions (lines) F, derived from PG(2,s) is weakly homogeneous of order 2.

Proof.

Let $M = (f_i(a_j))$ be the search matrix of the strategy based on the system F. Then F will be a weakly homogeneous system of order 2 if R_2, which is the number of functions in F for which

$$[illegible] = f(a_{j'}), \qquad a_j \neq a_{j'},$$

is constant. That is, the number of functions in F for which $f(a_j) = f(a_{j'}) = 1$ or $f(a_j) = f(a_{j'}) = 0$, $a_j \neq a_{j'}$ is constant.

But, the number of functions in F for which $f(a_j) = f(a_{j'}) = 1$ is

$$\sum_{i=1}^{s^2+s+1} f_i(a_j)\, f_i(a_{j'}), \qquad a_j \neq a_{j'}$$

and the number of functions in F for which $f(a_j) = f(a_{j'}) = 0$ is

$$\sum_{i=1}^{s^2+s+1} (1-f_i(a_j))(1-f_i(a_{j'})).$$

Thus,

$$R_2 = \sum_{i=1}^{s^2+s+1} \left[f_i(a_j)f_i(a_{j'}) + (1-f_i(a_j))(1-f_i(a_{j'})) \right]$$

$$= \sum_{i=1}^{s^2+s+1} \left[1- f_i(a_j) - f_i(a_{j'}) + 2f_i(a_j)f_i(a_{j'}) \right]$$

$$= (s^2+ s + 1) - 2(s + 1) + 2$$

$$= s^2 - s + 1. \qquad (3.6)$$

which is a constant as required. Hence F is a weakly homogeneous system of order 2.

Lemma 3.2:- The system of functions (lines) F, derived from PG(2,s) is weakly homogeneous of order 3.

Proof

The system F will be weakly homogeneous of order 3 if R_3, which is the number of functions in F for which

$$f(a_j) = f(a_{j'}) = f(a_{j''}), \qquad a_j \neq a_{j'} \neq a_{j''}$$

is constant. That is, if the number of functions in F for which $f(a_j) = f(a_{j'}) = f(a_{j''}) = 1$ or $f(a_j) = f(a_{j'}) = f(a_{j''}) = 0$, $a_j \neq a_{j'} \neq a_{j''}$ is constant.

But the number of functions in F for which $f(a_j) = f(a_{j'}) = f(a_{j''}) = 1$ is

$$\sum_{i=1}^{s^2+s+1} f_i(a_j)f_i(a_{j'})f_i(a_{j''}), \quad a_j \neq a_{j'} \neq a_{j''}$$

and the number of functions in F for which $f(a_j) = f(a_{j'}) = f(a_{j''}) = 0$ is

$$\sum_{i=1}^{s^2+s+1} (1-f_i(a_j))(1-f_i(a_{j'}))(1-f_i(a_{j''})).$$

Thus,

$$R_3 = \sum_{i=1}^{s^2+s+1} \Big[f_i(a_i)(f_i(a_{j'})(a_{j''})$$

$$+ (1-f_i(a_j))(1-f_i(a_{j'}))(1-f_i(a_{j''}))\Big]$$

$$= \sum_{i=1}^{s^2+s+1} \Big[1-f_i(a_j) - f_i(a_{j'}) - f_i(a_{j''})$$

$$+ f_i(a_j)f_i(a_{j'}) + f_i(a_j)f_i(a_{j''})$$

$$f_i(a_{j'})f_i(a_{j''})\Big]$$

$$= (s^2 + s + 1) - 3(s + 1) + 3$$

$$= s^2 - 2s + 1 = (s-1)^2. \qquad (3.7)$$

which is a constant as required. Hence the proof of the lemma.

Example 3.2:- Consider the incidence matrix of PG(2,3) given as follows;

$$M = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \\ \ell_7 \\ \ell_8 \\ \ell_9 \\ \ell_{10} \\ \ell_{11} \\ \ell_{12} \\ \ell_{13} \end{array} \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Identifying the points of this geometry with the elements of the set S_{13} and the lines with functions of $F = \{f_1, f_2, \ldots, f_{13}\}$, the incidence matrix M forms a search matrix.

Taking any two points, say p_1 and p_2 which correspond to elements $a_1, a_2 \in S_{13}$, R_2 is equal to the number of functions in F for which $f(a_1) = f(a_2) = 1$ or $f(a_1) = f(a_2) = 0$. But, the number of functions in F for which $f(a_1) = f(a_2) = 1$ is

$$\sum_{i=1}^{13} f_i(a_1) f_i(a_2) = 1$$

and the number of functions in F for which $f(a_1) = f(a_2) = 0$ is;

$$\sum_{i=1}^{13} \left[1-f_i(a_1)\right]\left[1-f_i(a_2)\right]$$

$$= \sum_{i=1}^{13} \left[1-f_i(a_1)-f_i(a_2)+f_i(a_1)f_i(a_2)\right]$$

$$= 13 - \sum_{i=1}^{13} f_i(a_1) - \sum_{i=1}^{13} f_i(a_2)$$

$$+ \sum_{i=1}^{13} f_i(a_1) f_i(a_2)$$

$$= 13 - 4 - 4 + 1 = 6$$

Thus,

$$R_2 = 1 + 6 = 7$$

which is constant for any pair of elements $(a_j, a_{j'})$, implying that the system of functions (lines) is weakly homogeneous of order 2.

Theorem 3.1:- The expected duration of the search process based on the incidence matrix of PG(2,s) for detecting one unknown element x, denoted by $E_1(x)$, satisfies the inequality:

$$E_1(x) \le \frac{(s+1)(s^2+s+1)}{2}\left[1 + \frac{(s^2+s-1)(s^2-2s+1)}{9s}\right]$$

$$- \frac{s(s+1)}{2(s^2+s+1)}\left(4s + (s^2+s-1)(s^2-2s-1)\right)$$

Proof.

From Equations (3.6) and (3.7), $R_2 = s^2 - s + 1$ and $R_3 = s^2 - 2s + 1$ respectively. Substituting these values and $R_1 = s^2 + s + 1$ in (3.2) and (3.3) we obtain:

$$P_1(N,x) \le 1 - (s^2+s)\left(\frac{s^2-s+1}{s^2+s+1}\right)^N + \binom{s^2+s}{2}\left(\frac{s^2-2s+1}{s^2+s+1}\right)^N$$

and

$$P_1(N-1,x) \geq 1 - (s^2+s)\left(\frac{s^2-s+1}{s^2+s+1}\right)^{N-1}$$

But

$$P_1(N,x) = P_1(N,x) - P_1(N-1,x)$$

$$\leq \left[1-(s^2+s)\left(\frac{s^2-s+1}{s^2+s+1}\right)^{N} + \binom{s^2+s}{2}\left(\frac{s^2-2s+1}{s^2+s+1}\right)^{N}\right]$$

$$- \left[1-(s^2+s)\left(\frac{s^2-s+1}{s^2+s+1}\right)^{N-1}\right]$$

$$= s(s+1)\left(\frac{s^2-s+1}{s^2+s+1}\right)^{N-1}\left(1-\frac{s^2-s+1}{s^2+s+1}\right) + \frac{(s^2+s)(s^2+s-1)}{2}$$

$$x \; \frac{s^2-2s+1}{s^2+s+1}\left(\frac{s^2-2s+1}{s^2+s+1}\right)^{N-1}$$

$$= \frac{2s^2(s+1)}{s^2+s+1}\left(\frac{s^2-s+1}{s^2+s+1}\right)^{N-1}$$

$$+ \frac{s(s^2+1)(s^2+s-1)(s^2-2s+1)}{2(s^2+s+1)}\left(\frac{s^2-2s+1}{s^2+s+1}\right)^{N-1}, \quad \text{for } N > 2 \tag{3.8}$$

and

$$P_1(1,x) = 0, \quad \text{for } N = 1$$

since no single function derived from PG(2,s) can detect th[e] unknown element.The expected duration of the search [...] is;

$$E_1(x) = \sum_{N=2}^{x} N.P_1(N,x) \qquad \text{c.f(1.[illegible]}$$

which implies that

$$E_1(x) \leq \sum_{N=2}^{x} N\left[\frac{2s^2(s+1)}{s^2+s+1}\left(\frac{s^2-s+1}{s^2+s+1}\right)^{N-1}\right.$$

$$\left. + \frac{s(s+1)(s^2+s-1)(s^2-2s-1)}{2(s^2+s+1)}\left(\frac{s^2-2s+1}{s^2+s+1}\right)^{N-1}\right]$$

$$= \frac{2s^2(s+1)}{(s^2+s+1)}\left(1-\frac{s^2-s+1}{s^2+s+1}\right)^{-2} - \frac{2s^2(s+1)}{s^2+s+1}$$

$$+ \frac{s(s+1)(s^2+s-1)(s^2-2s+1)}{2(s^2+s+1)}\left(1-\frac{s^2-2s+1}{s^2+s+1}\right)^{-2}$$

$$- \frac{s(s+1)(s^2+s-1)(s^2-2s+1)}{2(s^2+s+1)}$$

$$= \frac{2s^2(s+1)}{(s^2+s+1)} \times \frac{(s^2+1)^2}{4s^2} + \frac{s(s+1)(s^2+s-1)(s^2-2s+1)(s^2+s+1)^2}{2(s^2+s+1).9s^2}$$

$$- \frac{s(s+1)}{2(s^2+s+1)}(4s + (s^2+s-1)(s^2-2s-1))$$

$$= \frac{(s+1)(s^2+s+1)}{2}\left(1 + \frac{s^2+s-1)(s^2-2s+1)}{9s}\right)$$

$$- \frac{s(s+1)}{2(s^2+s+1)}(4s + (s^2+s-1)(s^2-2s-1)) \qquad (3.9)$$

which is the required result. The upper bound of the expected duration of the search process given in (3.9) clearly increases with increase in s.

Example 3.3: Consider the incidence matrix of PG(2,3) given as;

$$M = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \\ \ell_7 \\ \ell_8 \\ \ell_9 \\ \ell_{10} \\ \ell_{11} \\ \ell_{12} \\ \ell_{13} \end{array} \left[\begin{array}{ccccccccccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} & P_{11} & P_{12} & P_{13} \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \text{[illegible]} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \text{[illegible]} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Identifying the points of PG(2,3) with the elements $a_1, a_2, \ldots\ldots\ldots\ldots, a_{13}$ of the set S_{13} and the lines with functions $f_1, f_2, \ldots, f_{13}$ of the system F, the incidence matrix M of PG(2,3) forms a search matrix. The system $F = \{f_1, f_2, \ldots, f_{13}\}$ would detect any unknown element x in S_{13} since the columns of the incidence matrix of PG(2,3) which is a search matrix M are distinct and thus, F is a separating system in S_{13}.

Using the incidence matrix of PG(2,3) as a search matrix, the probability that the search process for detecting one unknown element, x terminates in exactly N steps, $p_1(N,x)$ and the duration of the search process $E_1(x)$ satisfy the inequalities 3.8 and 3.9 respectively. That is,

$$p_1(N,x) \le \frac{2x9x4}{13}\left(\frac{7}{13}\right)^{N-1} + \frac{3x4x12x4}{2x13}\left(\frac{4}{13}\right)^{N-1}$$

$$= \frac{72}{13}\left(\frac{7}{13}\right)^{N-1} + \frac{576}{26}\left(\frac{4}{13}\right)^{N-1}$$

and

$$E_1(x) \le 72.2.$$

The exact values for $p_1(N,x)$ and $E_1(x)$ have been computed by Chakravarti and Manglik (1972) and found to be:

$$p_1(N,x) = \frac{72}{13}\left(\frac{7}{13}\right)^{N-1} - \frac{66x9}{13}\left(\frac{4}{13}\right)^{N-1} + \frac{72x10}{13}\left(\frac{3}{13}\right)^{N-1} - \frac{23x12}{13}\left(\frac{1}{13}\right)^{N-1}$$

and

$$E_1(x) = 49.64$$

Clearly, the exact expected duration of the search process given above satisfies the inequality given in (3.9).

We note here that the formula given above gives an upper bound far from the exact value, thus an improvement on this bound is necessary.

3.3 RANDOM SEARCH MODELS BASED ON FINITE PLANE EUCLIDEAN GEOMETRIES: EG(2, s).

Again we recall that the incidence matrix of EG(2,s) is a mxn matrix $M = ((a_{ij}))$ where $m = s^2+s$, $n = s^2$ and $a_{ij} = 0$ or 1 depending on whether the j-th point is incident with the i-th line or not ($i = 1,2,..,m$; $j = 1,2,...,n$)

Identifying the points of EG(2,s) with elements of the set S_n and the lines with functions of F, we see that the incidence matrix M of EG(2,s) is a search matrix.

Lemma 3.3: The system of functions (lines) F, derived from EG(2,s) is a weakly homogeneous system of order 2.

Proof.

Let $M = (f_i(a_j))$ be the search matrix of the strategy based on the system F. Then F will be a weakly homogeneous system of order 2 if R_2, which is the number of functions in F for which

$$f(a_j) = f(a_{j'}), \qquad a_j \neq a_{j'};$$

is constant. That is, the number of functions in F for which $f(a_j) = f(a_{j'}) = 1$ or $f(a_j) = f(a_{j'}) = 0$, $a_j \neq a_{j'}$ is constant.

But, the number of functions in F for which $f(a_j) = f(a_{j'}) = 1$ is

$$\sum_{\iota=1}^{s^2+s} f_\iota(a_j) f_\iota(a_{j'}), \qquad a_j \neq a_{j'}$$

and the number of functions in F, for which $f(a_j) = f(a_{j'}) = 0$ is

$$\sum_{\iota=1}^{s^2+s} (1 - f_\iota(a_j))(1 - f_\iota(a_{j'})).$$

Thus,

$$R_2 = \sum_{\iota=1}^{s^2+s} \Big[f_\iota(a_j)\, f_\iota(a_{j'}) + (1 - f_\iota(a_j))(1 - f_\iota(a_{j'})) \Big]$$

$$= \sum_{\iota=1}^{s^2+s} \Big[1 - f_\iota(a_j) - f_\iota(a_{j'}) + 2\, f_\iota(a_j) f_\iota(a_{j'}) \Big]$$

$$= (s^2 + s) - 2(s + 1) + 2$$

$$= s^2 - s. \qquad (3.12)$$

which is a constant as required. Hence F is a weakly homogeneous system of order 2.

Lemma 3.4: The system of functions (lines) F derived from EG(2,s) is weakly homogeneous of order 3.

Proof

The system F will be weakly homogeneous of order 3 if R_3, which is the number of functions in F for which

$$f(a_j) = f(a_{j'}) = f(a_{j''}) \qquad a_j \neq a_{j'} \neq a_{j''}$$

is constant. That is, the number of functions in F for which $f(a_j) = f(a_{j'}) = f(a_{j''}) = 1$ or $f(a_j) = f(a_{j'}) = f(a_{j''}) = 0$, $a_j \neq a_{j'} \neq a_{j''}$ is constant.

But, the number of functions in F for which $f(a_j) = f(a_{j'}) = f(a_{j''}) = 1$ is;

$$\sum_{i=1}^{s^2+s} f_i(a_j)f_i(a_{j'})f_i(a_{j''}), \quad a_j \neq a_{j'} \neq a_{j''}$$

and the number of functions in F for which $f(a_j) = f(a_{j'}) = f(a_{j''}) = 0$ is;

$$\sum_{i=1}^{s^2+s} \Big[(1-f_i(a_j))(1-f_i(a_{j'}))(1-f_i(a_{j''}))\Big].$$

Thus,

$$R_3 = \sum_{i=1}^{s^2+s} \Big[f_i(a_j)f_i(a_{j'})f_i(a_{j''})$$

$$+ (1-f_i(a_j))(1-f_i(a_{j'}))(1-f_i(a_{j''}))\Big]$$

$$= \sum_{i=1}^{s^2+s} \Big[1-f_i(a_j) - f_i(a_{j'}) - f_i(a_{j''})$$

$$+ f_i(a_j)f_i(a_{j'}) + f_i(a_j)f_i(a_{j''})$$

$$+ f_i(a_{j'})f_i(a_{j''})\Big]$$

$$= (s^2 + s) - 3(s + 1) + 3$$

$$= s^2 - 2s \qquad (3.13)$$

which is a constant as required. Hence the proof of the lemma.

Example 3.4: Consider the incidence matrix of EG(2,3) given as follows:

M =

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
l_1	1	0	0	1	0	1	1	0	0
l_2	1	1	1	0	0	0	0	0	1
l_3	1	0	0	0	0	1	0	0	0
l_4	1	0	0	1	1	0	0	0	1
l_5	0	1	0	0	1	0	0	1	0
l_6	0	0	1	0	0	1	0	1	1
l_7	0	0	0	1	1	1	0	0	0
l_8	0	0	1	1	0	0	0	1	0
l_9	0	0	1	1	0	1	0	1	0
l_{10}	0	0	0	0	0	0	1	1	1
l_{11}	0	1	0	0	0	1	1	0	0
l_{12}	0	0	1	0	1	0	1	0	1

Identifying the points of this geometry with the elements of the set $S_9 = \{a_1, a_2, .., a_9\}$ and the lines with functions of $F = \{f_1, f_2, .., f_{12}\}$ the incidence matrix M forms a search matrix.

Taking any two points, say p_1 and p_2 which correspond to the elements a_1, a_2 of S_9, R_2 is equal to the number of functions in F for which $f(a_1) = f(a_2) = 1$ or $f(a_1) = f(a_2) = 0$. But, the number of functions in F for which $f(a_1) = f(a_2) = 1$ is;

$$\sum_{i=1}^{12} f_i(a_1)f_i(a_2) = 1$$

and the number of functions in F which $f(a_1) = f(a_2) = 0$ is;

$$\sum_{i=1}^{12}(1-f_i(a_1))(1-f_i(a_2)) = \sum_{i=1}^{12}\Big[1-f_i(a_1)-f_i(a_2) + f_i(a_1)f_i(a_2)\Big]$$

$$= 12 - \sum_{i=1}^{12} f_i(a_1) - \sum_{i=1}^{12} f_i(a_2) - \sum_{i=2}^{12} f_i(a_1)f_i(a_2)$$

$$= 12 - 4 - 4 + 1 = 5.$$

Thus

$$R_2 = 1 + 5 = 6$$

which is constant for any pair of elements $(a_j, a_{j'})$ implying that the system of functions F is weakly homogeneous of order 2.

Theorem 3.2: The expected duration of the search process based on the incidence matrix of EG(2,s) for detecting unknown element x, $E_1(x)$ satisfies the inequality

$$E_1(x) \leq \Big[(s-1)(s+1)^2/2\Big]\Big[1+2(s^2-2)(s-2)/9\Big].$$

Proof

From (3.12) and (3.13), $R_2 = s^2 - s$ and $R_3 = s^2 - 2s$. Substituting these values and $R_1 = s^2 + s$ in (3.2) and (3.3) we obtain:

$$P_1(N,x) \leq 1 - (s^2-1)\left(\frac{s^2-s}{s^2+s}\right)^N + \binom{s^2-1}{2}\left(\frac{s^2-2s}{s^2+s}\right)^N$$

and

$$P_1(N-1,x) \geq 1 - (s^2-1)\left(\frac{s^2-s}{2}\right)^{N-1}.$$

But

$$\begin{aligned} p_1(N,x) &= P_1(N,x) - P_1(N-1,x) \\ &\leq (s^2-1)\left(\frac{s^2-s}{s^2+s}\right)^{N-1}\left(1-\frac{s^2-s}{2}\right) \\ &\quad + \binom{s^2-1}{2}\left(\frac{s^2-2s}{s^2+s}\right)^N \\ &= \frac{(s^2-1).2s}{s^2+s}\left(\frac{s^2-s}{s^2+s}\right)^{N-1} \\ &\quad + \frac{s(s^2-1)(s^2-2)(s-2)}{2s(s+1)}\left(\frac{s^2-2s}{s^2+s}\right)^{N-1}. \end{aligned}$$

That is,

$$\begin{aligned} p_1(N,x) &\leq 2(s-1)\left(\frac{s^2-s}{s^2+s}\right)^{N-1} \\ &\quad + \frac{(s-1)(s^2-2)(s-2)}{2}\left(\frac{s^2-2s}{s^2+s}\right)^{N-1} \end{aligned}$$

The expected duration of the search process is given by:

$$E_1(x) = \sum_{N=0}^{\infty} N \cdot p_1(N,x) \qquad \text{c.f(1.7)}$$

which implies

$$E_1(x) \le \sum_{N=0}^{\infty} N\left[2(s-1)\left(\frac{s^2-s}{s^2+s}\right)^{N-1} + \frac{(s-1)(s^2-2)(s-2)}{2}\left(\frac{s^2-2s}{s^2+s}\right)^{N-1}\right]$$

$$= 2(s-1)\left(1-\frac{s^2-s}{s^2+s}\right)^{-2} - 2(s-1) + \frac{(s-1)(s^2-2)(s-2)}{2}\left(1-\frac{s^2-2s}{s^2+s}\right)^{-2} - \frac{(s-1)(s^2-2)(s-2)}{2}$$

$$= \frac{2(s-1)s^2(s+1)^2}{4s^2} + \frac{(s-1)(s^2-2)(s-2)s^2(s+1)^2}{9s^2} - \frac{s-1}{2}(4 + (s^2-2)(s-2))$$

$$= \frac{(s-1)(s+1)^2}{2}\left[1 + 2\frac{(s^2-2)(s-2)}{9}\right] - \frac{s-1}{2}(4 + (s^2-2)(s-2)) \qquad (3.15)$$

which is the required result. Hence, the proof of the theorem. From (3.15) we see that the expected duration of the search process based on the incidence matrix of EG(2,s) increases with increase in s.

Example 3.5: Consider the incidence matrix of EG(2,2) given as follows;

$$M = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \end{array} \begin{array}{c} \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Identifying the points of EG(2,2) with the elements a_1, a_2, a_3, a_4 of set S_4 and the lines with functions $f_1, f_2, \ldots\ldots\ldots\ldots, f_6$ of the system F, the incidence matrix M of EG(2,2) forms a search matrix. The system $F = \{f_1, f_2, \ldots, f_6\}$ would detect any unknown element x in S_4 since the columns of the search matrix M are distinct and thus, F is a separating system on S_4. Using the incidence matrix of EG(2,2) as a search matrix, the probability $p_1(N,x)$, that the search process for detecting one unknown element x, terminates in exactly N steps, and the duration, $E_1(x)$ of the search process satisfy the inequalities (3.14) and (3.15) respectively. That is,

$$p_1(N,x) \leq 2\left(\frac{1}{3}\right)^{N-1} \qquad (3.16)$$

and

$$E_1(x) \leq \frac{5}{2} \qquad (3.17)$$

To obtain the exact value of $P_1(N,x)$ and $E_1(x)$, we substitute $R_1 = s^2 + s$, [illegible] and $R_3 = s^2 - 2s$ in (3.2) and (3.3) to get

$$P_1(N,x) \geq 1 - (s^2 - 1)\left(\frac{s^2 - s}{s^2 + s}\right)^N \qquad (3.18)$$

$$P_1(N,x) \leq 1 - (s^2 - 1)\left(\frac{s^2 - s}{s^2 + s}\right)^N + \binom{s^2 - 1}{2}\left(\frac{s^2 - 2s}{s^2 + s}\right)^N \qquad (3.19)$$

For s = 2, the expressions (3.18) and (3.19) reduce to;

$$P_1(N,x) \geq 1 - 3\left(\frac{1}{3}\right)^N$$

and

$$P_1(N,x) \leq 1 - 3\left(\frac{1}{3}\right)^N$$

Thus

$$P_1(N,x) = 1 - 3\left(\frac{1}{3}\right)^N$$

and from

$$p_1(N,x) = P_1(N,x) - P_1(N-1,x)$$

we get

$$p_1(N,x) = \left[1 - 3\left(\frac{1}{3}\right)^N\right] - \left[1 - 3\left(\frac{1}{3}\right)^{N-1}\right]$$

$$= 3\left(\frac{1}{3}\right)^{N-1}\left(1 - \frac{1}{3}\right) = 2\left(\frac{1}{3}\right)^{N-1}, N \geq 2$$

and

$$p_1(1,x) = 0.$$

since no single function (line) can detect the unknown element.

The expected duration of the search process $E_1(x)$ is then given by;

$$E_1(x) = 2\sum_{N=2}^{x} N\left(\frac{1}{3}\right)^{N-1}$$

$$= 2\left(\frac{1}{(1-\frac{1}{3})^2} - 1\right)$$

$$= 2.5$$

Thus, the expected duration of the search process

for detecting one unknown element using the incidence matrix of EG(2,2) as a search matrix is 2.5 test-functions. This expected duration of the search process satisfies the inequality given in (3.17).

We note here that although the formula given in section 3.2 gives an upper bound far from the exact value in example 3.4, the above formula gives an upper bound which concides with the exact value in this example.

3.4 SEARCH MODELS BASED ON RANDOM 0-1 MATRICES.

Consider an mxn matrix $M = ((a_{ij}))$, whose entries a_{ij} $(i = 1,2,...,m;\ j = 1,2,...,n)$ take only two values 0 and 1 with equal probabilities. That is

$$\text{Prob.}(a_{ij} = 0) = \text{prob.}(a_{ij} = 1) = \frac{1}{2}$$

Then the matrix M is called a *random 0-1 matrix.*

Identifying the ith column of the matrix M with the element a_i of the set $S_n = \{a_1, a_2, .., a_n\}$ and the jth row with the function f_j of the system $F = \{f_1, f_2, ..., f_m\}$, the random 0 - 1 matrix M gives a search matrix of the system F.

Let $x \in S_n$, be the unknown element whose identity we wish to determine by choosing a sequence of functions $f_1, f_2, .., f_N$ from the system F and observing the values of these functions at the unknown element x, until enough information is obtained to determine the unknown element. The unknown element, x would then be determined in any of the following mutual exclusive cases:

(i) only one function in F, say f_j is selected. The unknown element would be determined if in the submatrix consisting of the jth row of M there exists "1" in the xth column and 0's in the remaining (n-1) columns or "0" on the xth column and 1's on the remaining (n-1) columns. The probability of such arrangement is

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n-1}$$

and the expected number of such functions (rows) is

$$m\left(\frac{1}{2}\right)^{n-1}.$$

(ii) Two functions in F, say f_{j_1} and f_{j_2} are selected. Then the unknown element would be determined if in the submatrix consisting of the j_1th and j_2th rows of M, the xth column is different from any other column. Possible columns of the submatrix consisting of the j_1th and j_2th rows of M are:

$$\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix} \text{ and } \begin{pmatrix}0\\0\end{pmatrix}$$

$$\text{Prob.}\begin{pmatrix}1\\1\end{pmatrix} = \text{Prob.}\begin{pmatrix}1\\0\end{pmatrix} = \text{Prob.}\begin{pmatrix}0\\1\end{pmatrix} = \text{Prob.}\begin{pmatrix}0\\0\end{pmatrix} = \frac{1}{4}.$$

Thus, the probability that a column will be different from all other columns is

$$\left(\frac{1}{4}\right)\cdot\left(\frac{3}{4}\right)^{n-1} + \left(\frac{1}{4}\right)\cdot\left(\frac{3}{4}\right)^{n-1} + \left(\frac{1}{4}\right)\cdot\left(\frac{3}{4}\right)^{n-1} + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1} = \left(\frac{3}{4}\right)^{n-1}$$

And the expected number of pairs of rows with a column different from all other columns is

$$\binom{m}{2}\cdot\left(\frac{3}{4}\right)^{n-1}.$$

(iii) Generally a sequence of k functions $f_1, f_2, f_3, \ldots, f_k$ will determine the unknown element if in the submatrix consisting of k rows of M, the xth column is different from any other column, possible columns of such a submatrix of M are:

$$\begin{pmatrix}1\\1\\\vdots\\\vdots\\1\end{pmatrix}, \begin{pmatrix}1\\0\\\vdots\\\vdots\\1\end{pmatrix}, \ldots\ldots\ldots\ldots\ldots\ldots, \begin{pmatrix}0\\0\\\vdots\\\vdots\\0\end{pmatrix}$$

with

$$\text{Prob.}\begin{pmatrix}1\\1\\\vdots\\\vdots\\1\end{pmatrix} = \text{Prob.}\begin{pmatrix}1\\0\\\vdots\\\vdots\\1\end{pmatrix} = \ldots\ldots\text{Prob.}\begin{pmatrix}0\\0\\\vdots\\\vdots\\0\end{pmatrix} = \frac{1}{2^k}.$$

Thus, the probability that a column will be different from all other columns is

$$\left(\frac{1}{2^k}\right)\left(1-\frac{1}{2^k}\right)^{n-1} + \frac{1}{2^k}\left(1-\frac{1}{2^k}\right)^{n-1} + \ldots\ldots\ldots$$

$$= \text{[illegible]}\left(\frac{\text{[illegible]}}{2^k}\right)\text{[illegible]}^{n-1} \quad \left(1-\frac{1}{2^k}\right)^{n-1}$$

and the expected number of k rows of M with a column different from all other columns is

$$\binom{m}{k}\cdot\left(1-\frac{1}{2^k}\right)^{n-1}. \qquad (3.20)$$

Termination of the search process.

To determine the probability that the search process will terminate at the Nth step, we consider the complementary event that the search process will not terminate in N steps. To do this we require the following counting lemma.

Lemma 3.5: Let t be the number of ways of placing N balls in m cells such that all the m cells are occupied, then;

$$t = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^N .$$

For proof of this lemma see Renyi (1970).

Lemma 3.6: Let $P_1^c(N,x)$ be the probability that the sequence

$$f_{i_1}, f_{i_1}, \ldots, f_{i_1}, f_{i_2}, f_{i_2}, \ldots, f_{i_2}, f_{i_3}, \ldots, f_{i_3}, \ldots, f_{i_\ell}, \ldots, f_{i_\ell}$$

of length N will not detect the unknown element x . Then,

$$P_1^c(N,x) = \left\{ \binom{n}{\ell} \left[1 - \left(1 - \frac{1}{2^\ell} \right)^{n-1} \right] \sum_{i=1}^{\ell} (-1)^i \binom{\ell}{i} (\ell - i)^N \right\} \Big/ m^N .$$

Proof

Taking the functions $f_{i_1}, f_{i_2}, \ldots, f_{i_\ell}$ to be cells and the length N to be number of balls in Lemma (3.5), we find that the number of ways of arranging the sequence

$$f_{i_1}, f_{i_1}, \ldots, f_{i_1}, f_{i_2}, \ldots, f_{i_2}, f_{i_3}, \ldots, f_{i_3}, \ldots, f_{i_\ell}, \ldots, f_{i_\ell}$$

of functions such that all the functions $f_{i_1}, f_{i_2}, \ldots, f_{i_\ell}$ appear at least once in each sequence is

$$\sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} (\ell-i)^N .$$

But from (3.20) the expected number of ℓ rows of matrix M with a column different from all other columns and thus detect the unknown element is

$$\binom{m}{\ell}\left(1-\frac{1}{2^\ell}\right)^{n-1}$$

so the expected number of ℓ rows of matrix M which do not detect the unknown element is

$$\binom{m}{\ell} - \binom{m}{\ell}\left(1-\frac{1}{2^\ell}\right)^{n-1}.$$

Thus, the expected number of sequences

$$f_{i_1}, f_{i_1}, \ldots, f_{i_1}, f_{i_2}, \ldots, f_{i_2}, f_{i_3}, \ldots, f_{i_3}, \ldots, f_{i_\ell}, \ldots, f_{i_\ell}$$

which do not detect the unknown element is

$$\binom{m}{\ell}\left\{1-\left(1-\frac{1}{2^\ell}\right)^{n-1}\right\}\sum_{i=1}^{\ell} (-1)^i \binom{\ell}{i} (\ell-i)^N$$

and the probability that the sequence

$$f_{i_1}, f_{i_1}, \ldots, f_{i_1}, f_{i_2}, \ldots, f_{i_2}, f_{i_3}, \ldots, f_{i_3}, \ldots, f_{i_\ell}, \ldots, f_{i_\ell}$$

will not detect the unknown element is therefore;

$$P_1^c(N,x) = \left[\binom{m}{\ell}\left\{1-\left(1-\frac{1}{2^\ell}\right)^{n-1}\right\}\sum_{i=1}^{\ell} (-1)^i \binom{\ell}{i} (\ell-i)^N\right] \Big/ m^N ,$$

which completes the proof of Lemma 3.6. We illustrate this Lemma by computing the expected probability that the search process will not terminate within N steps for $\ell = 3$. That is, a sequence of three functions

$$f_{\iota_1}, f_{\iota_1}, \ldots, f_{\iota_1}, f_{\iota_2}, \ldots, f_{\iota_2}, f_{\iota_3}, \ldots, f_{\iota_3}$$

in $F = \left\{f_1, f_2, \ldots, f_m\right\}$ will not detect the unknown element.

Now, the unknown element x will be detected by three functions $f_{\iota_1}, f_{\iota_2}, f_{\iota_3}$ if in the submatrix consisting of the ι_1-th, ι_2-th and ι_3-th rows of M, the xth column is different from any other column. Possible columns of such a submatrix of M consisting of three rows are:

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}$$

with

$$Pr.\begin{pmatrix}1\\1\\1\end{pmatrix} = Pr.\begin{pmatrix}1\\1\\0\end{pmatrix} = Pr.\begin{pmatrix}1\\0\\0\end{pmatrix} = Pr.\begin{pmatrix}0\\1\\1\end{pmatrix} = Pr.\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$= Pr.\begin{pmatrix}0\\1\\0\end{pmatrix} = Pr.\begin{pmatrix}0\\0\\1\end{pmatrix} = Pr.\begin{pmatrix}0\\0\\0\end{pmatrix} = \frac{1}{2^3}.$$

Thus, the probability that a column will be different from all other columns is;

$$\left(\frac{1}{2^3}\right)\left(1-\frac{1}{2^3}\right)^{n-1} + \left(\frac{1}{2^3}\right)\left(1-\frac{1}{2^3}\right)^{n-1}$$

$$+ \frac{1}{2^3}\left(1-\frac{1}{2^3}\right)^{n-1} + \frac{1}{2^3}\left(1-\frac{1}{2^3}\right)^{n-1} + \left(\frac{1}{2^3}\right)\left(1-\frac{1}{2^3}\right)^{n-1}$$

$$+ \left(\frac{1}{2^3}\right)\left(1-\frac{1}{2^3}\right)^{n-1} + \left(\frac{1}{2^3}\right)\left(1-\frac{1}{2^3}\right)^{n-1}$$

$$+ \left(\frac{1}{2^3}\right)\left(1-\frac{1}{2^3}\right)^{n-1} = \left(1-\frac{1}{2^3}\right)^{n-1}.$$

The expected number of three rows of M with a column different from all other columns is;

$$\binom{m}{3}\left(1-\frac{1}{2^3}\right)^{n-1}. \qquad (3.21)$$

Taking the functions f_{i_1}, f_{i_2} and f_{i_3} to be cells and the length N to be number of balls in lemma 3.5 we find that the number of ways of arranging the sequence

$$f_{i_1}, f_{i_1}, \ldots, f_{i_1}, f_{i_2}, \ldots, f_{i_2}, f_{i_3}, \ldots, f_{i_3}$$

of functions such that all the functions $f_{i_1}, f_{i_2}, f_{i_3}$ appear at least once in each sequence is

$$\sum_{j=0}^{3} (-1)^j \binom{3}{j} (3-j)^N = 3^N - 3.2^N - 3.$$

But from (3.21) the expected number of three rows of M with a column different from all other columns and thus detect the unknown element is

$$\binom{m}{3}\left(1-\frac{1}{2^3}\right)^{n-1}$$

so the expected number of three rows of matrix M which do not detect the unknown element is

$$\binom{m}{3} - \binom{m}{3}\left(1-\frac{1}{2^3}\right).$$

Thus, the expected number of sequences

$$f_{i_1}, f_{i_1}, \ldots, f_{i_1}, f_{i_2}, \ldots, f_{i_2}, f_{i_3}, \ldots, f_{i_3}$$

which do not detect the unknown element is

$$\binom{m}{3}\left[1-\left(1-\frac{1}{2^3}\right)^{n-1}\right]\left(3^N - 3.2^N + 3\right),$$

and the probability that the sequence

$$f_{i_1}, f_{i_1}, \ldots, f_{i_1}, f_{i_2}, \ldots, f_{i_2}, f_{i_3}, \ldots, f_{i_3}$$

will not detect the unknown element is therefore:

$$P_1^{c}(N,x) = \left[\binom{m}{3}\left\{1-(1-\frac{1}{2^3})\right]^{n-1}\right\}\left[3^N - 3.2^N + 3\right]\Big/ m^N$$

Remark: The probability that the search process does not terminate within N steps, $P_1^{c}(N,x)$ given in lemma 3.6 is the average of the probabilities that the search processes do not terminate within N steps. That is, if a number of random 0-1 matrices are considered then the average of the probabilities that the search processes do not terminate within N steps is given in lemma 3.6.

Corollary 3.1: The probability that the search process terminates in N or less steps is

$$P_1(N,x)=1-\left[\sum_{\ell=1}^{m}\binom{m}{\ell}\left\{1-\left[1-\frac{1}{2^{\ell}}\right]^{n-1}\right\}\sum_{i=1}^{\ell}(-1)^{i}\binom{\ell}{i}(\ell-i)^N\right]\Big/ m^N$$

Proof

From lemma 3.6 the probability that the search process will not terminate in N or less steps is

$$\left[\sum_{\ell=1}^{m} \binom{m}{\ell}\left\{1-\left(1-\frac{1}{2^{\ell}}\right)^{n-1}\right\}\sum_{\iota=1}^{\ell}(-1)^{\iota}\binom{\ell}{\iota}(\ell-\iota)^{N}\right]\Big/ m^{N} .$$

Thus, the probability that the search process will terminate in N or less steps is

$$P_1(N,x)= 1-\left[\frac{\sum_{\ell=1}^{m} \binom{m}{\ell}\left\{1-\left(1-\frac{1}{2^{\ell}}\right)^{n-1}\right\} \sum_{\iota=1}^{\ell} (-1)^{\iota}\binom{\ell}{\iota}(\ell-\iota)^{N}}{m^{N}}\right] .$$

Hence the proof of Corollary 3.1.

Example 3.6:- Let $M = ((a_{\iota j}))$ be a 5x5 matrix constructed from five rows and five columns of random numbers such that:

$$a_{\iota j} = \begin{cases} 1 \text{ if } (\iota,j)\text{th random number is even} \\ 0 \text{ if } (\iota,j)\text{th random number is odd.} \end{cases}$$

Then one possible such matrix is:

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} .$$

Now, since random numbers are even or odd with equal probabilities, the prob.$(a_{\iota j} = 1)$= prob.$(a_{\iota j} = 0)$= $\frac{1}{2}$.

Thus, the matrix M is a random 0-1 matrix.

Identifying the columns of the matrix M with the elements of the set $S_5 = \{a_1, a_2, a_3, a_4, a_5\}$, that is, the ith column corresponds to the element $a_i \in S_5$, and the functions f_1, f_2, f_3, f_4, f_5 with the rows of M, that is, the jth row corresponds to the function f_j, the random 0-1 matrix M gives a search matrix of the functions f_1, f_2, f_3, f_4, f_5.

Let $a_1 \in S_5$, be the unknown element whose identity we wish to determine by choosing a sequence of functions $f_1, f_2, \ldots\ldots\ldots, f_5$ from the system $F = \{f_1, f_2, f_3, f_4, f_5\}$ and observe the values of these functions at a_i until enough information is obtained to determine it.

To determine the probability of termination of the search process, we consider the complementary event, that is, the event that the search process does not terminate in N steps. We will use lemma 3.5 to get the number of sequences of length N which do not detect the unknown element.

The search process will not terminate in N steps if any of the following sequences occur:

(i) Only one function $f \in F$ is selected N times. The unkn[illegible] will not be detected because there is no row with "1" in the first column and 0's in the remaining 4 columns or 0 in the first column and 1's in the remaining 4 columns. The number of possible sequences is five, viz:

$f_1, f_1, \ldots, f_1;\ f_2, f_2, \ldots, f_2;\ f_3, f_3, \ldots, f_3;$

$f_4, f_4, \ldots, f_4;\ f_5, f_5, \ldots, f_5.$

(ii) Two functions f_1 and f_2 are selected x_1 and x_2 times respectively, where $x_1+x_2 = N$. In this case the unknown element a_1 will not be detected because the first and second columns of the submatrix consisting of the 1st and 2nd rows of matrix M are the same. The number of possible sequences of f_1 and f_2 is

$$\binom{2}{2}(2^N - 2) = 2^N - 2.$$

(iii) Two functions f_1 and f_3 are selected x_1 and x_2 times respectively, where $x_1+x_2 = N$. In this case the unknown element a_1 will not be detected because the first and second columns of the submatrix consisting of the 1st and 3rd rows of matrix M are the same. The number of possible sequences of f_1 and f_3 is

$$\binom{2}{2}(2^N - 2) = 2^N - 2.$$

Using similar a argument the sequences of the following functions will not detect the unknown element a_1.

(iv) Two functions f_1 and f_4 are selected x_1 and x_2 times respectively, where

$x_1 + x_2 = N$; the number of possible sequences of f_1 and f_4 is

$$\binom{2}{2}(2^N - 2) = 2^N - 2.$$

(v) Two functions f_2 and f_4 are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; the number of possible sequences of f_2 and f_4 is

$$\binom{2}{2}(2^N - 2) = 2^N - 2.$$

(vi) Two functions f_2 and f_5 are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; the number of possible sequences of f_2 and f_5 is

$$\binom{2}{2}(2^N - 2) = 2^N - 2.$$

(vii) Two functions f_2 and f_3 are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; the number of possible sequences of f_2 and f_3 is

$$\binom{2}{2}(2^N - 2) = 2^N - 2.$$

(viii) Two functions f_3 and f_4 are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; the number of possible sequences of f_3 and f_4 is

$$\binom{2}{2}(2^N - 2) = 2^N - 2.$$

(ix) Two functions f_3 and f_5 are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; the number of possible sequences of f_3 and f_5 is

$$\binom{2}{2}(2^N - 2) = 2^N - 2.$$

(x) Three functions f_1, f_2, f_3 are selected x_1, x_2, x_3 times respectively, where $x_1 + x_2 + x_3 = N$; the number of possible sequences of f_1, f_2, f_3 is

$$\binom{3}{3}(3^N - 3.2^N + 3) = 3^N - 3.2^N + 3.$$

(xi) Three functions f_1, f_2, f_4 are selected x_1, x_2, x_3 times respectively, where $x_1 + x_2 + x_3 = N$; the number of possible sequences of f_1, f_2, f_4 is

$$\binom{3}{3}(3^N - 3.2^N + 3) = 3^N - 3.2^N + 3.$$

(xii) Three functions f_1, f_3, f_4 are selected x_1, x_2, x_3 times respectively, where $x_1 + x_2 + x_3 = N$; the number of possible sequences of f_1, f_3, f_4 is

$$\binom{3}{3}(3^N - 3.2^N + 3) = 3^N - 3.2^N + 3.$$

(xiii) Three functions f_2, f_3, f_4 are selected x_1, x_2, x_3 times respectively, where $x_1 + x_2 + x_3 = N$; the number of possible sequences of f_2, f_3, f_4 is

$$\binom{3}{3}(3^N - 3.2^N + 3) = 3^N - 3.2^N + 3.$$

(xiv) Three functions f_2, f_3, f_5 are selected x_1, x_2, x_3 times respectively, where $x_1 + x_2 + x_3 = N$; the number of possible sequences of f_2, f_3, f_5 is

$$\binom{3}{3}(3^N - 3.2^N + 3) = 3^N - 3.2^N + 3.$$

(xv) Four functions f_1, f_2, f_3, f_4 are selected x_1, x_2, x_3, x_4 times respectively, where $x_1 + x_2 + x_3 + x_4 = N$; the number of possible sequences of f_1, f_2, f_3, f_4 is

$$\binom{4}{4}(4^N - 4.3^N + 6.2^N - 4) = (4^N - 4.3^N + 6.2^N - 4).$$

Thus, the probability of the search process terminating in N or less steps is;

$$P_1(N,x) = 1 - \left[\frac{8(2^N - 2) + 5(3^N - 3.2^N + 3) + (4^N - 4.3^N + 6.2^N - 4) + 5}{5^N}\right]$$

$$= 1 - \left[\frac{-2^N + 3^N + 4^N}{5^N}\right]$$

i.e

$$P_1(N,x) = 1 - \left(\frac{4}{5}\right)^N - \left(\frac{3}{5}\right)^N + \left(\frac{2}{5}\right)^N$$

and the probability of the search process terminating in exactly N steps is

$$p_1(N,x) = P_1(N,x) - P_1(N-1,x)$$

$$= \left\{1-\left[\left(\frac{4}{5}\right)^{N} + \left(\frac{3}{5}\right)^{N} - \left(\frac{2}{5}\right)^{N}\right]\right\}$$

$$- \left\{1-\left[\left(\frac{4}{5}\right)^{N-1} + \left(\frac{3}{5}\right)^{N-1} - \left(\frac{2}{5}\right)^{N-1}\right]\right\}$$

$$= \left(\frac{4}{5}\right)^{N-1}\left(1-\frac{4}{5}\right) + \left(\frac{3}{5}\right)^{N-1}\left(1-\frac{3}{5}\right)$$

$$+ \left(\frac{2}{5}\right)^{N-1}\left(\frac{2}{5}-1\right)$$

$$= \frac{1}{5}\left(\frac{4}{5}\right)^{N-1} + \frac{2}{5}\left(\frac{3}{5}\right)^{N-1} - \frac{3}{5}\left(\frac{2}{5}\right)^{N-1}.$$

The expected duration of the search process is

$$E_1(x) = \sum_{N=1}^{\infty} N.\ p_1(N,x) \qquad \text{c .f (1.7)}$$

$$= \frac{1}{5}\sum_{N=1}^{\infty} N\left(\frac{4}{5}\right)^{N-1} + \frac{2}{5}\sum_{N=1}^{\infty} N\left(\frac{3}{5}\right)^{N-1} - \frac{3}{5}\sum_{N=1}^{\infty} N.\left(\frac{2}{5}\right)^{N-1}$$

$$= \frac{1}{5}\left(\frac{1}{(1-4/5)^2}\right) + \frac{2}{5}\left(\frac{1}{(1-3/5)^2}\right) - \frac{3}{5}\left(\frac{1}{(1-2/5)^2}\right)$$

$$= \frac{1}{5} \times 25 + \frac{2}{5} \times \frac{25}{4} - \frac{3}{5} \times \frac{25}{9}$$

$$= 5.83.$$

Thus, an average of 5.83 test-functions required to detect the unknown element a_1.

Note that the probability of termination of the search process and expected duration of the search process given here are for a specific example. If a number of random 0-1 matrices are considered then the average of the probabilities and durations of the search processes would be given by Lemma 3.6.

Remarks:- To compare search systems derived from incidence matrices of PG(2,s) and EG(2,s) with search systems derived from random 0-1 matrices, we first note the following:

(i) In a search system derived from the incidence matrix of PG(2,s) or EG(2,s) the number of functions is always greater than or at least equal to the number of elements.

(ii) The search systems derived from the incidence matrices of PG(2,s) and EG(2,s) are always separating systems.

(iii) In a search system derived from the random 0-1 matrices the number of functions can be less than, equal to or greater than the number of elements.

(iv) The search systems derived from the random 0-1 matrices are not always separating systems.

Now, since not all search systems derived from random 0-1 matrices are separating systems, one would prefer to use search systems derived from incidence matrices of PG(2,s) or EG(2,s) since such search systems are always separating systems.

CHAPTER 4

DETECTING MORE THAN ONE UNKNOWN ELEMENT.

4.1 INTRODUCTION.

In this Chapter, we study two different strategies for detecting more than one unknown element from a set S_n consisting of n distinguishable elements $a_1, a_2, \ldots\ldots, a_n$. We will first study strategies for detecting two unknown elements. These strategies are described below.

2-Complete search designs.

Using the definition of t-complete search design given in Section 1.2 of Chapter 1, we define a 2-complete search design as a system $\{A_1, A_2, \ldots\ldots, A_m; S_n\}$ consisting of m subsets $A_1, A_2, \ldots, A_m$ of a finite set S_n, in which for any pair of elements $a_\ell, a_{\ell'}$ in S_n, there exist subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$, $\{i_1, i_2, \ldots, i_k\} \subset \{1,2,\ldots,m\}$ such that $a_\ell, a_{\ell'} \in A_{i_j}$ for $j = 1,2,\ldots,k$ and $\bigcap_{j=1}^{k} A_{i_j} = \{a_\ell, a_{\ell'}\}$. Without any loss of generality we will assume that the subsets $A_{i_1}, A_{i_2}, \ldots\ldots, A_{i_k}$, are the only subsets in the set $\{A_1, A_2, \ldots\ldots\ldots, A_m\}$ which contain the pair $a_\ell, a_{\ell'}$.

To identify two unknown elements, say $u, v \in S_n$, we determine subsets $A_{i_1}, A_{i_2}, \ldots\ldots\ldots\ldots A_{i_k}$, $\{i_1, i_2, \ldots, i_k\} \subset \{1,2,\ldots\ldots,m\}$, such that $u, v \in A_{i_j}$

for $j = 1,2,\ldots,k$. The identity of the two unknown elements is then given by the intersection of these subsets, that is $\bigcap_{j=1}^{k} A_{i_j} = \{u,v\}$.

The following example illustrates this strategy.

Example 4.1: Suppose the system $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7;\ S_7\}$ constitutes a 2-Complete search design for separating the elements of the set $S_7 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$. Then one possible configuration of the subsets $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ is the following:

$$A_1 = \{a_4, a_5, a_6, a_7\},$$

$$A_2 = \{a_2, a_3, a_6, a_7\},$$

$$A_3 = \{a_2, a_3, a_4, a_5\},$$

$$A_4 = \{a_1, a_3, a_5, a_7\},$$

$$A_5 = \{a_1, a_3, a_4, a_6\},$$

$$A_6 = \{a_1, a_2, a_4, a_7\},$$

$$A_7 = \{a_1, a_2, a_5, a_6\}.$$

This design will detect any arbitrary pair of elements of S_7. That is, for any distinct pair $(a_\ell, a_{\ell'})$ of elements of the set S_7, there exists a pair of subsets A_{i_1}, A_{i_2} such that $a_\ell, a_{\ell'} \in A_{i_j}$, $j = 1,2$ and $A_{i_1} \cap A_{i_2} = \{a_\ell, a_{\ell'}\}$. Thus, to detect any pair of unknown elements in the set S_7 using this design, we determine subsets amongst

$A_1, A_2, A_3, A_4, A_5, A_6, A_7$ which contain the unknown pair of elements. The intersection of these subsets gives the identity of the unknown pair of elements.

More explicity, we have the following display of detectable pairs of elements and the associated subsets.

Subsets	Elements	Subsets	Elements
A_6, A_7	a_1, a_2	A_3, A_5	a_3, a_4
A_4, A_5	a_1, a_3	A_3, A_4	a_3, a_5
A_5, A_6	a_1, a_4	A_2, A_5	a_3, a_6
A_4, A_7	a_1, a_5	A_2, A_4	a_3, a_7
A_5, A_7	a_1, a_6	A_1, A_3	a_4, a_5
A_4, A_6	a_1, a_7	A_1, A_5	a_4, a_6
A_2, A_3	a_2, a_3	A_1, A_6	a_4, a_7
A_3, A_6	a_2, a_4	A_1, A_7	a_5, a_6
A_3, A_7	a_2, a_5	A_1, A_4	a_5, a_7
A_2, A_7	a_2, a_6	A_1, A_2	a_6, a_7

The display shows that every pair of the seven elements can be detected by a unique pair of subsets. For example, if (a_1, a_5) is the unknown pair of elements, then we determine subsets amongst $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ which contain both a_1 and a_5. The intersection of these subsets gives the identity of the unknown pair of elements. In this case, the subsets which contain both a_1 and a_5 are A_4 and A_7. The intersection of these subsets, A_4 and A_7 gives the identities of the unknown elements. That is, $A_4 \cap A_7 = \{a_1, a_5\}$.

We can further characterize this arrangement in terms of the incidence matrix of the search design.

This is an mxn matrix $N = ((n_{ij}))$ such that if $a_1, a_2, \ldots, a_n$ are the elements of the set S_n and $A_1, A_2, \ldots, A_m$ are the subsets, then:

$$n_{ij} = \begin{cases} 1 & \text{if } a_i \in A_j \qquad i = 1,2,\ldots,n \\ 0 & \text{if } a_i \notin A_j \qquad j = 1,2,\ldots,m. \end{cases}$$

In the above example we therefore have:

$$N = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad (4.1)$$

From this matrix, we notice that every element of S_n appears in four subsets, every pair of elements appears in two subsets and any three elements appear in at most one subset. Now, for any pair of elements to be uniquely detectable the number of subsets in which they appear must be strictly more than the number of subsets in which any three elements appear. This is because if the number of subsets in which any three elements appear is the same as the number of subsets in which any pair of elements appears, then the intersection of these subsets will consist of three elements, not two as required for the identification of the unknown pair of elements. This requirement is satisfied in this example, and so any pair of elements can be uniquely detected.

Partition search design.

Here the strategy is to determine m subsets $A_1, A_2, \ldots, A_m$ of S_n such that for any pair of elements a_i, a_j ($a_i \neq a_j$) in S_n, there exist two disjoint subsets A_ℓ and A_k with $a_i \in A_\ell$ and $a_j \in A_k$. The composite set $\{A_1, A_2, \ldots, A_m; S_n\}$ is then called a *partition search design.*

To detect two unknown elements, say $u, v \in S_n$, we determine two disjoint subsets A_{ι_1} and A_{ι_2}, $\{\iota_1, \iota_2\} \subset \{1, 2, \ldots, m\}$ such that $u \in A_{\iota_1}$ and $v \in A_{\iota_2}$. The two unknown elements are then identified separately from the subsets A_{ι_1} and A_{ι_2} by separating systems described in Chaper 2.

The following example illustrates this strategy.

Example 4.2: Consider the set $S_8 = \{a_1, a_2, \ldots, a_8\}$ and the subsets $A_1, A_2, A_3, A_4, A_5, A_6$ described below:

$$A_1 = \{a_1, a_2, a_3, a_4\},$$

$$A_2 = \{a_5, a_6, a_7, a_8\},$$

$$A_3 = \{a_1, a_2, a_7, a_8\}$$

$$A_4 = \{a_3, a_4, a_5, a_6\}$$

$$A_5 = \{a_1, a_3, a_5, a_7\},$$

$$A_6 = \{a_2, a_4, a_6, a_8\}.$$

Then, the system $\{A_1, A_2, A_3, A_4, A_5, A_6; S_8\}$ constitutes a partition search design, since for every distinct pair (a_i, a_j) of elements of S_8, there exists a pair of disjoint subsets A_ℓ and A_k such that $a_i \in A_\ell$ and $a_j \in A_k$. The configuration of the elements of the partition search design can be more explicitly displayed as follows:

Disjoint subsets	Identifiable pairs of elements
A_1, A_2	$(a_1,a_5),(a_1,a_6),(a_1,a_7),(a_1,a_8)$ $(a_2,a_5),(a_2,a_6),(a_2,a_7),(a_2,a_8)$ $(a_3,a_5),(a_3,a_6),(a_3,a_7),(a_3,a_8)$ $(a_4,a_5),(a_4,a_6),(a_4,a_7),(a_4,a_8)$
A_3, A_4	$(a_1,a_3),(a_1,a_4),(a_2,a_3),(a_2,a_4)$ $(a_5,a_7),(a_6,a_7),(a_5,a_8),(a_6,a_8)$
A_5, A_6	$(a_1,a_2),(a_3,a_4),(a_5,a_6),(a_7,a_8)$

The display shows that every pair of the eight elements can be separated into two disjoint subsets.

To detect two unknown elements, say a_5 and a_7, we determine two disjoint subsets A_{i_1} and A_{i_2}, $\{i_1, i_2\} \subset \{1,2,3,4,5,6\}$ such that $a_5 \in A_{i_1}$ and $a_7 \in A_{i_2}$. In this case, the two disjoint subsets are A_3 and A_4. That is, $a_5 \in A_3$ and $a_7 \in A_4$. The two unknown elements a_5 and a_7 are then identified separately from the subsets A_3 and A_4 respectively.

Again we can characterise this design by its incidence matrix M, given as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \end{array} . \qquad (4.2)$$

From the incidence matrix, we notice that in any two columns of the matrix M, there exist two rows such that the 2x2 submatrix formed by the intersection of these columns and rows is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the subsets corresponding to the rows are disjoint. That is, to say, for any two distinct elements a_i, a_j $(i \neq j)$ there exists two disjoint subsets A_{i_1}, A_{i_2}, $\{i_1, i_2\} \subset \{1,2,3,4,5,6\}$, such that $a_i \in A_{i_1}$ and $a_j \in A_{i_2}$.

4.2 2-COMPLETE SEARCH DESIGNS.

Let $N = ((n_{ij}))$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$ be the incidence matrix of a search design $\{A_1, A_2, \ldots, A_m; S_n\}$ of the set S_n. Further, let the elements in A_i correspond to the entries of 1's in the i-th row of the incidence matrix M and T_j be a set consisting of all the subsets A_i's which are not incident with the j-th element, a_j of S_n. That is, T_j corresponds to the entries of 0's in the j-th column of the matrix M. For example, in the incidence matrix (4.1) of a 2-Complete search design given in Section (4.1) of this Chapter,

$$T_1 = \{A_1, A_2, A_3\}.$$

The following theorem gives a necessary and sufficient condition for the existence of a 2-Complete search design.

Theorem 4.1: A necessary and sufficient condition for the existence of a 2- Complete search design $\{A_1, A_2, \ldots\ldots, A_m;\ S_n\}$ for detecting an arbitrary pair of elements (a_i, a_j) in S_n is that

$$T_k \nsubseteq T_i \cup T_j$$

for $k = 1, 2, \ldots\ldots, n;\qquad k \neq i \neq j$.

Proof

Let the system $\{A_1, A_2, \ldots\ldots\ldots, A_m; S_n\}$ be a 2-Complete search design. Then, consider two pairs of elements (a_i, a_j) and (a_r, a_k). Since $\{A_1, A_2, \ldots, A_m; S_n\}$ is a 2-Complete search design, there exist subsets $A_{h_1}, A_{h_2}, \ldots\ldots, A_{h_\ell}$, $\{h_1, h_2, \ldots\ldots\ldots, h_\ell\} \subset \{1, 2, \ldots\ldots\ldots, m\}$ such that

$a_i, a_j \in A_{h_g}$ for $g = 1,2,\ldots\ldots\ldots\ldots\ldots\ldots,\ell$ and $\bigcap_{g=1}^{\ell} A_{h_g} = \{a_i, a_j\}$. That is, the subsets A_{h_1}, $A_{h_2},\ldots,A_{h_\ell}$ are incident with both a_i and a_j.

But from our definiton of T_j, as a set consisting of all the subsets A_i's which are not incident with the jth element a_j, it follows that T_j^c is a set consisting of all the subsets A_i's which are incident with the j-th element a_j.

Thus

$$\{A_{h_1}, A_{h_2}, \ldots\ldots\ldots, A_{h_\ell}\} = T_i^c \cap T_j^c . \qquad (4.3)$$

That is, the subsets $A_{h_1}, A_{h_2}, \ldots\ldots, A_{h_\ell}$ which detect the two unknown elements a_i and a_j are given by

$$T_i^c \cap T_j^c . \qquad (4.4)$$

Similarly, the subsets $A_{h'_1}, A_{h'_2}, \ldots\ldots, A_{h'_\ell}$ which detect the two unknown elements a_r and a_k are given by

$$T_r^c \cap T_k^c .$$

Now, since $\{A_1, A_2, \ldots\ldots, A_m; S_n\}$ is a 2-Complete search design

$$\bigcap_{g=1}^{\ell} A_{h_g} = \{a_i, a_j\} \quad \text{and} \quad \bigcap_{g=1}^{\ell'} A_{h'_g} = \{a_r, a_k\}$$

and so

$$\{A_{h_1}, A_{h_2}, \ldots, A_{h_\ell}\} \nsubseteq \{A_{h'_1}, A_{h'_2}, \ldots, A_{h'_{\ell'}}\} \qquad (4.5)$$

That is,

$$T_i^c \cap T_j^c \nsubseteq T_r^c \cap T_k^c \qquad (4.6)$$

which implies that

$$T_r \cup T_k \nsubseteq T_i \cup T_j . \qquad (4.7)$$

In particular, if the pairs were (a_i, a_j) and (a_i, a_k) then (4.7) reduces to

$$T_i \cup T_k \nsubseteq T_i \cup T_j$$

which implies that

$$T_k \nsubseteq T_i \cup T_j . \qquad (4.8)$$

Conversely, suppose that $T_k \nsubseteq T_i \cup T_j$, then we have to show that the system $\{A_1, A_2, \ldots, A_m; S_n\}$ is a 2-Complete search design. That is, for any pair of elements (a_i, a_j) there exist subsets $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_\tau}$, $\{\alpha_1, \alpha_2, \ldots, \alpha_\tau\} \in \{1,2,\ldots,m\}$ such that $a_i, a_j \in A_{\alpha_t}$, for $t = 1,2,\ldots,\tau$ and $\bigcap_{t=1}^{\tau} A_{\alpha_t} = \{a_i, a_j\}$.

Now, $T_k \nsubseteq T_i \cup T_k$ implies that

$$T_r \cup T_k \nsubseteq T_i \cup T_j \qquad (4.9)$$

for any other set T_r $(r \neq k)$, $r = 1,2,\ldots,n$; and from (4.6) it follows that

$$T_i^c \cap T_j^c \nsubseteq T_r \cap T_j^c . \qquad (4.10)$$

Now, $T_i^c \cap T_j^c$ gives subsets of S_n, which are incident with both a_i and a_j $(i \neq j, i,j = 1,2,\ldots,n)$ say, A_{α_1} $A_{\alpha_2}, \ldots\ldots, A_{\alpha_\tau}$.

Thus, for any pair of elements (a_i, a_j) there exists subsets of S_n, $A_{\alpha_1}, A_{\alpha_2}, \ldots\ldots, A_{\alpha_\tau}$ such that $a_i, a_j \in A_{\alpha_t}$ for $t = 1, 2, \ldots\ldots, \tau$. To complete the proof we need to show that $\bigcap_{t=1}^{\tau} A_{\alpha_t} = \{a_i, a_j\}$.

Now suppose,

$$\bigcap_{t=1}^{\tau} A_{\alpha_t} \neq \{a_i, a_j\}.$$

That is, $\bigcap_{t=1}^{\tau} A_{\alpha_t} = \emptyset$ or a set consisting of one element or a set consisting of a_i, a_j and some other element(s). Now $\bigcap_{t=1}^{\tau} A_{\alpha_t}$ cannot be an empty set or a set consisting of one element since $a_i, a_j \in A_{\alpha_t}$, for $t = 1, 2, \ldots, \tau$. Thus, we are left with the possibility that $\bigcap_{t=1}^{\tau} A_{\alpha_t}$ is a set consisting of a_i, a_j and some other element(s). To investigate this possibility we let $a_i, a_j, a_{j'} \in \bigcap_{t=1}^{\tau} A_{\alpha_t}$. That is, $a_j, a_{j'} \in A_{\alpha_t}$ for $t = 1, 2, \ldots, \tau$ and so $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_\tau}\}$ is a subset of the set of subsets which are incident with both a_j and $a_{j'}$. The set of subsets which are incident with both a_j and $a_{j'}$ is given by $T_j^{c} \cap T_{j'}^{c}$. Thus

$$\{A_{\alpha_1}, A_{\alpha_2}, \ldots\ldots, A_{\alpha_\tau}\} = T_i^{c} \cap T_j^{c} \leq T_j^{c} \cap T_{j'}^{c}.$$

This contradicts (4.10), hence $\bigcap_{t=1}^{\tau} A_{\alpha_t}$ is not a set consisting of a_i, a_j and some other element(s). We therefore, conclude that $\bigcap_{i=1}^{T} A_{\alpha_i} = \{a_i, a_j\}$ which completes the proof.

Corollary 4.1: Let the cardinality of the set $T_i (i = 1,2,..,n)$ be p and the cardinality of the intersection of any two sets T_i and T_j, $i \neq j$, be less than p/2. Then the system $\{A_i, A_2, ..., A_m; S_n\}$, where S_n is a finite set and $\{A_1, A_2, ..., A_m\}$ is a collection of all the elements of the set T_i^c $(i = 1,2,....,n)$ is a 2-Complete search design.

Proof

We are given that for any distinct indices i and j, $|T_i \cap T_j| < \frac{p}{2}$ Where $||$ denotes the cardinality of the set concerned.

That is,

$$|T_k \cap T_i| < \frac{p}{2} \quad (4.11)$$

and

$$|T_k \cap T_j| < \frac{p}{2} \quad (4.12)$$

Then

$$|T_k \cap (T_i \cup T_j)| < p \quad (4.13)$$

Therefore,

$$T_k \nsubseteq T_i \cup T_j \quad (4.14)$$

since $|T_k| = p$. Thus, it follows from theorem (4.1) that the system $\{A_1, A_2, \ldots\ldots, A_m; S_n\}$ is a 2-Complete search design.

Theorem 4.2: Suppose the system $\{A_1, A_2, \ldots, A_m; S_n\}$ is a 2-Complete search design and suppose that T_i ($i = 1,2,\ldots\ldots,n$) consists of $2p + 1$ elements and that $|T_i \cap T_j| \leq p$ for any other set T_j ($j \neq i$), $j = 1,2,\ldots,n$; then n,m and p are related by the equation:

$$n = \frac{m\,!\;p\,!}{(2p+1)\,!\;(m-p-1)!}$$

Proof

Each set T_i consists of 2p+1 elements and so the possible number of such subsets out of m is

$$\binom{m}{2p+1}. \qquad (4.15)$$

Out of these subset any p+1 arbitrary elements appear in

$$\binom{m-(p+1)}{2p+1-(p+1)} = \binom{m-p-1}{p} \qquad (4.16)$$

subsets. This is obtained by considering (p+1) elements to have already been chosen, thus we are left to choose ((2p+1) - (p+1)) elements from m - (p+1).

But, we are given that $|T_i \cap T_j| \leq p$, thus, any p+1 elements must appear in only one subset. So the number of subsets T_i which satisfy the condition of the theorem, is given by;

$$n = \frac{\binom{m}{2p+1}}{\binom{m-p-1}{p}} \qquad (4.17)$$

$$= \frac{m\,!}{(2p+1)\,!\,(m-2p-1)\,!} \times \frac{p\,!\,(m-2p-1)\,!}{(m-p-1)\,!}$$

$$= \frac{m\,!\,p\,!}{(2p+1)\,!\,(m-p-1)!} \qquad (4.18)$$

which completes the proof.

4.3 CONSTRUCTION OF 2-COMPLETE SEARCH DESIGNS.

In the construction of a 2-Complete search design, we will make use of the properties of a t-(v,k,λ_t) design and a balanced incomplete block design which are defined in Chapter 1.

We recall that, a t-(v,k,λ_t) design is a family **B** of subsets B_i, called blocks, of a finite set X containing v points, such that every B_i has the same cardinality k and every t elements of X are contained in exactly λ_t blocks of **B**. A balanced incomplete block design is a special case of t- (v,k,λ_t) design with $t = 2$.

Taking the subset B_j to represent a set consisting of all the subsets A_i's which are not incident with the j-th element of S_n, $T_{\gamma\dots}$ $(j = 1,2,..,n)$, we see that a t- (v,k,λ_t) design with parameters $t = p+1$, $v = m$, $k = 2p+1$ and $\lambda_t = 1$ is a 2-Complete search design. This is because each set

T_j (B_j) consists of 2p+1 elements and since every p+1 elements is contained in exactly one ($\lambda_t = 1$) subset, it follows that $|T_i \cap T_j| \le p$, which are the requirements for the exsistence of a 2-Complete search design, according to Corollary (4.1).

A necessary and sufficient condition for a t- (v,k,λ_t) design to exist states that the quantity

$$\lambda_t \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s}$$

be an integer for s=0,1,2,..,(t-1); see Renyi (1970)

In the following theorem we give a necessary condition for exsistence for a 2-Complete search design.

Theorem 4.3: Suppose the system $\{A_1, A_2, \ldots, A_m; S_n\}$ is a 2-Complete search design and suppose that T_i ($i = 1,2,\ldots,n$) consists of 2p+1 elements and that $|T_i \cap T_j| \le p$ for any other set $T_j (j \ne i)$, $j = 1,2,\ldots,n$; then the quantity

$$\binom{m-p-1+s}{p+s} \Big/ \binom{m-p-1}{p}$$

is an integer for $s = 0,1,2,\ldots,p$.

Proof

Let λ_i be the number of subsets in which $p+1-i$ elements appear, that is λ_0 is the number of subsets in which p+1 elements appear, λ_1 is the number of subsets in which p elements appear and so on.

Then from Theorem 4.2 together with the fact that any (p+1) elements appear in λ_o sets and p elements appear in λ_1 sets and so on, we have

$$n = \frac{\binom{m}{2p+1}}{\binom{m-p-1}{p}}\lambda_o = \frac{\binom{m}{2p+1}}{\binom{m-p}{p+1}}\lambda_1$$

$$= \frac{\binom{m}{2p+1}}{\binom{m-p+1}{p+2}}\lambda_2 = \ldots\ldots = \frac{\binom{m}{2p+1}}{\binom{m-p-1+s}{p+s}}\lambda_s .$$

But $\lambda_o = 1$, since $|T_i \cap T_j| \leq p$, so

$$\lambda_s = \frac{\binom{m}{2p+1}}{\binom{m-p-1}{p}} \times \frac{\binom{m-p-1+s}{p+s}}{\binom{m}{2p+1}}$$

$$= \binom{m-p-1+s}{p+s} \Big/ \binom{m-p-1}{p} \qquad (4.19)$$

which must be an integer. Hence the proof.

Example 4.3: Let the cardinality of the set T_i ($i = 1,2,\ldots,n$) be 2p+1 and the cardinality of the intersection of any two sets T_i and T_j ($i \neq j$) be less than or equal to p. Then for p = 1, every pair of elements appears in exactly one subset and each subset consists of three elements. The system $\{T_1,T_2,\ldots,T_n\}$ forms a simple triple system and thus a BIB design in which k = 3, λ = 1, b = n and

$v = m$.

As a particular case, consider the system $\{A_1, A_2, \ldots, A_9, S_{12}\}$ then with $p = 1$, the set $\{T_1, T_2, \ldots, T_{12}\}$ forms a simple triple system with $k = 3$, $\lambda = 1$, $b = 12$ and $v = 9$. One possible configuration of the simple triple system with these parameters is;

$$B_1 = \{1,2,3\} \qquad B_2 = \{1,4,5\}$$
$$B_3 = \{1,6,7\} \qquad B_4 = \{1,8,9\}$$
$$B_5 = \{2,4,7\} \qquad B_6 = \{2,6,9\}$$
$$B_7 = \{2,5,8\} \qquad B_8 = \{3,5,6\}$$
$$B_9 = \{3,7,8\} \qquad B_{10} = \{3,4,9\}$$
$$B_{11} = \{4,6,8\} \qquad B_{12} = \{5,7,9\}$$

If we let the block B_i to correspond to the set T_i and points in the blocks to correspond to the subsets A_i's, that is, the j-th point corresponds to the subset A_j, then the sets T_i's are as follows:

$$T_1 = \{A_1, A_2, A_3\} \qquad T_7 = \{A_2, A_5, A_8\}$$
$$T_2 = \{A_1, A_4, A_5\} \qquad T_8 = \{A_3, A_5, A_6\}$$
$$T_3 = \{A_1, A_6, A_7\} \qquad T_9 = \{A_3, A_7, A_8\}$$
$$T_4 = \{A_1, A_8, A_9\} \qquad T_{10} = \{A_3, A_4, A_9\}$$
$$T_5 = \{A_2, A_4, A_7\} \qquad T_{11} = \{A_4, A_6, A_8\}$$
$$T_6 = \{A_2, A_6, A_9\} \qquad T_{12} = \{A_5, A_7, A_9\}$$

Now, the cardinality of the sets

T_i ($i = 1,2,...,12$) is three and the cardinality of the intersection of any two set T_i and T_j ($i \neq j$) is at most one. Thus, using Corollary 4.1, the system $\{A_1, A_2, ..., A_9; S_{12}\}$ is a 2-Complete search design.

From our definition of the set T_i given earlier, as a set consisting of all the subsets A_i's which are not incident with the j-th element, a_j of S_n, we see that the subsets A_1, A_2, A_3, for example are not incident with $a_1 \in S_n$, and $T_1^c = \{A_4, A_5, A_6, A_7, A_8, A_9\}$ consists of subsets which are incident with a_1. Using this information, provided by $T_1, T_2, ...T_{12}$ we get subsets $A_1, A_2, .., A_9$ as follows

$$A_1 = \{a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\},$$
$$A_2 = \{a_2, a_3, a_4, a_8, a_9, a_{10}, a_{11}, a_{12}\},$$
$$A_3 = \{a_2, a_3, a_4, a_5, a_6, a_7, a_{11}, a_{12}\},$$
$$A_4 = \{a_1, a_3, a_4, a_6, a_7, a_8, a_9, a_{12}\},$$
$$A_5 = \{a_1, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{11}\},$$
$$A_6 = \{a_1, a_2, a_4, a_5, a_7, a_9, a_{10}, a_{12}\},$$
$$A_7 = \{a_1, a_2, a_4, a_6, a_7, a_8, a_{10}, a_{11}\},$$
$$A_8 = \{a_1, a_2, a_3, a_5, a_6, a_8, a_{10}, a_{12}\},$$
$$A_9 = \{a_1, a_2, a_3, a_5, a_7, a_8, a_9, a_{11}\}.$$

The incidence matrix of this design is therefore;

$$N = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}. \qquad (4.20)$$

Now, any pair of elements will be detected according to the following scheme:

Subsets	Elements	Subsets	Elements
A_6, A_7, A_8, A_9	a_1, a_2	A_2, A_4, A_7	a_4, a_8
A_4, A_5, A_8, A_9	a_1, a_3	A_2, A_4, A_5, A_6	a_4, a_9
A_4, A_5, A_6, A_7	a_1, a_4	A_2, A_5, A_6, A_7	a_4, a_{10}
A_5, A_6, A_8, A_9	a_1, a_5	A_2, A_3, A_5, A_7	a_4, a_{11}
A_4, A_5, A_7, A_8	a_1, a_6	A_2, A_3, A_4, A_6	a_4, a_{12}
A_4, A_6, A_7, A_9	a_1, a_7	A_1, A_3, A_5, A_8	a_4, a_6
A_4, A_7, A_8, A_9	a_1, a_8	A_1, A_3, A_6, A_9	a_5, a_7
A_4, A_5, A_6, A_9	a_1, a_9	A_1, A_8, A_9	a_5, a_8
A_5, A_6, A_7, A_8	a_1, a_{10}	A_1, A_5, A_6, A_9	a_5, a_9
A_5, A_7, A_8	a_1, a_{11}	A_1, A_5, A_6, A_8	a_5, a_{10}
A_4, A_6, A_8	a_1, a_{12}	A_1, A_3, A_5, A_9	a_5, a_{11}
A_2, A_3, A_8, A_9	a_2, a_3	A_1, A_3, A_6, A_8	a_5, a_{12}
A_2, A_3, A_6, A_7	a_2, a_4	A_1, A_3, A_4, A_7	a_6, a_7
A_3, A_6, A_8, A_9	a_2, a_5	A_1, A_4, A_7, A_8	a_6, a_8
A_3, A_7, A_8	a_2, a_6	A_1, A_4, A_5	a_6, a_9
A_3, A_6, A_7, A_8	a_2, a_7	A_1, A_5, A_7, A_8	a_6, a_{10}
A_2, A_7, A_8, A_9	a_2, a_8	A_1, A_3, A_5, A_7	a_6, a_{11}
A_2, A_6, A_9	a_2, a_9	A_1, A_3, A_4, A_8	a_6, a_{12}
A_2, A_6, A_7, A_8	a_2, a_{10}	A_1, A_3, A_7, A_8	a_7, a_8

Subsets	Elements	Subsets	Elements
A_2, A_5, A_7, A_9	a_2, a_{11}	A_1, A_4, A_6, A_9	a_7, a_9
A_2, A_3, A_6, A_8	a_2, a_{12}	A_1, A_6, A_7	a_7, a_{10}
A_2, A_3, A_4, A_5	a_3, a_4	A_1, A_3, A_7, A_9	a_7, a_{11}
A_3, A_5, A_8, A_9	a_3, a_5	A_1, A_3, A_4, A_6	a_7, a_{12}
A_3, A_4, A_5, A_8	a_3, a_6	A_1, A_2, A_4, A_9	a_8, a_9
A_3, A_4, A_9	a_3, a_7	A_1, A_2, A_7, A_8	a_8, a_{10}
A_2, A_4, A_8, A_9	a_3, a'_8	A_1, A_2, A_7, A_9	a_8, a_{11}
A_2, A_4, A_5, A_9	a_3, a_9	A_1, A_2, A_4, A_8	a_8, a_{12}
A_2, A_5, A_8	a_3, a_{10}	A_1, A_2, A_5, A_6	a_9, a_{10}
A_2, A_3, A_5, A_9	a_3, a_{11}	A_1, A_2, A_5, A_9	a_9, a_{11}
A_2, A_3, A_4, A_6	a_3, a_{12}	A_1, A_2, A_4, A_6	a_9, a_{12}
A_3, A_5, A_6	a_4, a_5	A_1, A_2, A_5, A_7	a_{10}, a_{11}
A_3, A_4, A_5, A_7	a_4, a_6	A_1, A_2, A_6, A_7	a_{10}, a_{12}
A_3, A_4, A_6, A_7	a_4, a_7	A_1, A_2, A_3	a_{11}, a_{12}

We can also construct 2-Complete search designs from the theorem given by Bush and Federer (1984). Before using this theorem we state it and give an alternative proof.

Theorem 4.4. A BIB design with parameters (v, r, k, b, λ) is a 2-Complete search design if

[illegible] $- 2\lambda >$ [illegible]

Proof

Let M be the incidence matrix of a BIB design with v objects $a_1, a_2, \ldots, a_v$ and b blocks $B_1, B_2, \ldots, B_b$. That is, $M = ((n_{ij}))$, $i = 1, 2, \ldots, v$; $j = 1, 2, \ldots, b$, where;

$$n_{ij} = \begin{cases} 1 & \text{if } a_i \in B_j \\ 0 & \text{if } a_i \notin B_j . \end{cases}$$

Now, let the complements of the blocks $B_1, B_2, \ldots\ldots, B_b$ of this BIB design correspond to the subsets $A_1, A_2, \ldots\ldots, A_b$, of the finite set S_n, that is, B_i^c corresponds to A_i for $i = 1,2,\ldots\ldots,b$ and T_j is as defined earlier, that is, a set of all subsets A_i's (B_i^c's) which are not incident with the element $a_j \in S_n$. Then $|T_j|$ is the number of subsets A_i's which are not incident with the element a_j, that is, the number of complements of the subsets A_i's which are incident to a_j. But the complements of the subsets A_i's correspond to the blocks B_i's so $|T_i|$ is the number of blocks which are incident with a particular object (element). That is $|T_i| = r$ and $|T_i \cap T_j| = \lambda$ $(i \neq j)$. Using Corollary 4.1, this design will be a 2-Complete search design if $\lambda < r/2$, that is $r-2\lambda > 0$.

A BIB design with 2-Complete property will have $b \geq r$. In search problems we need designs with $b < r$. These designs could be obtained from BIB designs by deleting q objects (treatments) and all the blocks in which these objects occur.

Theorem 4.5: Suppose a BIB design has the 2-Complete property, that is $r-2\lambda > 0$. Then the number of objects (treatments) q which could be deleted together with all the blocks in which they occur without affecting the 2-Complete property satisfies the inequality:

$$q < \frac{r}{\lambda} - 2.$$

Proof

If q objects are deleted together with all the blocks in which they occur, then the minimum number of blocks in which any object can occur is $r-\lambda q$. Therefore, for a design to retain the 2-complete property after deleting q objects $r - \lambda q - 2\lambda > 0$ That is,

$$q < \frac{r}{\lambda} - 2. \qquad (4.21)$$

Example 4.4: Consider the symmetric BIB design (13,4,4,13,1). Here $r = 4$ and $\lambda = 1$, thus the number of objects which can be deleted without affecting the 2-Complete property is less than $4 - 2 = 2$. That is, only one treatment and the blocks in which it occurs can be deleted without affecting the 2-Complete property.

Consider the BIB design (13,4,4,13,1) whose blocks are:

$B_1 = \{1,2,4,10\}$ $\qquad$ $B_8 = \{4,5,7,13\}$

$B_2 = \{1,3,9,13\}$ $\qquad$ $B_9 = \{4,8,9,11\}$

$B_3 = \{1,7,11,12\}$ $\qquad$ $B_{10} = \{3,4,6,12\}$

$B_4 = \{1,5,6,8\}$ $\qquad$ $B_{11} = \{6,10,11,13\}$

$B_5 = \{2,3,5,11\}$ $\qquad$ $B_{12} = \{5,9,10,12\}$

$B_6 = \{2,8,12,13\}$ $\qquad$ $B_{13} = \{3,7,8,10\}$

$B_7 = \{2,6,7,9\}$

If one treatment, say 13 is deleted with all the blocks in which it occurs, we obtain:

$B_1 = \{1,2,4,10\}$

$B_3 = \{1,7,11,12\}$

$B_4 = \{1,5,6,8\}$

$B_5 = \{2,3,5,11\}$

$B_7 = \{2,6,7,9\}$

$B_9 = \{4,8,9,11\}$

$B_{10} = \{3,4,6,12\}$

$B_{12} = \{5,9,10,12\}$

$B_{13} = \{3,7,8,10\}$

Let the element a_j correspond to the j-th treatment, then the subsets A_i's which are the complements of the blocks B_i's are:

$$A_1 = \{a_3, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}\},$$
$$A_3 = \{a_2, a_3, a_4, a_5, a_6, a_8, a_9, a_{10}\},$$
$$A_4 = \{a_2, a_3, a_4, a_7, a_9, a_{10}, a_{11}, a_{12}\},$$
$$A_5 = \{a_1, a_4, a_6, a_7, a_8, a_9, a_{10}, a_{12}\},$$
$$A_7 = \{a_1, a_3, a_4, a_5, a_8, a_{10}, a_{11}, a_{12}\},$$
$$A_9 = \{a_1, a_2, a_3, a_5, a_6, a_7, a_{10}, a_{12}\},$$
$$A_{10} = \{a_1, a_2, a_5, a_7, a_8, a_9, a_{10}, a_{11}\},$$
$$A_{12} = \{a_1, a_2, a_3, a_4, a_6, a_7, a_8, a_{11}\},$$
$$A_{13} = \{a_1, a_2, a_4, a_5, a_6, a_9, a_{11}, a_{12}\}.$$

And the sets T_j, that is, sets of all subsets A_i's, which are not incident with the element a_j are:

$T_1 = \{A_1, A_3, A_4\}$

$T_2 = \{A_1, A_5, A_7\}$

$T_3 = \{A_5, A\ , A_{13}\}$

$T_4 = \{A_1, A_9, A_{10}\}$

$T_5 = \{A_4, A_5, A_{12}\}$

$T_6 = \{A_4, A_7, A_{10}\}$

$T_7 = \{A_3, A_7, A_{13}\}$

$T_8 = \{A_4, A_9, A_{13}\}$

$T_9 = \{A_7, A_9, A_{12}\}$

$T_{10} = \{A_1, A_{12}, A_{13}\}$

$T_{11} = \{A_3, A_5, A_9\}$

$T_{12} = \{A_3, A_{10}, A_{12}\}$

Each subset T_i ($i = 1,2,\ldots\ldots,12$) consists of three elements and $|T_i \cap T_j| \le 1$ ($i \ne j$). Using Corollary 4.1, which states that if the sets T_i ($i = 1,2,\ldots,n$) contain the same number of elements p and if the intersection of any two sets contains less than p/2, then the system $\{A_1, A_2, \ldots\ldots, A_m; S_n\}$ is a 2-Complete search design, we conclude that the system $\{A_1, A_3, A_4, A_5, A_7, A_9, A_{10}, A_{12}, A_{13}; S_{12}\}$ is a 2-Complete search design. The incidence matrix, N, of this design is

$$N = \begin{array}{c} \\ A_1 \\ A_3 \\ A_4 \\ A_5 \\ A_7 \\ A_9 \\ A_{10} \\ A_{12} \\ A_{13} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{array} \\ \left[\begin{array}{cccccccccccc} 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}$$

A given pair of elements will be detected according to the following scheme

Subsets	Elements	Subsets	Elements
$A_9, A_{10}, A_{12}, A_{13}$	a_1, a_2	A_3, A_4, A_5, A_{13}	a_4, a_6
A_7, A_9, A_{12}	a_1, a_3	A_3, A_4, A_5, A_{13}	a_4, a_9
A_5, A_7, A_{12}, A_{13}	a_1, a_4	A_3, A_4, A_5, A_7	a_4, a_{10}
A_7, A_9, A_{10}, A_{13}	a_1, a_5	A_4, A_7, A_{12}, A_{13}	a_4, a_{11}
A_5, A_9, A_{12}, A_{13}	a_1, a_6	A_4, A_5, A_7, A_{13}	a_4, a_{12}
A_5, A_9, A_{10}, A_{12}	a_1, a_7	A_1, A_3, A_9, A_{13}	a_5, a_6
A_5, A_7, A_{10}, A_{12}	a_1, a_8	A_1, A_9, A_{10}	a_5, a_7

Subsets	Elements	Subsets	Elements
A_5, A_{10}, A_{13}	a_1, a_9	A_1, A_3, A_7, A_{10}	a_5, a_8
A_5, A_7, A_9, A_{10}	a_1, a_{10}	A_1, A_3, A_{10}, A_{13}	a_5, a_9
$A_7, A_{10}, A_{12}, A_{13}$	a_1, a_{11}	A_3, A_7, A_9, A_{10}	a_5, a_{10}
A_5, A_7, A_9, A_{13}	a_1, a_{12}	A_1, A_7, A_{10}, A_{13}	a_5, a_{11}
A_3, A_4, A_9, A_{12}	a_2, a_3	A_1, A_7, A_9, A_{13}	a_5, a_{12}
A_3, A_4, A_{12}, A_{13}	a_2, a_4	A_1, A_5, A_9, A_{12}	a_6, a_7
A_3, A_9, A_{10}, A_{13}	a_2, a_5	A_1, A_5, A_{12}	a_6, a_8
A_3, A_9, A_{12}, A_{13}	a_2, a_6	A_1, A_3, A_5, A_{13}	a_6, a_9
A_4, A_9, A_{10}, A_{12}	a_2, a_7	A_3, A_5, A_9	a_6, a_{10}
A_3, A_{10}, A_{13}	a_2, a_8	A_1, A_{12}, A_{13}	a_6, a_{11}
A_3, A_4, A_{10}, A_{13}	a_2, a_9	A_1, A_5, A_9, A_{13}	a_6, a_{12}
A_3, A_4, A_9, A_{10}	a_2, a_{10}	A_1, A_5, A_{10}, A_{12}	a_7, a_8
$A_4, A_{10}, A_{12}, A_{13}$	a_2, a_{11}	A_1, A_4, A_5, A_{10}	a_7, a_9
A_4, A_9, A_{13}	a_2, a_{12}	A_4, A_5, A_9, A_{10}	a_7, a_{10}
A_3, A_4, A_7, A_{12}	a_3, a_4	A_1, A_4, A_{10}, A_{12}	a_7, a_{11}
A_1, A_3, A_7, A_9	a_3, a_5	A_1, A_4, A_5, A_9	a_7, a_{12}
A_1, A_3, A_9, A_{12}	a_3, a_6	A_1, A_3, A_5, A_{10}	a_8, a_9
A_1, A_4, A_9, A_{12}	a_3, a_7	A_3, A_5, A_7, A_{10}	a_8, a_{10}
A_1, A_3, A_7, A_{12}	a_3, a_8	A_1, A_7, A_{10}, A_{12}	a_8, a_{11}
A_1, A_3, A_4	a_3, a_9	A_1, A_5, A_7	a_8, a_{12}
A_3, A_4, A_7, A_9	a_3, a_{10}	A_3, A_4, A_5, A_{10}	a_9, a_{10}
A_1, A_4, A_7, A_{12}	a_3, a_{11}	A_1, A_4, A_{10}, A_{13}	a_9, a_{11}
A_1, A_4, A_7, A_9	a_3, a_{12}	A_1, A_4, A_5, A_{13}	a_9, a_{12}
A_3, A_7, A_{13}	a_4, a_5	A_4, A_7, A_{10}	a_{10}, a_{11}
A_3, A_5, A_{12}, A_{13}	a_4, a_6	A_4, A_5, A_7, A_9	a_{10}, a_{12}
A_4, A_5, A_{12}	a_4, a_7	A_1, A_4, A_7, A_{13}	a_{11}, a_{12}

The display shows that every pair of the twelve elements can be detected by a unique set of subsets. For example, if (a_3, a_9) are the unknown elements, then the intersection of A_1, A_3, A_4 gives the identity of the unknown pair, that is, $A_1 \cap A_3 \cap A_4 = \{a_3, a_9\}$.

4.4 PARTITION SEARCH DESIGNS.

Suppose the set S_n consists of n elements $\{a_1, a_2, \ldots, a_n\}$. Then in a partiton search design, we determine m subsets $\{A_1, A_2, \ldots, A_m\}$ of S_n such that for any two distinct elements $a_{i_1}, a_{i_2} \in S_n$, there exists two disjoint subsets A_{i_1} and A_{i_2} such that $a_{i_1} \in A_{i_1}$ and $a_{i_2} \in A_{i_2}$.

We describe here a procedure for constructing the subsets $A_1, A_2, \ldots, A_m$. We start by partitioning the set S_n into two sets A_{i_1} and A_{i_2}.
That is,

$$A_{i_1} \cup A_{i_2} = S_n \qquad (4.23)$$

and

$$A_{i_1} \cap A_{i_2} = \emptyset.$$

We proceed to obtain other subsets by considering the subsets A_{i_1} and A_{i_2} as the set S_n and then partition each into two. The union of the first part of A_{i_1} and the first part A_{i_2} forms the third subset A_{i_3} and the union of the second part of A_{i_1} and the second part A_{i_2} forms the fourth subset A_{i_4}. This process is repeated until all pairs of the elements of the set S_n have been separated into disjoint subsets. This procedure of partitioning a set into two is called *halving procedure*.

Example 4.5: Consider the set S_{16} = $\{a_1, a_2, a_3, \ldots, a_{16}\}$ then applying the halving procedure, we obtain the following subsets of S_n, which will separate all pairs of elements of S_n:

$A_1 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$.

$A_2 = \{a_9, a_{10}, a_{11}, a_{13}, a_{14}, a_{15}, a_{16}\}$.

$A_3 = \{a_1, a_2, a_3, a_4\} \cup \{a_9, a_{10}, a_{11}, a_{12}\}$.

$A_4 = \{a_5, a_6, a_7, a_8\} \cup \{a_{13}, a_{14}, a_{15}, a_{16}\}$.

Other subsets obtained in a similar manner as the above subsets A_3 and A_4 are:

$A_5 = \{a_1, a_2, a_5, a_6, a_9, a_{10}, a_{13}, a_{14}\}$.

$A_6 = \{a_3, a_4, a_7, a_8, a_{11}, a_{12}, a_{15}, a_{16}\}$.

$A_7 = \{a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}, a_{15}\}$.

$A_8 = \{a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}\}$.

To detect two unknown elements, say a_4 and a_7, we determine two disjoint subsets A_{ι_1} and A_{ι_2}, $\{\iota_1, \iota_2\} \subset \{1, 2, \ldots, 8\}$ such that $a_4 \in A_{\iota_1}$ and $a_7 \in A_{\iota_2}$. In this example, the two disjoint subsets are A_3 and A_4. That is, $a_4 \in A_3$ and $a_7 \in A_4$ and $A_3 \cap A_4 = \emptyset$. The unknown elements are then identified separately from the subsets A_3 and A_4

using separating systems.

More explicitly, we have the following display of detectable pairs of elements and the corresponding subsets:

Subsets	Identifiable pairs of elements
A_1, A_2	$(a_1,a_9),(a_1,a_{10}),(a_1,a_{11}),(a_1,a_{12})$ $(a_1,a_{13}),(a_1,a_{14}),(a_1,a_{15}),(a_1,a_{16})$ $(a_2,a_9),(a_2,a_{10}),(a_2,a_{11}),(a_2,a_{12})$ $(a_2,a_{13}),(a_2,a_{14}),(a_2,a_{15}),(a_2,a_{16})$ $(a_3,a_9),(a_3,a_{10}),(a_3,a_{11}),(a_3,a_{12})$ $(a_3,a_{13}),(a_3,a_{14}),(a_3,a_{15}),(a_3,a_{16})$ $(a_4,a_9),(a_4,a_{10}),(a_4,a_{11}),(a_4,a_{12})$ $(a_4,a_{12}),(a_4,a_{13}),(a_4,a_{14}),(a_4,a_{15})$ $(a_4,a_{16}),(a_5,a_9),(a_5,a_{10}),(a_5,a_{11})$ $(a_5,a_{12}),(a_5,a_{13}),(a_5,a_{14}),(a_5,a_{15})$ $(a_5,a_{16}),(a_6,a_9),(a_6,a_{10}),(a_6,a_{11})$ $(a_6,a_{12}),(a_6,a_{13}),(a_6,a_{14}),(a_6,a_{15})$ $(a_6,a_{16}),(a_7,a_9),(a_7,a_{10}),(a_7,a_{11})$ $(a_7,a_{12}),(a_7,a_{13}),(a_7,a_{14}),(a_7,a_{15})$ $(a_7,a_{16}),(a_8,a_9),(a_8,a_{10}),(a_8,a_{11})$ $(a_8,a_{12}),(a_8,a_{13}),(a_8,a_{14}),(a_8,a_{15})$ (a_8,a_{16})
A_3, A_4	$(a_1,a_5),(a_1,a_6),(a_1,a_7),(a_1,a_8)$ $(a_2,a_5),(a_2,a_6),(a_2,a_7),(a_2,a_8)$ $(a_3,a_5),(a_4,a_6),(a_4,a_7),(a_4,a_8)$ $(a_4,a_5),(a_5,a_6),(a_4,a_7),(a_4,a_8)$

A_3, A_4 $(a_9,a_{15}),(a_9,a_{16}),(a_{10},a_{13}),$
$(a_{10},a_{14}),(a_{10},a_{15}),(a_{10},a_{16})$
$(a_{11},a_{13}),(a_{11},a_{14}),(a_{11},a_{15})$
$(a_{11},a_{16}),(a_{12},a_{13}),(a_{12},a_{14})$
$(a_{12},a_{15}),(a_{12},a_{16}),(a_9,a_{13})$
(a_9,a_{14})

A_5, A_6 $(a_1,a_3),(a_1,a_4),(a_2,a_3),(a_2,a_4)$
$(a_5,a_7),(a_5,a_8),(a_6,a_7),(a_6,a_8)$
$(a_8,a_{11}),(a_9,a_{11}),(a_9,a_{12})$
$(a_{10},a_{11}),(a_{10},a_{12}),(a_{13},a_{15})$
$(a_{13},a_{16}),(a_{14},a_{15})$

A_7, A_8 $(a_1,a_2),(a_3,a_4),(a_5,a_6),(a_7,a_8)$
$(a_9,a_{10}),(a_{11},a_{12}),(a_{13},a_{14})$
(a_{15},a_{16}).

Suppose that the set S_n consists of n elements. Then we construct the subsets $A_1, A_2, \ldots, A_m$ by partitioning the set S_n into x equal parts for $n = x^k$ and x parts not all equal but with a maximum size difference of one for $n \neq x^k$. The parts formed in the partitioning of the set S_n form the subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_x}$.

That is,

$$S_n = A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_x}$$

and

$$A_{i_j} \cap A_{i_{j'}} = \emptyset, \qquad j \neq j'.$$

We proceed to obtain other subsets by considering each of subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_x}$ as the set S_n and then partition each into x parts. The union of the first parts of each of the subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_x}$ forms (x + 1)-th subset, that is A_{x+1}; the union of the second parts of each of the subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_x}$ forms the (x+2)-th subset, and so on. This process is repeated until all pairs of the elements of the set S_n have been separated into disjoint subsets. This procedure of partitioning a set into x parts is called $\frac{1}{x}$ - *procedure*.

Theorem 4.6 The number of subsets, m in the $\frac{1}{x}$ - procedure is;

$$m = x\{\log_x n\}$$

where {y} denotes the least interger greater than or equal to y.

Proof

Suppose the set S_n consists of $n = x^k$ elements, then the set [illegible] titioned into x equal parts. Each partition produces x subsets consisting of n/x elements. Suppose the first subset is A_1 then n/x^2 elements are taken from it to form part of subset A_{x+1}, n/x^3 elements taken to form part of subset A_{2x+1}. This process is repeated until the elements taken from A_1 to form part of a new subset is $n/x^{\ell} = 1$, that is $\ell = \log_2 n$. But each partition produces x subsets, thus the total number of subsets is,

$$m = x\log_x n.$$

For $n \neq x^k$, the set S_n is partitioned into x parts not all equal but have maximum size difference of 1. Let k_1 be the size of the largest part, then k_1-1 is the size of the smallest part.The following inequality therefore holds:

$$(k_1 - 1)x < n < k_1 x. \qquad (4.21)$$

Next, we partition the largest part (size k_1) into x parts, again not all equal but have a maximum size difference of one. Let k_2 be the size of the largest part, then $k_2 - 1$ is the size of the smallest part. Again the inequality

$$(k_2 - 1)x < k_1 < k_2 x \qquad (4.22)$$

holds.

But

$$k_1 > (k_2 - 1) x$$

implies that

$$(k_1 - 1) \geq (k_2 - 1) x \qquad (4.23)$$

since k_1, $(k_2 - 1)$ and x are all integers. Thus, from (4.21) and (4.22), we have

$$n < k_1 x < k_2 x^2 \qquad (4.24)$$

and

$$n > (k_1 - 1)x \geq (k_2 - 1)\ x^2 \qquad (4.25)$$

This process is repeated until, we have the inequalities

$$n < k_1 x < k_2 x^2 < \ldots\ldots\ldots\ldots\ldots\ldots < k_{m'-1}\ x^{m'-1}$$

$$n > (k_1 - 1)x \geq (k_2 - 1)x^2 > \ldots > (k_{m'-1} - 1)x^{m'-1}$$

and

$$2 < k_{m'-1} \leq x.$$

That is,

$$n < x^{m'}$$

and

$$n > x^{m'-1}$$

which implies

$$m' > \log_x n$$

and

$$m' < \log_x n + 1.$$

But m′ is an integer so

$$m' = \{\log_x n\}.$$

Each partition produces x subsets, so the total number of subsets is

$$m = xm' = x\{\log_x n\}$$

which completes the proof.

Theorem 4.7: An 1/x procedure, with x = 3 gives the minimum number of subsets, m.

Proof

From Theorem 4.6, m is given by

$$m = x \log_x n.$$

Differentiating with respect to x and equating to zero, we get

$$\frac{dm}{dx} = \log_e n\left(\frac{1}{\log_e x} - \frac{1}{(\log_e x)^2}\right) = 0$$

which gives,

$$\log_e x = 1 \qquad \text{or} \qquad x = e;$$

and

$$\frac{d^2 m}{dx^2} = \log_e n\left[\frac{\frac{1}{x}(\log_e x)^2 - 2(\log_e x - 1)(\log_e x)}{(\log_e x)^4}\right]$$

which is positive when $x = e$.

Thus, $x = e$ gives minimum number of subsets, m. But x must be an integer so $x = 3$ would give minimum m, that is

$$m = 3\log_3 n. \qquad (4.26)$$

4.5 DETECTING t (t > 2) UNKNOWN ELEMENTS.

In this section we study a strategy for detecting t $(t > 2)$ unknown elements from a finite set S_n. The strategy we propose to study is called a *t-complete search design* defined in Section 1.2 of Chapter 1. For purposes of our study of the t-complete search design we define it in terms of the intersection of the subsets $A_1, A_2, \ldots\ldots, A_m$ of the finite set S_n, as a system $\{A_1, A_2, \ldots, A_m; \quad S_n\}$ in which for any arbitrary set of t elements $\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\} \in S_n$ there exists a set of indices $\{i_1, i_2, \ldots, i_k\} \subset \{1,2,\ldots,m\}$ such that $\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\} \in A_{i_\ell}$ for $\ell = 1,2,\ldots,k$ and $\bigcap_{\ell=1}^{k} A_{i_\ell} = \{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$. Without loss of generality we will assume that the subsets A_{i_ℓ} $(\ell = 1,2,\ldots,k)$ are the only subsets which contain the set $\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$.

To identify any t unknown elements, say $a_1, a_2, \ldots, a_t$, we determine subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ $\{i_1, i_2, \ldots, i_k\} \subset \{1,2,3,\ldots,m\}$ such that $a_1, a_2, \ldots, a_t \in A_{i_j}$ for $j = 1,2,\ldots,k$. The identity of the t unknown elements is then given by the intersection of these subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$, that is $\bigcap_{j=1}^{k} A_{i_j} = \{a_1, a_2, \ldots, a_t\}$.

The following example illustrates this strategy.

Example 4.6:- Suppose the system $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13};\ S_{13}\}$ constitutes a 3-complete search design for separating the elements of the set $S_n = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}\}$. Then one possible configuration of the subsets $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}\}$ is the following:

$$A_1 = \{a_3, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{13}\},$$

$$A_2 = \{a_1, a_4, a_6, a_7, a_8, a_9, a_{10}, a_{12}, a_{13}\},$$

$$A_3 = \{a_1, a_2, a_5, a_7, a_8, a_9, a_{10}, a_{11}, a_{13}\},$$

$A_4 = \{a_1, a_2, a_3, a_6, a_8, a_9, a_{10}, a_{11}, a_{12}\}$,

$A_5 = \{a_2, a_3, a_4, a_7, a_9, a_{10}, a_{11}, a_{12}, a_{13}\}$,

$A_6 = \{a_1, a_3, a_4, a_5, a_8, a_{10}, a_{11}, a_{12}, a_{13}\}$,

$A_7 = \{a_1, a_2, a_4, a_5, a_6, a_9, a_{11}, a_{12}, a_{13}\}$,

$A_8 = \{a_1, a_2, a_3, a_5, a_6, a_7, a_{10}, a_{12}, a_{13}\}$,

$A_9 = \{a_1, a_2, a_3, a_4, a_6, a_7, a_8, a_{11}, a_{13}\}$,

$A_{10} = \{a_1, a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{12}\}$,

$A_{11} = \{a_2, a_3, a_4, a_5, a_6, a_8, a_9, a_{10}, a_{11}\}$,

$A_{12} = \{a_1, a_3, a_4, a_5, a_6, a_7, a_9, a_{10}, a_{11}\}$,

$A_{13} = \{a_2, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{11}, a_{12}\}$.

This design will detect any arbitrary group of three elements from S_{13}. That is, for any group of three elements $a_\ell, a_{\ell'}, a_{\ell''}$ $(\ell \neq \ell' \neq \ell'')$ of the set S_{13}, there exist subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$, $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, m\}$, such that $a_\ell, a_{\ell'}, a_{\ell''} \in A_{i_j}$, for $j = 1, 2, \ldots, k$ and $\bigcap_{j=1}^{k} A_{i_j} = \{a_\ell, a_{\ell'}, a_{\ell''}\}$. Thus, to detect any three unknown elements of the set S_{13}, we determine subsets amongst $A_1, A_2, \ldots, A_{13}$ which contain the three unknown elements. The intersection of these subsets gives the identity of the three unknown elements.

More explicitly, we have the following display of detectable pairs of elements and the associated subsets.

Subsets	Elements	Subsets	Elements
A_4, A_8, A_9, A_{10}	a_1, a_2, a_3	A_6, A_7, A_8, A_{10}	a_1, a_5, a_{12}
A_7, A_9, A_{10}	a_1, a_2, a_4	A_3, A_6, A_7, A_8	a_1, a_5, a_{13}
A_3, A_7, A_8, A_{10}	a_1, a_2, a_5	A_2, A_8, A_9, A_{12}	a_1, a_6, a_7
A_4, A_7, A_8, A_9	a_1, a_2, a_6	A_2, A_4, A_9, A_{11}	a_1, a_6, a_8
A_3, A_8, A_9, A_{10}	a_1, a_2, a_7	A_2, A_4, A_7, A_{12}	a_1, a_6, a_9
A_3, A_4, A_9, A_{10}	a_1, a_2, a_8	A_2, A_4, A_8, A_{12}	a_1, a_6, a_{10}
A_3, A_4, A_7, A_{10}	a_1, a_2, a_9	A_4, A_7, A_9, A_{12}	a_1, a_6, a_{11}
A_3, A_4, A_8	a_1, a_2, a_{10}	A_2, A_7, A_8	a_1, a_6, a_{12}
A_3, A_4, A_7, A_9	a_1, a_2, a_{11}	A_2, A_7, A_8, A_9	a_1, a_6, a_{13}
A_4, A_7, A_8, A_{10}	a_1, a_2, a_{12}	A_2, A_3, A_9, A_{10}	a_1, a_7, a_8
A_3, A_7, A_8, A_9	a_1, a_2, a_{13}	A_2, A_3, A_{10}, A_{12}	a_1, a_7, a_9
A_6, A_8, A_9, A_{10}	a_1, a_3, a_4	A_2, A_3, A_8, A_{12}	a_1, a_7, a_{10}
A_6, A_8, A_{10}, A_{12}	a_1, a_3, a_5	A_3, A_9, A_{12}	a_1, a_7, a_{11}
A_4, A_8, A_9, A_{12}	a_1, a_3, a_6	A_2, A_8, A_{10}	a_1, a_7, a_{12}
A_8, A_9, A_{10}, A_{12}	a_1, a_3, a_7	A_2, A_3, A_8, A_9	a_1, a_7, a_{13}
A_4, A_8, A_9, A_{10}	a_1, a_3, a_8	A_2, A_3, A_4, A_{10}	a_1, a_8, a_9
A_4, A_{10}, A_{12}	a_1, a_3, a_9	A_2, A_3, A_4, A_6	a_1, a_8, a_{10}
A_4, A_6, A_8, A_{12}	a_1, a_3, a_{10}	A_3, A_4, A_6, A_9	a_1, a_8, a_{11}
A_4, A_6, A_9, A_{12}	a_1, a_3, a_{11}	A_2, A_4, A_6, A_{10}	a_1, a_8, a_{12}
A_6, A_8, A_{10}	a_1, a_3, a_{12}	A_2, A_3, A_6, A_9	a_1, a_8, a_{13}
A_4, A_6, A_8, A_9	a_1, a_3, a_{13}	A_2, A_3, A_4, A_{12}	a_1, a_9, a_{10}
A_6, A_7, A_{10}, A_{12}	a_1, a_4, a_5	A_3, A_4, A_7, A_{12}	a_1, a_9, a_{11}
A_2, A_7, A_9, A_{12}	a_1, a_4, a_6	A_2, A_4, A_7, A_{10}	a_1, a_9, a_{12}
A_2, A_9, A_{10}, A_{12}	a_1, a_4, a_7	A_2, A_3, A_7	a_1, a_9, a_{13}
A_2, A_6, A_9, A_{10}	a_1, a_4, a_8	A_3, A_4, A_6, A_{12}	a_1, a_{10}, a_{11}
A_2, A_3, A_{10}, A_{12}	a_1, a_4, a_9	A_2, A_4, A_6, A_8	a_1, a_{10}, a_{12}
A_2, A_6, A_{12}	a_1, a_4, a_{10}	A_2, A_3, A_6, A_8	a_1, a_{10}, a_{13}
A_6, A_7, A_9, A_{12}	a_1, a_4, a_{11}	A_4, A_6, A_7	a_1, a_{11}, a_{12}
A_2, A_6, A_7, A_{10}	a_1, a_4, a_{12}	A_3, A_6, A_7, A_9	a_1, a_{11}, a_{13}
A_2, A_6, A_7, A_9	a_1, a_4, a_{13}	A_2, A_6, A_7, A_9	a_1, a_{12}, a_{13}
A_7, A_8, A_{12}	a_1, a_5, a_6	A_5, A_9, A_{10}, A_{11}	a_2, a_3, a_4
A_3, A_8, A_{10}, A_{12}	a_1, a_5, a_7	A_8, A_{10}, A_{11}	a_2, a_3, a_5
A_3, A_6, A_{10}	a_1, a_5, a_8	A_4, A_8, A_9, A_{11}	a_2, a_3, a_6
A_3, A_7, A_{10}, A_{11}	a_1, a_5, a_9	A_5, A_8, A_9, A_{10}	a_2, a_3, a_7
A_3, A_6, A_8, A_{12}	a_1, a_5, a_{10}	A_4, A_9, A_{10}, A_{11}	a_2, a_3, a_8
A_3, A_6, A_7, A_{12}	a_1, a_5, a_{11}	A_4, A_5, A_{10}, A_{11}	a_2, a_3, a_9

Subsets	Elements	Subsets	Elements
A_4, A_5, A_{11}	a_2, a_3, a_{10}	A_4, A_8, A_{11}, A_{13}	a_2, a_6, a_{10}
A_4, A_5, A_9	a_2, a_3, a_{11}	A_4, A_7, A_9, A_{13}	a_2, a_6, a_{11}
A_4, A_5, A_8, A_{10}	a_2, a_3, a_{12}	A_7, A_8, A_9, A_{11}	a_2, a_6, a_{12}
A_5, A_8, A_9, A_{11}	a_2, a_3, a_{13}	A_7, A_8, A_9, A_{11}	a_2, a_6, a_{13}
$A_7, A_{10}, A_{11}, A_{13}$	a_2, a_4, a_5	$A_3, A_5, A_{10} A_{11}$	a_2, a_7, a_8
A_7, A_9, A_{11}, A_{13}	a_2, a_4, a_6	A_3, A_5, A_{10}	a_2, a_7, a_9
A_5, A_9, A_{10}, A_{13}	a_2, a_4, a_7	A_3, A_5, A_8, A_{13}	a_2, a_7, a_{10}
$A_9, A_{10}, A_{11}, A_{13}$	a_2, a_4, a_8	A_3, A_5, A_9, A_{13}	a_2, a_7, a_{11}
A_5, A_{10}, A_{11}	a_2, a_4, a_9	A_5, A_8, A_{10}, A_{13}	a_2, a_7, a_{12}
A_5, A_{11}, A_{13}	a_2, a_4, a_{10}	A_3, A_5, A_8, A_9	a_2, a_7, a_{13}
A_5, A_7, A_9, A_{13}	a_2, a_4, a_{11}	A_3, A_7, A_{10}, A_{11}	a_2, a_8, a_9
A_5, A_7, A_{10}, A_{13}	a_2, a_4, a_{12}	A_3, A_4, A_{11}, A_{13}	a_2, a_8, a_{10}
A_5, A_7, A_9, A_{11}	a_2, a_4, a_{13}	A_3, A_4, A_9, A_{13}	$a_2 a_8, a_{11}$
A_7, A_8, A_{11}, A_{13}	a_2, a_5, a_6	A_4, A_{10}, A_{13}	a_2, a_8, a_{12}
A_3, A_8, A_{10}, A_{13}	a_2, a_5, a_7	A_3, A_7, A_{11}	a_2, a_8, a_{13}
$A_3, A_{10}, A_{11}, A_{13}$	a_2, a_5, a_8	A_3, A_4, A_5, A_{11}	a_2, a_9, a_{10}
A_3, A_7, A_{10}, A_{11}	a_2, a_5, a_9	A_3, A_4, A_5, A_7	a_2, a_9, a_{11}
A_3, A_8, A_{11}, A_{13}	a_2, a_5, a_{10}	A_4, A_5, A_7, A_{10}	a_2, a_9, a_{12}
A_3, A_7, A_{13}	a_2, a_5, a_{11}	A_3, A_5, A_7, A_{11}	a_2, a_9, a_{13}
A_7, A_8, A_{10}, A_{13}	a_2, a_5, a_{12}	A_3, A_4, A_5, A_{13}	a_2, a_{10}, a_{11}
A_3, A_7, A_8, A_{11}	a_2, a_5, a_{13}	A_4, A_5, A_8, A_{13}	a_2, a_{10}, a_{12}
A_8, A_9, A_{13}	a_2, a_6, a_7	A_3, A_5, A_8, A_{11}	a_2, a_{10}, a_{13}
A_4, A_9, A_{11}, A_{13}	a_2, a_6, a_8	A_4, A_5, A_7, A_{13}	a_2, a_{11}, a_{12}
A_4, A_7, A_{11}	a_2, a_6, a_9	A_3, A_5, A_7, A_9	a_2, a_{11}, a_{13}
A_6, A_9, A_{11}, A_{12}	a_3, a_4, a_5	A_4, A_5, A_6, A_{12}	a_3, a_{10}, a_{11}
$A_1, A_8, A_{10} A_{11}$	a_3, a_5, a_7	A_4, A_5, A_6, A_8	a_3, a_{10}, a_{12}
$A_1, A_6, A_{10} A_{11}$	a_3, a_5, a_8	A_5, A_6, A_8, A_{11}	a_3, a_{10}, a_{13}
$A_1, A_{10} A_{11} A_{12}$	a_3, a_5, a_9	A_1, A_4, A_5, A_6	a_3, a_{11}, a_{12}
$A_6, A_8, A_{11} A_{12}$	a_3, a_5, a_{10}	A_1, A_5, A_6, A_9	a_3, a_{11}, a_{13}
A_1, A_6, A_7, A_{10}	a_3, a_5, a_{11}	A_1, A_5, A_6, A_8	a_3, a_{12}, a_{13}
A_1, A_6, A_8, A_{10}	a_3, a_5, a_{12}	$A_7, A_{11}, A_{12}, A_{13}$	a_4, a_5, a_6
A_1, A_6, A_8, A_{11}	a_3, a_5, a_{13}	$A_{10} A_{12} A_{13}$	a_4, a_5, a_7
A_1, A_8, A_9, A_{12}	a_3, a_6, a_7	$A_6, A_{10}, A_{11}, A_{13}$	$a_4 a_5, a_8$
A_1, A_4, A_9, A_{11}	a_3, a_6, a_8	$A_7, A_{10}, A_{11}, A_{12}$	a_4, a_5, a_9
$A_1, A_4, A_{11} A_{12}$	a_3, a_6, a_9	$A_6, A_{11}, A_{12}, A_{13}$	a_4, a_5, a_{10}
$A_4, A_8, A_{11} A_{12}$	a_3, a_6, a_{10}	A_6, A_7, A_{12}, A_{13}	a_4, a_5, a_{11}

Subsets	Elements	Subsets	Elements
A_1, A_4, A_9, A_{12}	a_3, a_6, a_{11}	A_6, A_7, A_{10}, A_{13}	a_4, a_5, a_{12}
A_1, A_4, A_8	a_3, a_6, a_{12}	A_6, A_7, A_{11}	a_4, a_5, a_{13}
A_1, A_8, A_9, A_{11}	a_3, a_6, a_{13}	A_2, A_9, A_{12}, A_{13}	a_4, a_6, a_7
A_1, A_9, A_{10}	a_3, a_7, a_8	A_2, A_9, A_{11}, A_{13}	a_4, a_6, a_8
A_1, A_5, A_{10}, A_{12}	a_3, a_7, a_9	A_2, A_7, A_{11}, A_{12}	a_4, a_6, a_9
A_5, A_8, A_{12}	a_3, a_7, a_{10}	$A_2, A_{11}, A_{12}, A_{13}$	a_4, a_6, a_{10}
A_1, A_5, A_9, A_{12}	a_3, a_7, a_{11}	A_7, A_9, A_{12}, A_{13}	a_4, a_6, a_{11}
A_1, A_5, A_8, A_{10}	a_3, a_7, a_{12}	A_2, A_7, A_{13}	a_4, a_6, a_{12}
A_1, A_5, A_8, A_9	a_3, a_7, a_{13}	A_2, A_7, A_9, A_{11}	a_4, a_6, a_{13}
$A_1, A_4, A_{10} A_{11}$	a_3, a_8, a_9	A_2, A_9, A_{10}, A_{13}	a_4, a_7, a_8
A_4, A_6, A_{11}	a_3, a_8, a_{10}	A_2, A_5, A_{10}, A_{12}	a_4, a_7, a_9
A_1, A_4, A_6, A_9	a_3, a_8, a_{11}	A_2, A_5, A_{12}, A_{13}	a_4, a_7, a_{10}
A_1, A_4, A_6, A_{10}	a_3, a_8, a_{12}	A_5, A_9, A_{12}, A_{13}	a_4, a_7, a_{11}
A_1, A_6, A_9, A_{11}	a_3, a_8, a_{13}	A_2, A_5, A_{10}, A_{13}	a_4, a_7, a_{12}
$A_4, A_5, A_{11} A_{12}$	a_3, a_9, a_{10}	A_2, A_5, A_9	a_4, a_7, a_{13}
A_1, A_4, A_5, A_{12}	a_3, a_9, a_{11}	A_2, A_{10}, A_{11}	a_4, a_8, a_9
A_1, A_4, A_5, A_{10}	a_3, a_9, a_{12}	A_2, A_6, A_{11}, A_{13}	a_4, a_8, a_{10}
A_1, A_5, A_{11}	a_3, a_9, a_{13}	A_6, A_9, A_{13}	a_4, a_8, a_{11}
A_2, A_6, A_{10}, A_{13}	a_4, a_8, a_{12}	A_1, A_4, A_9, A_{13}	a_6, a_8, a_{11}
A_2, A_6, A_9, A_{11}	a_4, a_8, a_{13}	A_1, A_2, A_4, A_{13}	a_6, a_8, a_{12}
$A_2, A_5, A_{11} A_{12}$	a_4, a_9, a_{10}	A_1, A_2, A_9, A_{11}	a_6, a_8, a_{13}
A_5, A_7, A_{12}	a_4, a_9, a_{11}	A_2, A_4, A_1, A_{12}	a_6, a_9, a_{10}
A_2, A_5, A_7, A_{10}	a_4, a_9, a_{12}	A_1, A_4, A_7, A_{12}	a_6, a_9, a_{11}
A_2, A_5, A_7, A_{11}	a_4, a_9, a_{13}	A_1, A_2, A_4, A_7	a_6, a_9, a_{12}
$A_5, A_6, A_{12} A_{13}$	a_4, a_{10}, a_{11}	A_1, A_2, A_7, A_{11}	a_6, a_9, a_{13}
A_2, A_5, A_6, A_{13}	a_4, a_{10}, a_{12}	A_4, A_{12}, A_{13}	a_6, a_{10}, a_{11}
A_2, A_5, A_6, A_{11}	a_4, a_{10}, a_{13}	A_2, A_4, A_8, A_{13}	a_6, a_{10}, a_{12}
A_5, A_6, A_7, A_{13}	a_4, a_{11}, a_{12}	A_2, A_8, A_1	a_6, a_{10}, a_{13}
A_5, A_6, A_7, A_9	a_4, a_{11}, a_{13}	A_1, A_4, A_7, A_{13}	a_6, a_{11}, a_{12}
A_2, A_5, A_6, A_7	a_4, a_{12}, a_{13}	A_1, A_7, A_9	a_6, a_{11}, a_{13}
A_1, A_8, A_{12}, A_{13}	a_5, a_6, a_7	A_1, A_2, A_7, A_8	a_6, a_{12}, a_{13}
A_1, A_{11}, A_{13}	a_5, a_6, a_8	A_1, A_2, A_3, A_{10}	a_7, a_8, a_9
A_1, A_7, A_{11}, A_{12}	a_5, a_6, a_9	A_2, A_3, A_{13}	a_7, a_8, a_{10}
A_8, A_{11}, A_{12}, A_1	a_5, a_6, a_{10}	A_1, A_3, A_4, A_{13}	a_7, a_8, a_{11}
A_1, A_7, A_{12}, A_{13}	a_5, a_6, a_{11}	$A_1, A_2, A_{10} A_{13}$	a_7, a_8, a_{12}
A_1, A_7, A_8, A_{13}	a_5, a_6, a_{12}	A_1, A_2, A_3, A_9	a_7, a_8, a_{13}

Subsets	Elements	Subsets	Elements
A_1, A_7, A_8, A_{11}	a_5, a_7, a_{13}	A_2, A_3, A_5, A_{12}	a_7, a_9, a_{10}
A_1, A_3, A_{10}, A_{13}	a_5, a_7, a_8	A_1, A_3, A_5, A_{12}	a_7, a_9, a_{11}
$A_1, A_3, A_{10} A_{12}$	a_5, a_7, a_9	A_1, A_2, A_5, A_{10}	a_7, a_9, a_{12}
A_3, A_8, A_{12}, A_{13}	a_5, a_7, a_{10}	A_1, A_2, A_3, A_5	a_7, a_9, a_{13}
A_1, A_3, A_{12}, A_{13}	a_5, a_7, a_{11}	A_3, A_5, A_{12}, A_{13}	a_7, a_{10}, a_{11}
A_1, A_8, A_{10}, A_{13}	a_5, a_7, a_{12}	A_2, A_5, A_8, A_{13}	a_7, a_{10}, a_{12}
A_1, A_3, A_8	a_5, a_7, a_{13}	A_2, A_3, A_5, A_8	a_7, a_{10}, a_{13}
A_1, A_3, A_{10}, A_1	a_5, a_8, a_9	A_1, A_5, A_{13}	a_7, a_{11}, a_{12}
A_3, A_6, A_{12}, A_{13}	a_5, a_8, a_{10}	A_1, A_3, A_5, A_9	a_7, a_{11}, a_{13}
A_1, A_3, A_6, A_{13}	a_5, a_8, a_{11}	A_1, A_2, A_5, A_8	a_7, a_{12}, a_{13}
A_1, A_6, A_{10}, A_{13}	a_5, a_8, a_{12}	A_2, A_3, A_4, A_{11}	a_8, a_9, a_{10}
A_1, A_3, A_6, A_{11}	a_5, a_8, a_{13}	A_1, A_3, A_4	a_8, a_9, a_{11}
A_3, A_{11}, A_{12}	a_5, a_9, a_{10}	A_1, A_2, A_4, A_{10}	a_8, a_9, a_{12}
A_1, A_3, A_7, A_{12}	a_5, a_9, a_{11}	A_1, A_2, A_3, A_{11}	a_8, a_9, a_{13}
A_2, A_7, A_{10}	a_5, a_9, a_{12}	A_3, A_4, A_6, A_{13}	a_8, a_{10}, a_{11}
A_1, A_3, A_7, A_{11}	a_5, a_9, a_{13}	A_2, A_4, A_6, A_{13}	a_8, a_{10}, a_{12}
A_3, A_6, A_{12}, A_{13}	a_5, a_{10}, a_{11}	A_2, A_3, A_6, A_{11}	a_8, a_{10}, a_{13}
A_6, A_8, A_{13}	a_5, a_{10}, a_{12}	A_1, A_4, A_6, A_{13}	a_8, a_{11}, a_{12}
A_3, A_6, A_9, A_{11}	a_5, a_{10}, a_{13}	A_1, A_3, A_6, A_9	a_8, a_{11}, a_{13}
A_1, A_6, A_7, A_{13}	a_5, a_{11}, a_{12}	A_1, A_2, A_6	a_8, a_{12}, a_{13}
A_1, A_3, A_6, A_7	a_5, a_{11}, a_{13}	A_3, A_4, A_5, A_{12}	a_9, a_{10}, a_{11}
A_1, A_6, A_7, A_8	a_5, a_{12}, a_{13}	A_2, A_4, A_5	a_9, a_{10}, a_{12}
A_1, A_2, A_9, A_{13}	a_6, a_7, a_8	A_2, A_3, A_5, A_{11}	a_9, a_{10}, a_{13}
A_1, A_2, A_{12}	a_6, a_7, a_9	A_1, A_4, A_5, A_7	a_9, a_{11}, a_{12}
A_2, A_8, A_{12}, A_{13}	a_6, a_7, a_{10}	A_1, A_3, A_5, A_7	a_9, a_{11}, a_{13}
A_1, A_9, A_{12}, A_{13}	a_6, a_7, a_{11}	A_1, A_2, A_5, A_7	a_9, a_{12}, a_{13}
A_1, A_2, A_8, A_{13}	a_6, a_7, a_{12}	$A_4, A_5, A_6 A_{13}$	a_{10}, a_{11}, a_{12}
A_1, A_2, A_8, A_9	a_6, a_7, a_{13}	A_3, A_5, A_6	a_{10}, a_{11}, a_{13}
$A_1, A_2 A_4, A_{11}$	a_6, a_8, a_9	A_2, A_5, A_6, A_8	a_{10}, a_{12}, a_{13}
A_2, A_4, A_{11}, A_{13}	a_6, a_8, a_{10}	A_1, A_5, A_6, A_7	a_{11}, a_{12}, a_{13}

The display shows that every set of three elements can be detected by a unique set of subsets. For example, if $\{a_1, a_5, a_{11}\}$ is the unknown set of elements, then we determine subsets amongst $A_1, A_2, \ldots\ldots, A_{13}$ which contain a_1, a_5 and a_{11}. The intersection of these subsets gives the identity of the three unknown elements. In this case, the subsets which contain a_1, a_5 and a_{11} are A_3, A_6, A_7, and A_{12}. The intersection of these subsets A_3, A_6, A_7 and A_{12} gives the identities of the elements. That is, $A_3 \cap A_6 \cap A_7 \cap A_{12} = \{a_1, a_5, a_{11}\}$

We can further characterise this arrangement in terms of the incidence matrix of the search design. This is an m x n matrix $M = ((n_{ij}))$, where;

$$n_{ij} = \begin{cases} 1 & \text{if } a_i \in A_j \ , i = 1,2,\ldots\ldots\ldots,m \\ 0 & \text{if } a_i \notin A_j \ , j = 1,2,\ldots\ldots\ldots,n. \end{cases}$$

In the above example, we therefore have:

$$M = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \end{array} \begin{array}{ccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{array} \tag{4.27}$$

From this matrix, we notice that every element of the set S_{13} appears in nine subsets, every pair of elements appears in six subsets, any three elements appear in four subsets and any four elements appear in at most three subsets. Now, for any three elements to be uniquely detectable, the number of subsets in which they appear must be strictly more than the number of subsets in which any four elements appear. This is because if the number of subsets in which any three elements appear is the same as the number of subsets in which any four elements appear, then the intersection of these subsets will consist of four elements, not three as required for correct-identification of the unknown three elements. This requirement is satisfied in this example, so any three unknown elements can be uniquely detected.

Suppose $N = ((n_{ij}))$, $i = 1,2,\ldots,m, j = 1,2,\ldots,n$ is the incidence matrix of a search design $\{A_1,A_2,\ldots,A_m;S_n\}$ consisting of m subsets $A_1,A_2,\ldots,A_m$ of the set S_n. Let the elements in A_i correspond to the entries of 1's in the i-th row of the incidence matrix N and T_j be a set consisting of all the subsets A_i's which are not incident with the j-th element, $a_j \in S_n$. That is, T_j corresponds to the entries Ø's in the j-th column of the matrix N. For example in the incidence matrix (4.27) of a 3-complete search design given in

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$$T_1 = \{A_1, A_5, A_{11}, A_{13}\}.$$

The following theorem gives a necessary and sufficient condition for the existence of a t-complete search design in terms of T_j's.

Theorem 4.8:- A necessary and sufficient condition for the existence of a t-complete search design $\{A_1, A_2, \ldots\ldots, A_m;\ S_n\}$ for detecting an arbitrary set $\{a_1, a_2, \ldots\ldots, a_t\}$ of t distinct unknown elements in S_n is that

$$T_k \nsubseteq \bigcup_{i=1}^{t} T_i .$$

Proof.

Let the system $\{A_1, A_2, \ldots, A_m; S_n\}$ be a t-complete search design. Then, consider two sets of elements $\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}$ and $\{a_{j_1}, a_{j_2}, \ldots, a_{j_t}\}$. Since $\{A_1, A_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, A_m;\ S_n\}$ is a t-complete search design there exist subsets $A_{h_1}, A_{h_2}, \ldots\ldots\ldots, A_{h_\ell}, \{h_1, h_2, \ldots\ldots, h_\ell\} \subset \{1,2,\ldots,m\}$ such that $\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\} \in A_{h_g}$ for $g = 1,2,\ldots\ldots,\ell$ and $\bigcap_{g=1}^{\ell} A_{h_g} = \{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}$. That is, the subsets $A_{h_1}, A_{h_2}, \ldots\ldots, A_{h_\ell}$ are incident with each of the points $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$.

But from the definition of T_j, we know that $T_j \subset$

is a set consisting of all the subsets A_i's which are incident with the jth element, and so

$$\{A_{h_1}, A_{h_2}, \ldots, A_{h_\ell}\} = T_{i_1}^{c} \cap T_{i_2}^{c} \ldots \cap T_{i_t}^{c}. \qquad (4.28)$$

That is, the subsets $A_{h_1}, A_{h_2}, \ldots\ldots, A_{h_\ell}$ which detect the set $\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$ of unknown elements are given by;

$$\bigcap_{k=1}^{t} T_{i_k}^{c}. \qquad (4.29)$$

Similarly, the subsets $A_{h'_1}, A_{h'_2}, \ldots\ldots, A_{h'_\ell}$ which detect the set $\{a_{j_1}, a_{j_2}, \ldots\ldots, a_{j_t}\}$ of unknown elements are given by;

$$\bigcap_{r=1}^{t} T_{j_r}^{c}. \qquad (4.30)$$

Now, since $\{A_1, A_2, \ldots, A_m; S_n\}$ is a t-complete search design

$$\bigcap_{g=1}^{\ell} A_{h_g} = \{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}$$

and

$$\bigcap_{g=1}^{\ell'} A_{h'_g} = \{a_{j_1}, a_{j_2}, \ldots, a_{j_t}\}, \qquad i \neq j$$

and so

$$\bigcap_{g=1}^{\ell} A_{h_g} \nsubseteq \bigcap_{g=1}^{\ell'} A_{h'_g}.$$

That is,

$$\bigcap_{k=1}^{t} T_{i_k}^{c} \nsubseteq \bigcap_{r=1}^{t} T_{j_r}^{c} \tag{4.31}$$

which implies that

$$\bigcup_{r=1}^{t} T_{j_r} \nsubseteq \bigcup_{k=1}^{t} T_{i_k}. \tag{4.32}$$

Inparticular, if the pair of sets are $\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}$ and $\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_{t-1}}, a_{j_t}\}$ then (4.32) reduces to:

$$\bigcup_{r=1}^{t-1} T_{i_r} \cup T_{j_t} \nsubseteq \bigcup_{k=1}^{t} T_{i_k}$$

which implies that

$$T_{j_t} \nsubseteq \bigcup_{k=1}^{t} T_{i_k}. \tag{4.33}$$

Conversely, suppose that $T_{j_t} \nsubseteq \bigcup_{k=1}^{t} T_{i_k}$, then we have to show that the system $\{A_1, A_2, \ldots\ldots, A_m; S_n\}$ is a t-complete search design. That is, for any set $\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}$ of unknown elements there exist subsets $A_{\alpha_1}, A_{\alpha_2}, \ldots\ldots, A_{\alpha_\tau}\}, \{\alpha_1, \alpha_2, \ldots\ldots, \alpha_\tau\} \subset \{1, 2, \ldots\ldots, m\}$ such that;

$$\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\} \subset A_{\alpha_r} \text{ for } r = 1, 2, \ldots, \tau$$

and

$$\bigcap_{r=1}^{\tau} A_{\alpha_r} = \{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_\tau}\}.$$

Now, $T_{j_t} \nsubseteq \bigcup_{k=1}^{t} T_{i_k}$, implies that

$$\bigcup_{r=1}^{t} T_{j_r} \nsubseteq \bigcup_{k=1}^{t} T_{i_k} \qquad (4.34)$$

for other sets $T_{j_1}, T_{j_2}, \ldots\ldots, T_{j_{t-1}}$, which in turn implies that

$$\bigcap_{k=1}^{t} T_{i_k}^{c} \nsubseteq \bigcap_{r=1}^{t} T_{j_r}^{c}. \qquad (4.35)$$

But, $\bigcap_{k=1}^{t} T_{i_k}^{c}$ gives subsets of S_n, which are incident with the points $\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$, say $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_\tau}$. Thus, for any set of t elements say $\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$ there exists subsets of S_n, say $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_\tau}\}$ such that $\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\} \subset A_{\alpha_r}$, for $r = 1, 2, \ldots\ldots, \tau$. To complete the proof we show that

$$\bigcap_{r=1}^{\tau} A_{\alpha_r} = \{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}.$$

Now, suppose that

$$\bigcap_{r=1}^{\tau} A_{\alpha_r} \neq \{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}.$$

That is, $\bigcap_{r=1}^{\tau} A_{\alpha_r} = \emptyset$ or a set consisting of one or more elements of the set $\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}$ or a set consisting of the set $\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}$ and some other element(s). Now, $\bigcap_{r=1}^{\tau} A_{\alpha_r}$ cannot be an empty set or a set consisting of one or more elements of the set $\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\}$ since

$\{a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}\} \subset A_{\alpha_r}$, $r = 1,2,\ldots,\tau$. Thus, we are left with the possibility that $\bigcap_{r=1}^{\tau} A_{\alpha_r}$ is a set consisting of $\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$ and some other element(s). To investigate this possibility we let

$$\{a_{i_1}, a_{i_2}, \ldots, a_{i_t}, a_{j_t}\} \subset A_{\alpha_r}.$$

That is,

$$\{a_{i_1}, a_{i_2}, \ldots, a_{i_{t-1}}, a_{j_t}\} \subset A_{\alpha_r}, \qquad r = 1,2,\ldots\ldots,\tau$$

and so

$$\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_\tau}\}$$

is a subset of the set of subsets which are incident with $a_{i_1}, a_{i_2}, \ldots, a_{i_{t-1}}, a_{j_t}$. This set of subsets which are incident with $a_{i_1}, a_{i_2}, \ldots, a_{i_{t-1}}, a_{j_t}$ is given by;

$$T^{c}_{i_1} \cap T^{c}_{i_2} \cap \ldots\ldots \cap T^{c}_{i_{t-1}} \cap T^{c}_{j_t}.$$

Thus,

$$\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_\tau}\} = \bigcap_{r=1}^{t} T^{c}_{i_r} \leq \bigcap_{r=1}^{t-1} T^{c}_{i_r} \cap T^{c}_{j_t}.$$

This contradicts (4.34), hence $\bigcap_{r=1}^{\tau} A_{\alpha_r}$ is not a set consisting of $a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_t}$ and some other element(s). We therefore, conclude that $\bigcap_{r=1}^{\tau} A_{\alpha_r} = \{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$ which completes the poof.

Corollary 4.2:- Let the cardinality of the set T_i $(i = 1,2,\ldots,n)$ be $tp + 1$ where t and p are positive integers, and let the cardinality of the intersection of any two sets T_i and T_j, $i \neq j$, be equal to or less than p. Then the system $\{A_1, A_2, A_3, \ldots, A_m; S_n\}$ is a t-complete search design.

Proof.

We are given that for any distinct indices i and j $|T_i \cap T_j| \leq p$, where $||$ denotes the cardinality of the set concerned.

That is,

$$\left|T_j \cap T_{i_k}\right| \leq p, \text{ for } k = 1,2,..,t.$$

Then,

$$\left|\left(T_j \cap T_{i_1}\right) \cup \left(T_j \cap T_{i_2}\right) \cup \ldots \cup \left(T_j \cap T_{i_t}\right)\right| \leq tp.$$

That is,

$$\left|T_j \cap \left(\bigcup_{k=1}^{t} T_{i_k}\right)\right| \leq tp.$$

But

$$\left|T_j\right| = tp + 1, \text{ for } j = 1,2,\ldots,t.$$

Therefore,

$$T_j \not\subseteq \bigcup_{k=1}^{t} T_{i_k}, \text{ for } k = 1,2,..,t.$$

Thus, from Theorem (4.8) and relation (4.33), the system $\{A_1, A_2, \ldots, A_m; S_n\}$ is a t-complete search design.

Example 4.7:- Consider the BIB design (16,4,4,20,1) whose blocks are given as follows:

$B_1 = \{4, 13, 8, 11\}$ $B_{11} = \{6, 15, 7, 10\}$

$B_2 = \{7, 1, 11, 14\}$ $B_{12} = \{9, 3, 10, 13\}$

$B_3 = \{10, 4, 14, 2\}$ $B_{13} = \{12, 4, 13, 1\}$

$B_4 = \{13, 7, 2, 5\}$ $B_{14} = \{15, 9, 1, 4\}$

$B_5 = \{1, 10, 5, 8\}$ $B_{15} = \{3, 12, 4, 7\}$

$B_6 = \{5, 14, 9, 12\}$ $B_{16} = \{16, 1, 2, 3\}$

$B_7 = \{8, 2, 12, 15\}$ $B_{17} = \{16, 4, 5, 6\}$

$B_8 = \{11, 5, 15, 13\}$ $B_{18} = \{16, 7, 8, 9\}$

$B_9 = \{14, 8, 3, 6\}$ $B_{19} = \{16, 10, 11, 12\}$

$B_{10} = \{2, 11, 6, 9\}$ $B_{20} = \{16, 13, 14, 15\}$

If we let the j-th block B_j to correspond to the set T_j and the points in the blocks to correspond to the subsets A_i's such the point j corresponds to the subset A_j, then we have:

$T_1 = \{A_4, A_{13}, A_8, A_{11}\}$ $T_5 = \{A_1, A_{10}, A_5, A_8\}$

$T_2 = \{A_7, A_1, A_{11}, A_{14}\}$ $T_6 = \{A_5, A_{14}, A_9, A_{12}\}$

$T_3 = \{A_{10}, A_4, A_{14}, A_2\}$ $T_7 = \{A_8, A_2, A_{12}, A_{15}\}$

$T_4 = \{A_{13}, A_7, A_2, A_5\}$ $T_8 = \{A_1, A_5, A_{15}, A_{13}\}$

$T_9 = \{A_{14}, A_8, A_3, A_6\}$ $T_{15} = \{A_3, A_{12}, A_4, A_7\}$

$T_{10} = \{A_2, A_{11}, A_6, A_9\}$ $T_{16} = \{A_{16}, A_1, A_2, A_3\}$

$T_{11} = \{A_6, A_{15}, A_7, A_{10}\}$ $T_{17} = \{A_{16}, A_4, A_5, A_6\}$

$T_{12} = \{A_9, A_3, A_{10}, A_{13}\}$, $T_{18} = \{A_{16}, A_7, A_8, A_9\}$

$T_{13} = \{A_{12}, A_4, A_{13}, A_1\}$ $T_{19} = \{A_{16}, A_{10}, A_{11}, A_{12}\}$

$T_{14} = \{A_{15}, A_9, A_1, A_4\}$ $T_{20} = \{A_{16}, A_{13}, A_{14}, A_{15}\}$

Now, the cardinality of the sets T_j ($j = 1,2,\ldots,20$) is four and the cardinality of the intersection of any two sets T_i and T_j ($i \neq j$) is at most one. Thus, using corollary 4.2, the system $\{A_1, A_2, \ldots\ldots, A_{16};\ S_{20}\}$ is a 3-Complete search design.

From our definition of the set T_i given earlier, as a set consisting of all the subsets A_i's which are not incident with the j-th element, a_j of S_{20}, we see that the subsets A_4, A_{13}, A_8, A_{11}, for example, are not incident with a_1 and $T_1^c = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{12}, A_{14}, A_{15}, A_{16}\}$ gives subsets which are incident with a_1. Using the information provided by $T_1, T_2, \ldots, T_{20}$, we get subsets $A_1, A_2, \ldots, A_{16}$ as follows:

$$A_1 = \{a_1, a_3, a_4, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{15}, a_{17}, a_{18}, a_{19}, a_{20}\}$$

$$A_2 = \{a_1, a_2, a_5, a_6, a_8, a_9, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{17}, a_{18}, a_{20}\}$$

$$A_3 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_{10}, a_{11}, a_{19}, a_{14}, a_{17}, a_{18}, a_{19}, a_{20}\}$$

$$A_4 = \{a_2, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{11}, a_{12}, a_{16}, a_{18}, a_{19}, a_{20}\}$$

$$A_5 = \{a_1, a_2, a_3, a_7, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{18}, a_{19}, a_{20}\}$$

$$A_6 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{12}, a_{13} a_{14}, a_{15}, a_{16}, a_{18}, a_{19}, a_{20}\}$$

$$A_7 = \{a_1, a_3, a_5, a_6, a_8, a_9, a_{19}, a_{12}, a_{13}, a_{14}, a_{16}, a_{17}, a_{19}, a_{20}\}$$

$$A_8 = \{a_2, a_3, a_4, a_6, a_7, a_8, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{16}, a_{17}, a_{19}, a_{20}\}$$

$$A_9 = \{a_1, a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{11}, a_{13}, a_{15}, a_{16}, a_{17}, a_{19}, a_{20}\}$$

$$A_{10} = \{a_1, a_2, a_4, a_6, a_7, a_8, a_9, a_{10}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{20}\}$$

$$A_{11} = \{a_3, a_4, a_5, a_6, a_7, a_9, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{20}\}$$

$A_{12} = \{a_1, a_2, a_3, a_4, a_5, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{14},$
$a_{16}, a_{17}, a_{18}, a_{20}\}$

$A_{13} = \{a_2, a_3, a_5, a_6, a_7, a_9, a_{10}, a_{11}, a_{14}, a_{15}, a_{16},$
$a_{17}, a_{18}, a_{19}\}$

$A_{14} = \{a_1, a_4, a_5, a_7, a_8, a_{10}, a_{11}, a_{12}, a_{13}, a_{15}, a_{16},$
$a_{17}, a_{18}, a_{19}\}$

$A_{15} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_8, a_{10}, a_{12}, a_{13}, a_{14},$
$a_{15}, a_{16}, a_{17}, a_{18}, a_{19}\}$

$A_{16} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12},$
$a_{13}, a_{14}, a_{15}\}$

The incidence matrix of this design is;

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}
A_1	1	0	1	1	0	1	1	1	1	1	1	1	0	0	1	0	1	1	1	1
A_2	1	1	0	0	1	1	0	1	1	0	1	1	1	1	1	0	1	1	1	1
A_3	1	1	1	1	1	1	1	0	0	1	1	0	1	1	0	0	1	1	1	1
A_4	0	1	0	1	1	1	1	1	1	1	1	1	0	0	0	1	0	1	1	1
A_5	1	1	1	0	0	0	1	0	1	1	1	1	1	1	1	1	0	1	1	1
A_6	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	0	1	1	1
A_7	1	0	1	0	1	1	1	0	1	1	1	1	1	1	0	1	1	0	1	1
A_8	0	1	1	1	0	1	1	1	0	1	1	1	1	1	0	1	1	0	1	1
A_9	1	1	1	1	1	0	1	1	1	0	1	0	1	0	1	1	1	0	1	[illegible]
A_{10}	1	1	0	1	0	1	1	1	1	0	0	1	1	1	1	1	1	1	0	1
A_{11}	0	0	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	0	1
A_{12}	1	1	1	1	1	0	0	1	1	1	1	1	0	1	0	1	1	1	0	1
A_{13}	0	1	1	0	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	0
A_{14}	1	0	0	1	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	0
A_{15}	1	1	1	1	1	1	0	0	1	1	0	1	1	0	1	1	1	1	1	0
A_{16}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0

This 3-complete search design can detect any three unknown elements. For example if a_1, a_5, a_9 are the three unknown elements, then these elements are detected by $A_2, A_7, A_9, A_{12}, A_{15}, A_{16}$. That is, $A_2 \cap A_7 \cap A_9 \cap A_{12} \cap A_{15} \cap A_{16} = \{a_1, a_5, a_9\}$.

CHAPTER 5

DURATION OF THE SEARCH PROCESS FOR DETECTING TWO UNKNOWN ELEMENTS

5.1 INTRODUCTION.

In this chapter we are interested in the duration of the search process for detecting two unknown elements using the subsets $A_1, A_2, .., A_m$ of a finite set S_n defined in Section 4.1 of Chapter 4. In the computation of the duration of the search process we will use the notations introduced in Chapter 1. That is, we shall use $P_1(N,u,v)$ to denote the probability that the sequence $A_{i_1}, A_{i_1}, .., A_{i_1}, A_{i_2}, .., A_{i_2}, A_{i_3}, .., A_{i_3}, .., A_{i_k}, .., A_{i_k}$ determines two unknown elements (u,v) within N steps and $p_1(N,u,v)$ to denote the probability that the process for detecting the two unknown elements terminates at exactly the Nth step.

The formula for computing the duration of the search process for detecting two unknown elements is

$$E_1(u,v) = \sum_{N=0}^{\infty} N.p_1(N,u,v). \qquad \text{c.f}(1.10)$$

5.2 SOME EXAMPLES.

Example 5.1:- In this example we illustrate the computation of the duration of the search process using a 2-Complete search design.

Now, consider the 2-Complete search design of

Example 4.1 in Section 4.1 of Chapter 4. The configuration of the subsets $A_1, A_2, \ldots, A_7$ of this 2-complete search design is

$$A_1 = \{a_4, a_5, a_6, a_7\},$$
$$A_2 = \{a_2, a_3, a_6, a_7\},$$
$$A_3 = \{a_2, a_3, a_4, a_5\},$$
$$A_4 = \{a_1, a_3, a_5, a_7\},$$
$$A_5 = \{a_1, a_3, a_4, a_6\},$$
$$A_6 = \{a_1, a_2, a_4, a_7\},$$
$$A_7 = \{a_1, a_2, a_5, a_6\}.$$

The incidence matrix of this design is;

$$M = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

Suppose the unknown pair of elements we wish to detect is (a_1, a_2), then since $\{A_1, A_2, \ldots, A_7; S_7\}$ is a 2-complete search design, we determine subsets of S_7 amongst $A_1, A_2, \ldots, A_7$ which contain the pair (a_1, a_2). The intersection of these subsets gives the identity of the unknown pair. In this example, the subsets which contain (a_1, a_2) are A_6 and A_7 with $A_6 \cap A_7 = \{a_1, a_2\}$. Thus, the unknown pair of elements (a_1, a_2) would be detected if and only if the subsets A_6 and A_7 are selected. It therefore follows that the pair (a_1, a_2) cannot be detected in one step. That is, the process of search cannot terminate at N = 1.

The process of search will terminate at N = 2, if the following sequences occur;

$$A_6, A_7 \quad \text{or} \quad A_7, A_6 .$$

Thus, the probability of terminating the search process after selection of two subsets is

$$p_1(x,u,v) = \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \cdot \frac{1}{7}$$

$$= \frac{2}{49} .$$

The search process will terminate at N = 3 if the following sequences occur:

Sequences	Number of possiblle ways
A_6, A_6, A_7	1 x 1 x 1 = 1
A_7, A_7, A_6	1 x 1 x 1 = 1
A_i, A_6, A_7	5 x 1 x 1 = 5
A_i, A_7, A_6	5 x 1 x 1 = 5
A_6, A_i, A_7	1 x 5 x 1 = 5
A_7, A_i, A_6	1 x 5 x 1 = 5

where i = 1,2,3,4,5. The probability of terminating the search process at N = 3 is, therefore,

$$p_1(3,u,v) = \frac{22}{7^3}$$

and the probability of terminating search process at $N \leq 3$ is

$$P_1(3,u,v) = \frac{2}{7^2} + \frac{22}{7^3}$$

$$= \frac{36}{7^3} .$$

For higher values of N, we consider the

complementary event; that is, the event that the search process does not terminate in N steps and use Lemma 3.5 which gives the number of ways of placing N balls in m cells such that all the m cells are occupied as

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}(m-k)^N.$$

The search process will terminate in N steps if and only if the subsets A_6 and A_7 are both selected. Thus the search process will not terminate in N steps if any of the following sequences occur:

(i) Only one subset A_ι is selected N times; in that case there are 7 sequences. These sequence are:

$A_1, A_1, \ldots\ldots, A_1$; $A_2, A_2, \ldots\ldots, A_2$;

$A_3, A_3, \ldots\ldots, A_3$; $A_4, A_4, \ldots\ldots, A_4$;

$A_5, A_5, \ldots\ldots, A_5$; $A_6, A_6, \ldots\ldots, A_6$;

$A_7, A_7, \ldots\ldots, A_7$.

(ii) Two subsets A_ι and A_α are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; $\iota \in \{6,7\}$ and $\alpha \in \{1,2,3,4,5\}$. Using the formula above, the number of such sequences is

$$2 \times 5(2^N - 2) = 10(2^N - 2).$$

(iii) Two subsets A_α and A_β are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; and $\alpha, \beta \in \{1,2,3,4,5\}$, $\alpha \neq \beta$. Number of such sequences is

$$\binom{5}{2}(2^N - 2) = 10(2^N-2).$$

(iv) Three subsets A_α, A_β and A_ι are selected x_1, x_2 and x_3 times respectively, where $x_1 + x_2 + x_3 = N$; $\alpha, \beta \in \{1,2,3,4,5\}$; $\alpha \neq \beta$ and $\iota \in \{6,7\}$. Number of such sequences is

$$\binom{5}{2} \cdot \binom{2}{1}(3^N-3.2^N+3) = 20(3^N-3.2^N+3).$$

(v) Three subsets A_α, A_β and A_λ are selected x_1, x_2 and x_3 times respectively, where $x_1 + x_2 + x_3 = N$; $\alpha, \beta, \lambda \in \{1,2,3,4,5\}$; $\alpha \neq \beta \neq \lambda$. Number of such sequences is

$$\binom{5}{3}.(3^N-3.2^N+3) = 10(3^N-3.2^N+3).$$

(vi) Four subsets A_α, A_β, A_λ and A_γ are selected x_1, x_2, x_3 and x_4 times respectively, where $x_1 + x_2 + x_3 + x_4 = N$: $\alpha, \beta, \lambda, \gamma \in \{1,2,3,4,5\}$; $\alpha \neq \beta \neq \lambda \neq \gamma$. Number of such sequences is

$$\binom{5}{4}(4^N - 4.3^N + 6.2^N - 4)$$

$$= 5(4^N - 4.3^N + 6.2^N - 4).$$

(vii) Four subsets A_α, A_β, A_λ and A_ι are selected x_1, x_2, x_3 and x_4 times respectively, where $x_1 + x_2 + x_3 + x_4 = N$; $\alpha, \beta, \lambda, \in \{1,2,3,4,5\}$. $\alpha \neq \beta \neq \lambda$ and $\iota \in \{6,7\}$. Number of such sequences is

$$\binom{5}{4}\binom{2}{1}(4^N-4.3^N+6.2^N-4)$$

$$= 20(4^N-4.3^N+6.2^N-4).$$

(viii) Five subsets A_1,A_2,A_3,A_4 and A_5 are selected x_1,x_2,x_3,x_4 and x_5 times respectively where $x_1+ x_2+ x_3+ x_4+ x_5 = N$; the number of such sequences is

$$\binom{5}{5}(5^N-5.4^N+10.3^N-10.2^N+5)$$

$$= (5^N-5.4^N+10.3^N-10.2^N+5).$$

(ix) Five subsets A_α, A_β, A_λ, A_γ and A_ι are selected x_1,x_2,x_3,x_4 and x_5 times respectively where $x_1+ x_2+ x_3+ x_4+ x_5 = N$; $\alpha,\beta,\lambda,\gamma, \in \{1,2,3,4,5,\}$ $\alpha \neq \beta \neq \lambda \neq \gamma$ and $\iota \in \{6,7\}$. Number of such sequences is

$$\binom{5}{5}\cdot\binom{2}{1}(5^N-5.4^N+10.3^N+10.2^N+5)$$

$$= 10(5^N-5.4^N+10.3^N-10.2^N+5).$$

(x) Six subsets A_1,A_2,A_3,A_4,A_5 and A_ι are selected x_1,x_2,x_3,x_4,x_5 and x_6 times respectively, where $x_1+x_2+x_3+x_4+x_5+x_6 = N$; and $\iota \in \{6,7\}$. Number of such sequences is

$$\binom{5}{5}\cdot\binom{2}{1}(6^N- 6.5^N+ 15.4^N$$

$$- 20.3^N+ 15.2^N- 6)$$

$$= 2(6^N-6.5^N+ 15.4^N$$

$$- 20.3^N+ 15.2^N- 6).$$

Therefore, the probability of terminating the search process in at most N steps is

$$P_1(N,u,v) = 1-\Big[7 + 20(2^N - 2) + 30(3^N - 3.2^N + 3)$$
$$+ 25(4^N - 4.3^N + 6.2^N - 4)$$
$$+ 11(5^N - 5.4^N + 10.3^N - 10.2^N + 5)$$
$$+ 2(6^N - 6.5^N + 15.4^N - 20.3^N$$
$$+ 15.2^N - 6)\Big] \Big/ 7^N$$
$$= 1-2\left(\frac{6}{7}\right)^N + \left(\frac{5}{7}\right)^N.$$

And the probability of terminating the search process in exactly N steps is

$$p_1(N,u,v) = P_1(N,u,v) - P_1(N-1,u,v)$$
$$= \left[1-2\left(\frac{6}{7}\right)^N + \left(\frac{5}{7}\right)^N\right] - \left[1-2\left(\frac{6}{7}\right)^{N-1} + \left(\frac{5}{7}\right)^{N-1}\right]$$
$$= \frac{2}{7}\left(\frac{6}{7}\right)^{N-1} - \frac{2}{7}\left(\frac{5}{7}\right)^{N-1}.$$

The expected number of tests required to detect the pair of unknown elements is

$$E_1(u,v) = \sum_{N=1}^{\infty} N.p_1(N,u,v)$$
$$= \frac{2}{7}\left[\sum_{N=1}^{\infty} N.\left(\frac{6}{7}\right)^{N-1} - \sum_{N=0}^{\infty} N.\left(\frac{5}{6}\right)^{N-1}\right]$$
$$= \frac{2}{7} \times 147/4$$
$$= 10.5.$$

Thus, to detect a pair of unknown elements (u,v) of S_n an average of 10.5 tests would be required.

Example 5.2:- In this example we illustrate the computation of the duration of the search process using partition search design.

Now, consider the partition search design of Example 4.2 in Section 4.1 of Chapter 4. The configuration of the subsets $A_1, A_2, \ldots\ldots, A_6$ of this partition search design is

$$A_1 = \{a_1, a_2, a_3, a_4\}, \qquad A_2 = \{a_5, a_6, a_7, a_8\},$$
$$A_3 = \{a_1, a_2, a_7, a_8\}, \qquad A_4 = \{a_3, a_4, a_5, a_6\},$$
$$A_5 = \{a_1, a_3, a_5, a_7\}, \qquad A_6 = \{a_2, a_4, a_6, a_8\}.$$

The incidence matrix of this design is

$$M = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \end{array}.$$

Suppose the unknown pair of elements we are to detect is (a_1, a_2). Then since $\{A_1, A_2, A_3, A_4, A_5, A_6; S_8\}$ is a partition search design, we determine two disjoint subsets A_{i_1} and A_{i_2} such that $a_1 \in A_{i_1}$ and $a_2 \in A_{i_2}$. In this case, the two disjoint subsets are A_5 and A_6. That is, $a_1 \in A_5$, $a_2 \in A_6$ and $A_5 \cap A_6 = \emptyset$. The unknown pair of elements (a_1, a_2) would then be separated if and only if the subsets A_5 and a_6 are selected. It therefore, follows that the pair (a_1, a_2) cannot be separated in one step. That is, the process of search cannot terminate at $N = 1$.

The process of search will terminate at N = 2, if the following sequences occur

A_5, A_6 or A_6, A_5.

Thus, the probability of terminating the search process after selection of two subsets is

$$P_1(2,4.v) = \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6}$$

$$= \frac{2}{36}$$

The search process will terminate at N = 3 if any of the following sequences occur

Sequences	Number of possible ways
A_5, A_5, A_6	1 x 1 x 1 = 1
A_6, A_6, A_5	1 x 1 x 1 = 1
A_i, A_5, A_6	4 x 1 x 1 = 4
A_i, A_6, A_5	4 x 1 x 1 = 4
A_5, A_i, A_6	1 x 4 x 1 = 4
A_6, A_i, A_5	1 x 4 x 1 = 4

where i = 1,2,3,4. The probability of terminating the search process at N = 3 is therefore,

$$P_1(3,u,v) = \frac{18}{6^3}$$

and the probability of terminating search process at $N \leq 3$ is

$$P_1(3,u,v) = \frac{2}{6^2} + \frac{18}{6^3}$$

$$= \frac{30}{6^3}.$$

For higher values of N, we consider the complementary events; that is, the event that the search process does not terminate in N steps. We will use Lemma 3.5 to get the number of sequences of length N which do not detect the unknown pair of elements.

The search process will not terminate in N steps if subsets A_5 and A_6 are both not selected. Thus, the search process will not terminate in N steps if any of the following sequences occur:

(i) Only one distinct subsets A_i is selected N times; in that case there are 6 sequences. These sequences are:

$$A_1, A_1, \ldots\ldots, A_1; \quad A_2, A_2, \ldots\ldots, A_2;$$
$$A_3, A_3, \ldots\ldots, A_3; \quad A_4, A_4, \ldots\ldots, A_4;$$
$$A_6, A_6, \ldots\ldots, A_6; \quad A_6, A_6, \ldots\ldots, A_6.$$

(ii) Two subsets A_ι and A_α are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; $\iota \in \{5,6\}$ and $\alpha \in \{1,2,3,4\}$. Possible number of such sequences is

$$\binom{2}{1}\cdot\binom{4}{1}(2^N - 2) = 8(2^N - 2).$$

(iii) Two subsets A_α and A_β are selected x_1 and x_2 times respectively, where $x_1 + x_2 = N$; $\iota \in \{5,6\}$ and $\alpha, \beta \in \{1,2,3,4\}$. Possible number of such sequences is

$$\binom{4}{2}(2^N - 2) = 6(2^N - 2).$$

(iv) Three subsets A_α, A_β and A_λ are selected x_1, x_2 and x_3 times respectively, where $x_1+x_2+x_3 = N$; $\alpha,\beta,\lambda \in \{1,2,3,4\}$. Possible number of such sequences is

$$\binom{4}{3}(3^N-3.2^N+3) = 4(3^N-3.2^N+3).$$

(v) Three subsets A_α, A_β and A_ι are selected x_1, x_2 and x_3 times respectively, where $x_1 + x_2 + x_3 = N$; $\alpha,\beta \in \{1,2,3,4\}$ and $\iota \in \{5,6\}$. Possible number of such sequences is

$$\binom{4}{2}\binom{2}{1}(3^N-3.2^N+3) = 12(3^N-3.2^N+3).$$

(vi) Four subsets $A_\alpha, A_\beta, A_\lambda$ and A_γ are selected x_1, x_2, x_3, and x_4 times respectively, where $x_1+x_2+x_3+x_4 = N$; and $\alpha,\beta,\lambda,\gamma \in \{1,2,3,4\}$. Possible number of such sequences is

$$\binom{4}{4}(4^N-4.3^N+6.2^N-4)$$

$$=(4^N-4.3^N+6.2^N-4).$$

(vii) Four subsets $A_\alpha, A_\beta, A_\lambda$ and A_ι are selected x_1, x_2, x_3, and x_4 times respectively, where $x_1+x_2+x_3+x_4 = N$; and $\alpha, \beta, \lambda, \in \{1,2,3,4\}$ and $\iota \in \{5,6\}$. Possible number of such sequences is

$$\binom{4}{3}\binom{2}{1}(4^N - 4.3^N + 6.2^N - 4).$$

$$= 8(4^N - 4.3^N + 6.2^N - 4).$$

(viii) Five subsets A_1, A_2, A_3, A_4 and A_ι are selected x_1, x_2, x_3, x_4 and x_5 times respectively, where $x_1+x_2+x_3+x_4+x_5 = N$; and $\iota \in \{,5,6\}$. Possible number of such sequences is

$$\binom{4}{4}\binom{2}{1}(5^N - 5.4^N + 10.3^N - 10.2^N + 5)$$

$$= 2(5^N - 5.4^N + 10.3^N - 10.2^N + 5).$$

Therefore, the probability of terminating the search process in at most N steps is

$$P_1(N,u,v) = 1 - \Big[6 + 14(2^N - 2) + 16(3^N - 3.2^N + 3)$$
$$+ 9(4^N - 4.3^N + 6.2^N - 4)$$
$$+ 2(5^N - 5.4^N + 10.3^N - 10.2^N + 5)\Big] \Big/ 6^N$$

$$= 1 - 2\left(\frac{5}{6}\right)^N + \left(\frac{4}{6}\right)^N,$$

and the probability of terminating the search process in exactly N steps is

$$p_1(N,u,v) = P_1(N,u,v) - P_1(N-1,u,v)$$

$$= \left[1 - 2\left(\frac{5}{6}\right)^N + \left(\frac{4}{6}\right)^N\right] - \left[1 - 2\left(\frac{5}{6}\right)^{N-1} + \left(\frac{4}{6}\right)^{N-1}\right]$$

$$= \frac{1}{3}\left[\left(\frac{5}{6}\right)^{N-1} - \left(\frac{4}{6}\right)^{N-1}\right].$$

The expected number of tests required to separate the two unknown elements into two disjoint subsets is

$$E_1(u,v = \sum_{N=0}^{\infty} N.P_1(N,u,v) \qquad \text{c.f(1.7)}$$

$$= \frac{1}{3}\left[\sum_{N=2}^{\infty} N\left(\frac{5}{6}\right)^{N-1} - \sum_{N=2}^{\infty} N.\left(\frac{4}{6}\right)^{N-1} \right]$$

$$= \frac{1}{3} \times 27 = 9.0 \; .$$

Thus, to separate the two unknown elements (a_1, a_2) into two disjoint subsets, an average of 9.0 tests would be required.

5.3 DURATION OF THE SEARCH PROCESS.

In Section 5.2 above we have seen two examples dealing with the computation of the duration of the search process for detecting two unknown elements using a 2-complete search design and duration of the search process for separating the two unknown elements into two disjoints subsets using a partition search design. In this Section we look at this problem in general.

Theorem 5.1: Let P_r be the probability that r subsets $A_{i_1}, A_{i_2}, \ldots\ldots\ldots, A_{i_r}$ selected from the set $\{A_1, A_2, \ldots\ldots\ldots, A_m\}$ will detect the unknown elements $(u,v) \in S_n$, in N or less steps. Then

$$P_r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left(\frac{m-i}{m}\right)^N .$$

Proof.

The search process will terminate in N or less steps if all the subsets, $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ which detect the unknown pair of elements, (u,v) are selected from the subsets $A_1, A_2, \ldots, A_m$. That is, if we select one or two or three or ...or (m-2) or (m-1) subsets from $A_1, A_2, \ldots, A_m$, which do not include all the subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$, then the unknown pair of elements will not be detected. Possible number of ways of selecting the subsets which do not detect the unknown elements in N steps, that is, under the sequences of one or two or three or............(m - 1) subsets of length N, which do not include all the subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ is obtained by applying Lemma 3.5 to be as follows:

$$\binom{m-r}{0}\left[\binom{r}{1} + \binom{r}{2}(2^N - 2) + \binom{r}{3}(3^N - 3.2^N + 3) + \ldots + \binom{r}{r-1}\left[(r-1)^N - (r-1)(r-2)^N + \ldots \pm (r-1)\right]\right]$$

$$+ \binom{m-r}{1}\left[\binom{r}{0} + \binom{r}{1}(2^N - 2) + \binom{r}{2}(3^N - 3.2^N + 3) + \ldots + \binom{r}{r-1}\left[r^N - r(r-1)^N + \binom{r}{2}(r-2)^N - \ldots \pm r\right]\right]$$

$$+ \binom{m-r}{2}\left[\binom{r}{0}(2^N - 2) + \binom{r}{1}(3^N - 3.2^N + 3) + \binom{r}{2}(4^N - 4.3^N + 6.2^N - 4) + , \ldots + \binom{r}{r-1}\left[(r+1)^N - (r+1)r^N + \ldots \pm (r+1)\right]\right]$$

$$+\cdots\cdots+\binom{m-r}{m-r}\left[\binom{r}{0}\left[(m-r)^N-(m-r)(m-r-1)^N+\cdots\cdots\right.\right.$$

$$\left.\pm(m-r)\right]+\binom{r}{1}\left[(m-r+1)^N-(m-r+1)(m-r)^N+\right.$$

$$\left.\cdots\cdots\cdots\pm(m-r+1)\right]+\cdots\cdots\cdots$$

$$\left.+\binom{r}{r-1}\left[(m-1)^N-(m-1)(m-2)^N+\cdots\pm(m-1)\right]\right].$$

The coefficient of $(m - 1)^N$ is

$$\binom{m-r}{m-r}\binom{r}{r-1}=r\ ;$$

the coefficient of $(m - 2)^N$ is

$$-\binom{m-r}{m-r}\binom{r}{r-1}(m-1)+\binom{m-r}{m-r}\binom{r}{r-2}+\binom{m-r}{m-r-r}$$

$$=\frac{-2(m-1)+r(r-1)\ (m-r)r}{2}$$

$$=-\ r\frac{(r-1)}{2}$$

$$=-\binom{r}{2};$$

the coefficient of $(m - 3)^N$ is

$$\binom{m-r}{m-r}\binom{r}{r-1}\binom{m-1}{2}-\binom{m-r}{m-r}\binom{r}{r-1}(m-2)$$

$$+\binom{m-r}{m-r}\binom{r}{r-3}-\binom{m-r}{m-r-1}\binom{r}{r-1}(m-2)+\binom{m-r}{m-r-1}\binom{r}{r-2}$$

$$+\binom{m-r}{m-r-2}\binom{r}{r-1}=\binom{r}{r-3}$$

$$=\binom{r}{3};$$

and in general, the coefficient of $(m-i)^N$ is

$(-1)^{i+1}\begin{bmatrix} r \\ i \end{bmatrix}$. It then follows that,

$$P_r = \left[m^N - \begin{bmatrix} r \\ 1 \end{bmatrix}(m-1)^N + \begin{bmatrix} r \\ 2 \end{bmatrix}(m-2)^N - \begin{bmatrix} r \\ 3 \end{bmatrix}(m-3)^N + \ldots\ldots \ldots\ldots\ldots \pm \begin{bmatrix} r \\ r \end{bmatrix}(m-r)^N\right] \Big/ m^N$$

$$= \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix} \left(\frac{m-i}{m}\right)^N$$

as required.

Corollary 5.1: Let p_r be the probability that r subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ selected from the collection $\{A_1, A_2, \ldots\ldots, A_m\}$ will detect the unknown elements (u,v) in exactly N steps. Then

$$p_r = \sum_{i=1}^{r} (-1)^{i+1} \left(\frac{i}{m}\right) \begin{bmatrix} r \\ i \end{bmatrix} \left(\frac{m-i}{m}\right)^{N-1}.$$

Proof.

From Theorem 5.1

$$P_r = P_1(N,u,v) = 1 - \begin{bmatrix} r \\ 1 \end{bmatrix}\left(\frac{m-1}{m}\right)^N + \begin{bmatrix} r \\ 2 \end{bmatrix}\left(\frac{m-1}{m}\right)^N - \ldots\ldots \ldots\ldots \pm \begin{bmatrix} r \\ r \end{bmatrix}\left(\frac{m-r}{m}\right)^N$$

and

$$P_1(N-1,u,v) = 1 - \begin{bmatrix} r \\ 1 \end{bmatrix}\left(\frac{m-1}{m}\right)^{N-1} + \begin{bmatrix} r \\ r \end{bmatrix}\left(\frac{m-2}{m}\right)^{N-1} - \begin{bmatrix} r \\ 3 \end{bmatrix}^{N-1} + \ldots\ldots\ldots \pm \begin{bmatrix} r \\ r \end{bmatrix}\left(\frac{m-r}{m}\right)^{N-1}.$$

Therefore,

$$
\begin{aligned}
P_r &= P_1(N,u,v) - P_1(N-1,u,v) \\
&= \Bigg[1-\binom{r}{1}\left(\frac{m-1}{m}\right)^{N} + \binom{r}{2}\left(\frac{m-2}{m}\right)^{N} - \binom{r}{3}\left(\frac{m-3}{m}\right)^{N} + \ldots \\
&\qquad \pm \binom{r}{r}\left(\frac{m-2}{m}\right)^{N}\Bigg] - \Bigg[1 - \binom{r}{1}\left(\frac{m-1}{m}\right)^{N-1} + \\
&\qquad + \binom{r}{2}\left(\frac{m-2}{m}\right)^{N-1} - \binom{r}{3}\left(\frac{m-3}{m}\right)^{N-1} + \ldots \\
&\qquad \ldots\ldots\ldots + \binom{r}{r}\left(\frac{m-r}{m}\right)^{N-1}\Bigg] \\
&= \binom{r}{1}\left(\frac{m-1}{m}\right)^{N-1}\left(1-\frac{m-1}{m}\right) \\
&\qquad + \binom{r}{2}\left(\frac{m-2}{m}\right)^{N-1}\left(\frac{m-2}{m}-1\right) \\
&\qquad + \binom{r}{3}\left(\frac{m-1}{m}\right)^{N-1}\left(1-\frac{m-3}{m}\right) \\
&\qquad + \ldots\ldots \pm \binom{r}{r}\left(\frac{m-r}{m}\right)^{N-1}\left(1-\frac{m-r}{m}\right) \\
&= \frac{1}{m}\binom{r}{1}\left(\frac{m-1}{m}\right)^{N-1} - \frac{2}{m}\binom{r}{2}\left(\frac{m-2}{m}\right)^{N-1} \\
&\qquad + \frac{3}{m}\binom{r}{3}\left(\frac{m-3}{m}\right)^{N-1} - \ldots \pm \frac{r}{m}\binom{r}{r}\left(\frac{m-r}{m}\right)^{N-1} \\
&= \sum_{i=1}^{r} (-1)^{i+1}\frac{i}{m}\binom{r}{i}\left(\frac{m-i}{m}\right)^{N-1}.
\end{aligned}
$$

Theorem 5.2: If r subsets $A_{i_1}, A_{i_2}, \ldots\ldots, A_{i_r}$ selected from the collection $\{A_1, A_2, \ldots\ldots\ldots, A_m\}$ are required to detect the unknown pair of elements, then the expected duration of the search process, $E_1(u,v)$ is given by

$$E_1(u,v) = \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \frac{m}{i} .$$

Proof.

From corollary 5.1, the probability that the process of search terminate in exactly N steps is given by

$$p_r = \frac{1}{m}\binom{r}{1}\left(\frac{m-1}{m}\right)^{N-1} - \frac{2}{m}\binom{r}{2}\left(\frac{m-2}{2}\right)^{N-1}$$

$$+ \frac{3}{m}\binom{r}{3}\left(\frac{m-3}{m}\right)^{N-1} - \dots \pm \frac{r}{m}\binom{r}{r}\left(\frac{m-r}{m}\right)^{N-1} .$$

But

$$E_1(u,v) = \sum_{N=0}^{\infty} N \; p_1(N,u,v) \qquad \text{c.f}(1.10)$$

$$= \sum_{N=0}^{\infty} N \; p_r$$

$$= \frac{1}{m}\binom{r}{1} \sum_{N=1}^{\infty} N \left(\frac{m-1}{m}\right)^{N-1} - \frac{2}{m}\binom{r}{2}\sum_{N=1}^{\infty} N . \left(\frac{m-2}{m}\right)^{N-1}$$

$$+ \dots\dots\dots \pm \frac{r}{m}\binom{r}{r}\sum_{N=1}^{\infty} N . \left(\frac{m-r}{m}\right)^{N-1}$$

$$= \frac{1}{m}\binom{r}{1}m^2 - \frac{2}{m}\binom{r}{2}\frac{m^2}{2^2} + \frac{3}{m}\binom{r}{3}\frac{m^2}{3^2}$$

$$- \dots\dots\dots \pm \frac{r}{m}\binom{r}{r}\frac{m^2}{r^2}$$

$$= \sum_{i=1}^{r} (-1)^{i+1}\binom{r}{i}\frac{m}{i}$$

as required.

Example 5.3:- Consider a partition search design $\{A_1, A_2, \ldots, A_m; S_n\}$. To separete two unknown elements (u,v) using this design we determine two disjoint subsets A_{i_1} and A_{i_2} such that $u \in A_{i_1}$ and $v \in A_{i_2}$. The two unknown elements u and v are then separately identified from A_{i_1} and A_{i_2} respectively. Thus, to separate the two unknown elements only two subsets A_{i_1} and A_{i_2} are required. That is, $r = 2$ The probability that the subsets A_{i_1} and A_{i_2} separate the unknown elements $(u,v) \in S_n$ in N or less steps is, therefore

$$P_2 = 1 - 2\left(\frac{m-1}{m}\right)^N + \left(\frac{m-2}{m}\right)^N;$$

the probability that the subsets A_{i_1} and A_{i_2} separate the unknown pair of elements in exactly N steps is

$$p_2 = \frac{2}{m}\left[\left(\frac{m-1}{m}\right)^{N-1} - \left(\frac{m-2}{m}\right)^{N-1}\right]$$

and the expected duration of the search process is

$$E_1(u,v) = \frac{3r}{2}$$

Remarks: Any strategy for detecting unknown elements will only be economical if the expected number of tests required for the identification is less than that of the number of elements in the set S_n. In the case of partition search design, the expected number of test is $3m/2$, which is more than the number of subsets m. The partition search design will be economical if the number of elements, n, in the finite set S_n is greater than $3m/2$. But $m=3\log_2 n$; that is, $n = 3^m$. Thus partition search design will be economical if $n = 3^m > 3m/2$. This inequality is true for $m \geq 7$. That is, a partition search design is economical for all $m \geq 7$.

The table below gives the number of elements n, the number of subsets m, given by the formula $m = 3\log_3 n$ and the expected number of tests required to separate any two unknown elements into two disjoint subsets, $E_1(u,v) = 3m/2$.

Number of elements n	Number of subsets $= 3\log_3 m$	The expected duration, $E_1(u,v) = 3m/2$
3	3	4.5
9	6	9.0
27	9	13.5
81	12	18.0
243	18	22.5
729	21	27.0
2187	24	31.5
6561	27	36.0
19683	30	40.5

Evidently, partition search design is very economical for large values of n.

CHAPTER 6

SEARCH IN THE PRESENCE OF NOISE

6.1 INTRODUCTION.

In the previous chapters we considered separating systems for determining the identity of one unknown element. We also considered two different strategies, namely 2-complete search design and partition search design for determining the identities of two unknown elements. In all cases, we assumed that the search process was performed in the absence of noise. That is, the observed values of the functions $f_1,f_2,...,f_m$ at the unknown element(s) were assumed to be free of any error. In this Chapter, we consider again, separating systems, 2-complete search and partition search designs except that we now assume that the search process is performed in the presence of noise.

For example, we are interested in problems like detecting an unknown element x in the set $S_n = \{a_1,a_2,......,a_n\}$ using a binary separating system $F = \{f_1,f_2,....,f_m\}$, whose observed values at x, may be in error. That is, it is possible to observe f(x) as 0 instead of the correct value 1, which leads to wrong identification of the unknown element(s).

Example 6.1:- Consider the set $S_8= \{a_1,a_2,....,a_8\}$, and suppose that we wish to determine one unknown element $x \in S_8$ using three functions f_1,f_2,f_3 whose search matrix M is.

$$M = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

The functions f_1, f_2, f_3 form a separating system since the columns of the matrix M are distinct.

Let the unknown element x be a_1. Then by observing f_1, f_2, f_3 at x we obtain subsets $A_1 = \{a_1, a_2, a_3, a_4\}$ $A_2 = \{a_1, a_2, a_5, a_6\}$ and $A_3 = \{a_1, a_3, a_5, a_7\}$ respectively, with

$$A_1 \cap A_2 \cap A_3 = \{a_1\}.$$

Suppose $f_2(x)$ is in error. That is, it is observed as 0 instead of the correct value 1. Then, the subset $A_2 = \{a_3, a_4, a_7, a_8\}$ would be specified by this incorrect observation, with

$$A_1 \cap A_2 \cap A_3 = \{a_3\}$$

which is wrong identification of the unknown element x.

In the next section we will consider separating systems which determine correctly the unknown element x in S_n in the presence of noise.

6.2 SEPARATING SYSTEMS WHICH DETERMINE ONE UNKNOWN ELEMENT IN THE PRESENCE OF NOISE.

In this section, we describe two types of separating systems. The first one detects an error in the search process for one unknown element without correcting it. The second system detects and corrects the error in determining the identity of the unknown element.

6.2.1. Single-error detecting system

Consider the set $S_n = \{a_1, a_2, \ldots\ldots, a_n\}$ and the system F of functions $f_1, f_2, \ldots\ldots, f_m$. Suppose that the unknown element x is searched for by observing the functions $f_1, f_2, \ldots, f_m$ successively at x. Further, let $A_i = f_i^{-1}(f_i(x))$, $i = 1,2,\ldots\ldots m$. Then the system $\{f_1, f_2, \ldots\ldots, f_m\}$ will be single-error detecting system if

$$A_1 \cap A_2 \cap \cdots\cdots\cdots \cap A_m = \{x\}$$

and

$$A_1 \cap A_2 \cap \cdots \cap A_{i-1} \cap \overset{\subset}{A_i} \cap A_{i+1} \cap \cdots \cap A_m = \emptyset$$

$$1 \leq i \leq m .$$

Examlpe 6.2:- Consider the set $S_6 = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and suppose the system $F = \{f_1, f_2, f_3, f_4\}$ has the following search matrix:

$$M = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array} .$$

The functions $\{f_1, f_2, f_3, f_4\}$ form a separating system, since the columns of the search matrix M are distinct. Let the unknown element x be a_1. Then by observing

f_1, f_2, f_3, f_4 at a_1 we obtain the subsets

$$A_1 = f_1^{-1}(f_1(a_1)) = \{a_1, a_2, a_3\},$$

$$A_2 = f_2^{-1}(f_2(a_1)) = \{a_1, a_4, a_5\},$$

$$A_3 = f_3^{-1}(f_3(a_1)) = \{a_1, a_3, a_5\},$$

and

$$A_4 = f_4^{-1}(f_4(a_1)) = \{a_1, a_2, a_4\};$$

with

$$A_1 \cap A_2 \cap A_3 \cap A_4 = \{a_1\},$$

$$A_1^c \cap A_2 \cap A_3 \cap A_4 = \emptyset,$$

$$A_1 \cap A_2^c \cap A_3 \cap A_4 = \emptyset,$$

$$A_1 \cap A_2 \cap A_3^c \cap A_4 = \emptyset.$$

and

$$A_1 \cap A_2 \cap A_3 \cap A_4^c = \emptyset.$$

Thus, the system $\{f_1, f_2, f_3, f_4\}$ is a single-error detecting system; it will detect if any function f_i is in error.

Lemma 6.1:- Let $\{f_1, f_2, \ldots\ldots, f_m\}$ be a system of functions defined on a finite set $S_n = \{a_1, a_2, \ldots\ldots, a_n\}$ and let $A_i = f_i^{-1}(f_i(x))$, where x is the unknown element in the set S_n. Then the system $\{f_1, f_2, \ldots\ldots, f_m\}$ is a single-error detecting system if and only if the intersection of any $(m-1)$ subsets $A_{i_1}, A_{i_2}, \ldots\ldots, A_{i_{m-1}}$, $\{i_1, i_2, \ldots\ldots, i_{m-1}\} \subset \{1, 2, \ldots\ldots, m\}$

is x. That is,

$$A_{i_1} \cap A_{i_2} \cap \cdots\cdots\cdots \cap A_{i_{m-1}} = \{x\}.$$

Proof.

Suppose the system $\{f_1, f_2, \ldots, f_m\}$ is a single-error detecting system and the unknown element x is a_ℓ. That is,

$$A_1 \cap A_2 \cap \cdots\cdots\cdots \cap A_m = \{a_\ell\}$$

and

$$A_1 \cap A_2 \cap \cdots \cap A_i^c \cap \cdots\cdots \cap A_m = \emptyset,$$

$$\text{for } 1 \leq i \leq m.$$

Then, we have to prove that

$$A_{i_1} \cap A_{i_2} \cap \cdots\cdots\cdots \cap A_{i_{m-1}} = \{a_\ell\} \quad (6.1)$$

where $\{i_1, i_2, \ldots\ldots, i_{m-1}\} \subset \{1,2,3,\ldots,m\}$

Now, suppose

$$A_{i_1} \cap A_{i_2} \cap \cdots\cdots\cdots \cap A_{i_{m-1}} \neq \{a_\ell\}. \quad (6.2)$$

That is, $A_{i_1} \cap A_{i_2} \cap \cdots\cdots\cdots \cap A_{i_{m-1}} = \emptyset$ or a set consisting of one element a_j (a [illegible] a set consisting of $a_{j_1}, a_{j_2}, \ldots\ldots, a_{j_\ell}$ [illegible] $j_1 \neq j_2 \neq \ldots\ldots \neq j_\ell \neq \ell$ or a set consisting of a_ℓ and some other elements. But, $A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_{m-1}}$ cannot be an empty set or a set consisting of one element a_j ($a_j \neq a_\ell$) or set consisting of $a_{j_1}, a_{j_2}, \ldots, a_{j_\ell}$, $j_1 \neq j_2 \neq j_3 \neq \ldots \neq j_\ell \neq \ell$ since $A_1 \cap A_2 \cap \cdots \cap A_m = \{a_\ell\}$

and so $a_\ell \in A_{i_k}$, for $k = 1,2,\ldots\ldots,(m-1)$. This leaves us with only one possibility that $A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_{m-1}}$ is a set consisting of a_ℓ and some other element(s). To investigate this possibility, we let $a_{j'}$ be one of these other elements in $A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_{m-1}}$. That is, $a_\ell, a_{j'} \in A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_{m-1}}$. This implies that $a_{j'} \notin A_{i_m}$ since $A_1 \cap A_2 \cap \cdots\cdots\cdots\cdots \cap A_m = (a_\ell)$ and $\{i_1, i_2, \ldots\ldots\ldots\ldots, i_m\} = \{1,2,\ldots\ldots\ldots m\}$, so $a_{j'} \in A_{i_m}^c$ and $A_{i_1} \cap A_{i_2} \cap \cdots\cdots\cdots\cdots\cdots \cap A_{i_{m-1}} \cap A_{i_m}^c = \{a_{j'}\}$. Thus, there exists j, $1 \leq j \leq m$, such that, $A_1 \cap A_2 \cap \cdots \cap A_j^c \cap \cdots \cap A_m = \{a_{j'}\}$, $j' \neq \ell$. This contradicts, the fact that $\{f_1, f_2, \ldots\ldots\ldots\ldots, f_m\}$ is a single-error detecting system; thus $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{m-1}}$ is not a set consisting of a_ℓ and some other element(s). We therefore, conclude that $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{m-1}} = \{a_\ell\}$.

Conversely, suppose

$$A_{i_1} \cap A_{i_2} \cap \cdots\cdots\cdots \cap A_{i_{m-1}} = \{a_\ell\}. \qquad (6.3)$$

Then, we are to prove that the system $\{f_1, f_2, \ldots\ldots\ldots, f_m\}$ is a single-error detecting system. That is, we have to show that

$$A_1 \cap A_2 \cap \cdots\cdots\cdots\cdots \cap A_m = \{a_\ell\} \qquad (6.4)$$

and

$$A_1 \cap A_2 \cap \cdots\cdots \cap A_\iota^c \cap \cdots \cap A_m = \emptyset \qquad (6.5)$$

for $1 \leq \iota \leq m$.

But

$$A_\iota = f_\iota^{-1}(f_\iota(a_\ell)),$$

and thus

$$A_{\iota_m} = f_{\iota_m}^{-1}(f_{\iota_m}(a_\ell))$$

$\iota_m \in \{1,2,\ldots\ldots\ldots m\}$ so that

$$a_\ell \in A_{\iota_m}.$$

That is,

$$a_\ell \notin A_{\iota_m}^c. \qquad (6.6)$$

From (6.3) and (6.6) it follows that:

$$A_{\iota_1} \cap A_{\iota_2} \cap \cdots \cap A_{\iota_{m-1}} \cap A_{\iota_m} = \{a_\ell\} \qquad (6.7)$$

and

$$A_{\iota_1} \cap A_{\iota_2} \cap \cdots \cap A_{\iota_{m-1}} \cap A_{\iota_m}^c = \emptyset. \qquad (6.8)$$

But, $\{\iota_1, \iota_2, \ldots, \iota_m\} = \{1,2,\ldots\ldots,m\}$; thus (6.7) and (6.8) imply that there exists j, $1 \leq j \leq m$, such that

$$A_1 \cap A_2 \cap \cdots\cdots\cdots\cdots \cap A_m = \{a_\ell\}$$

and

$$A_1 \cap A_2 \cap \cdots\cdots \cap A_j^c \cap \cdots \cap A_m = \emptyset$$

which is the required result.

Theorem 6.1:- Let the number of elements in the set S_n be n, then the minimum number of functions, m for which a single-error detecting system exists satisfies the inequality:

$$m \geq \log_2 n + 1.$$

Proof.

Let $\{f_1, f_2, \ldots, f_m\}$ be a single - error detecting system and $A_1, A_2, \ldots, A_m$ be subsets specified by the functions $f_1, f_2, \ldots, f_m$. Then, from Lemma 6.1, the intersection of any (m-1) subsets, $A_{i_1}, A_{i_2}, \ldots, A_{i_{m-1}}$ is the unknown element x. That is, any (m-1) functions identify the unknown element. The minimum number of functions which separate the unknown element is $\{\log_2 n\}$, see Renyi (1965). That is,

$$(m-1) \geq \log_2 n$$

$$m \geq \log_2 n + 1$$

which is the required result.

Example 6.3 -Let $S_8 = \{a_1, a_2, \ldots, a_8\}$, then a single-error detecting system for this set will contain at least $(\log_2 8)+1$, that is four, functions. One possible search matrix of a system of four functions f_1, f_2, f_3, f_4 which is a single-error detecting system is

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

The four functions f_1, f_2, f_3, f_4 can easily be shown to be a single-error detecting system by taking the unknown element x to be, say a_1. Then, the subsets A_1, A_2, A_3, A_4 specified by these functions are:

$$A_1 = \{a_1, a_2, a_3, a_7\},$$
$$A_2 = \{a_1, a_2, a_4, a_5\},$$
$$A_3 = \{a_1, a_3, a_4, a_6\},$$
$$A_4 = \{a_1, a_5, a_6, a_7\};$$

with

$$A_1 \cap A_2 \cap A_3 \cap A_4 = \{a_1\},$$
$$A_1^c \cap A_2 \cap A_3 \cap A_4 = \emptyset,$$
$$A_1 \cap A_2^c \cap A_3 \cap A_4 = \emptyset,$$
$$A_1 \cap A_2 \cap A_3^c \cap A_4 = \emptyset,$$
$$A_1 \cap A_2 \cap A_3 \cap A_4^c = \emptyset.$$

Thus, the system $\{f_1, f_2, f_3, f_4\}$ is a single-error detecting system on the set S_8.

Theorem 6.2: Let $M = ((m_{ij}))$ be an mxn matrix whose first and last columns are respectively $(1,1,\ldots\ldots,1)'$ and $(0,0,\ldots\ldots\ldots,0)'$ and the remaining columns consist of all possible combinations of $\left\{\frac{1}{2}m\right\}$ zeros (or ones) and $m-\left\{\frac{1}{2}m\right\}$ ones (or zeros), where $\{x\}$ is the least integer greater than or equal to x. Further, let the functions $f_1, f_2, \ldots\ldots, f_m$ be defined on a finite set $S_n = \{a_1, a_2, \ldots\ldots, a_n\}$ as follows:

$$f_i(a_j) = \begin{cases} 1 & \text{if} \quad m_{ij} = 1 \\ 0 & \text{if} \quad m_{ij} = 0. \end{cases}$$

Then, the system $\{f_1, f_2, \ldots\ldots, f_m\}$ of functions is a single-error detecting system.

Proof.

Identifying the columns of the matrix M with elements $a_1, a_2, \ldots, a_n$ of S_n and rows with the functions $f_1, f_2, \ldots\ldots, f_m$, we need to show that the intersection of any (m-1) subsets $A_{i_1}, A_{i_2}, \ldots\ldots, A_{i_{m-1}}$ specified by the functions $f_{i_1}, f_{i_2}, \ldots\ldots, f_{i_{m-1}}$ is a single element. That is ,

$$A_{i_1} \cap A_{i_2} \cap \cdots\cdots\cap A_{i_{m-1}} = \{x\}.$$

But,

$$A_{i_1} \cap A_{i_2} \cap \cdots\cdots\cap A_{i_{m-1}} = \{x\}$$

holds if and only if functions $f_{i_1}, f_{i_2}, \ldots\ldots, f_{i_{m-1}}$ form a separating system on $S_n = \{a_1, a_2, \ldots\ldots, a_n\}$. But we know that the functions $f_{i_1}, f_{i_2}, \ldots, f_{i_{m-1}}$ will form a separating system if the matrix N, defined by;

$$N = (f_{i_k}(a_j)), \qquad k = 1, 2, \ldots, m-1; j = 1, 2, \ldots, n;$$

has distinct columns.

Now, the columns of the matrix M defined above differ in at least two places and so if a row is deleted, the remaining columns will differ in at least one place. Thus, the matrix M′ obtained by deleting one row of M has distinct columns and so the functions $f_{i_1}, f_{i_2}, \ldots, f_{i_{m-1}}$ which correspond to the rows of M′ is a separating system. The system $f_1, f_2, \ldots, f_m$ is therefore a single-error detecting system.

Example 6.4: Consider a 6x22 matrix whose first and last columns are respectively (1,1,1,1,1,1)′ and (0,0,0,0,0,0)′ and the remaining 20 columns consist of all possible combinations of 3 zeros and 3 ones. The columns of this matrix are identified with the elements of a finite set $S_{22} = \{a_1, a_2, \ldots, a_{22}\}$ and the rows are identified with the functions $f_1, f_2, f_3, f_4, f_5, f_6$. The matrix M has the following form;

M =

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}
f_1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
f_2	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0
f_3	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	[illegible]	[illegible]	[illegible]	[illegible]
f_4	1	0	1	0	0	1	0	0	1	1	0	1	0	0	1	1	0	1	1	0	1	0
f_5	1	0	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1	1	0	1	1	0
f_6	1	0	0	0	1	0	0	1	0	1	1	0	0	1	0	1	1	0	1	[illegible]	[illegible]	0

Now, suppose the fourth row is deleted. The remaining matrix M′ will have the form,

$$M' = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_5 \\ f_6 \end{array} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} & a_{22} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

We notice that the columns of the matrix M′ are distinct; thus the system $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ is a single-error detecting system. We can easily, verify this by taking the unknown element x to be say a_1. Then,

$$A_1 = (a_1, a_2, a_3, a_4, a_5, a_6, a_7\ a_8, a_9, a_{10}, a_{11}\},$$
$$A_2 = \{a_1, a_2, a_3, a_4, a_5, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}\},$$
$$A_3 = \{a_1, a_2, a_6, a_7, a_8, a_{12}, a_{13}, a_{14}, a_{18}, a_{19}, a_{20}\},$$
$$A_4 = \{a_1, a_3, a_6, a_9, a_{10}, a_{12}, a_{15}, a_{16}, a_{18}, a_{19}, a_{20}\},$$
$$A_5 = \{a_1, a_4, a_7, a_9, a_{11}, a_{13}, a_{15}, a_{17}, a_{18}, a_{19}, a_{20}\},$$
$$A_6 = \{a_1, a_5, a_8, a_{10}, a_{11}, a_{13}, a_{16}, a_{17}, a_{19}, a_{20}, a_{21}\};$$

with

$$A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 = \{a_1\},$$
$$A_1^c \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 = \emptyset,$$
$$A_1 \cap A_2^c \cap A_3 \cap A_4 \cap A_5 \cap A_6 = \emptyset,$$
$$A_1 \cap A_2 \cap A_3^c \cap A_4 \cap A_5 \cap A_6 = \emptyset,$$
$$A_1 \cap A_2 \cap A_3 \cap A_4^c \cap A_5 \cap A_6 = \emptyset,$$
$$A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5^c \cap A_6 = \emptyset,$$

and

$$A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6^c = \emptyset.$$

Thus, $f_1, f_2, f_3, f_4, f_5, f_6$ is a single-error detecting system.

In our study of the next type of separating system, that is one which detects and corrects the error in determining the identity of one unknown element, we will require the basic concepts of coding theory introduced in Chapter 1. In addition, we will need the following property of block codes also given in Chapter 1.

A block code with distance d is capable of correcting all patterns of t or fewer errors and detecting all patterns of $t + j$, $0 < j < s$ errors if $2t + s < d, s > 0$.

6.2.2: Error-correcting system

Let $S_n = \{a_1, a_2, \ldots, a_n\}$ and $F = \{f_1, f_2, \ldots, f_m\}$ be a system of m functions. Further, let M be an mxn search matrix of the system F. That is;

$$M = (f_i(a_j)), \quad i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n.$$

Let $x \in S_n$ be the unknown element which is to be identified by observing the functions $f_1, f_2, \ldots, f_m$ successively at x. By these observations we obtain a $(f_1(x), f_2(x), \ldots, f_m(x))'$. The unknown element x, is then identified as a_ℓ $(\ell = 1,2,\ldots,n)$ if the vector $(f_1(x), f_2(x), \ldots, f_m(x))'$ is the ℓth column of the matrix M.

Now, suppose ρ functions are in error then the system $F = \{f_1, f_2, \ldots, f_m\}$ will be error-detecting and error-correcting if the vector $\underline{u}$ =

$(f_1(x), f_2(x), \ldots, f_m(x))'$ obtained by observing $f_1, f_2, \ldots, f_m$ at x is not one of the columns of the matrix M and the distance between any two columns of the matrix M say $\underline{v}_1$ and $\underline{v}_2$ is at least $2\rho + 1$. The ρ errors will be detected by the fact that none of the columns of the matrix M is the vector $\underline{u} = (f_1(x), f_2(x), \ldots, f_m(x))$ and corrected by identifying the vector $\underline{u}$ with a column $\underline{v}$ of the matrix M in which $d(\underline{u}, \underline{v}) = \rho$. The column $\underline{v}$ can easily be shown to be unique; for suppose that there exists another column $\underline{v}'$ of the matrix M, such that $d(\underline{u}, \underline{v}) = \rho$. Then,

$$\begin{aligned} d(\underline{v}, \underline{v}') &\leq d(\underline{v}, \underline{u}) + d(\underline{u}, \underline{v}') \\ &= d(\underline{u}, \underline{v}) + d(\underline{u}, \underline{v}) \\ &= \rho + \rho = 2\rho \end{aligned}$$

which implies that the distance between the two columns $\underline{v}$ and $\underline{v}'$ of the matrix M is less than or equal to 2ρ. This contradicts the earlier assumption that the distance between any two columns is at least $2\rho + 1$. Thus, the column $\underline{v}'$ does not exist.

Example 6.5:- Consider a block code which corrects ρ errors, that is, the distance between any two code words is at least $2\rho + 1$. Let these code words be the columns of the matrix M defined earlier. Then the system F of functions which correspond to the rows of the matrix M is error-correcting system since the distance between any two columns of the matrix M is at least $2\rho + 1$.

Special case

Consider, the following code words of the Hamming block codes that correct one error, (see Chakravarti (1976)):

0	0	0	0	0	0	0		0	1	1	0	1	1	0
1	0	0	0	1	1	0		0	1	0	1	0	1	0
0	1	0	0	1	0	1		0	0	1	1	1	0	0
0	0	1	0	0	1	1		1	1	1	0	0	0	0
0	0	0	1	1	1	1		1	1	0	1	1	0	0
1	1	0	0	0	1	1		1	0	1	1	0	1	0
1	0	1	0	1	0	1		0	1	1	1	0	0	1
1	0	0	1	0	0	1		1	1	1	1	1	1	1

Taking these code words to be the columns of the matrix M, we obtain the following search matrix,

$M =$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
f_1	0	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
f_2	0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1
f_3	0	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1
f_4	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1
f_5	0	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1
f_6	0	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1
f_7	0	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1

The distance between any two columns of the matrix M is at least three, thus the functions $f_1, f_2, \ldots, f_7$ form an error-correcting system which corrects at most one error.

6.3: DETERMINING TWO UNKNOWN ELEMENTS IN THE PRESENCE OF NOISE USING A 2-COMPLETE SEARCH DESIGN.

We first recall that a 2-complete search design is a system $\{A_1, A_2, \ldots, A_m; S_n\}$ consisting of m subsets $A_1, A_2, \ldots\ldots, A_m$ of a finite set S_n, in which for any pair of elements $a_\ell, a_{\ell'}$ in S_n, there exist subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}, \{i_1, i_2, \ldots, i_k\} \subset \{1,2,\ldots,m\}$ such that $a_\ell, a_{\ell'} \in A_{i_j}$ for $j = 1,2,\ldots,k$ and $\bigcap_{j=1}^{k} A_{i_j} = \{a_\ell, a_{\ell'}\}$.

Now, suppose that the unknown elements $(a_\ell, a_{\ell'})$ in S_n are to be determined in the presence of noise. That is, a subset A_i can be declared to contain the two unknown elements while it actually contains just one or none of them. Then, the intersection of the subsets $A_{i_1}, A_{i_2}, \ldots\ldots, A_{i_k}$ will not identify the two unknown elements.

In this section, we consider 2-complete search designs which detect the error in the search process for the two unknown elements without correcting it.

6.3.1: 2-Complete search design which detects an error in the search process.

Consider the set $S_n = \{a_1, a_2, \ldots\ldots, a_n\}$ and 2-complete search design $\{A_1, A_2, \ldots\ldots, A_m; S_n\}$. Next, consider the set of indices $\{i_1, i_2, \ldots, i_k\} \subset \{1,2,\ldots\ldots,m\}$ and let $A_{i_1}, A_{i_2}, \ldots\ldots\ldots, A_{i_k}$ be the subsets of S_n which contain the two unknown elements $(a_\ell, a_{\ell'})$. Then, the

2-complete search design $\{A_1, A_2, \ldots, A_m; S_n\}$ will detect an error in the search process if

$$A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}\}$$

and

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A^c_{i_j} \cap \cdots \cap A_{i_k} = \emptyset;$$

or

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A^c_{i_j} \cap \cdots \cap A_{i_k} = \{a\},$$

$$a \in S_n, \; 1 \le j \le k.$$

Example 6.6:- Consider the 2-Complete search design $\{A_1, A_2, \ldots, A_9; S_{12}\}$ of Example 4.3 given in Section 4.3 of Chapter 4. The subsets $A_1, A_2, \ldots\ldots, A_9$ were given as follows:

$$A_1 = \{a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\},$$
$$A_2 = \{a_2, a_3, a_4, a_8, a_9, a_{10}, a_{11}, a_{12}\},$$
$$A_3 = \{a_1, a_3, a_4, a_6, a_7, a_8, a_9, a_{12}\},$$
$$A_4 = \{a_1, a_3, a_4, a_6, a_7, a_8, a_9, a_{12}\},$$
$$A_5 = \{a_1, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{12}\},$$
$$A_6 = \{a_1, a_2, a_4, a_5, a_7, a_9, a_{10}, a_{12}\},$$
$$A_7 = \{a_1, a_2, a_4, a_6, a_7, a_8, a_{10}, a_{11}\},$$
$$A_8 = \{a_1, a_2, a_3, a_5, a_6, a_8, a_{10}, a_{12}\},$$
$$A_9 = \{a_1, a_2, a_3, a_5, a_7, a_8, a_9, a_{11}\}.$$

Suppose the two unknown elements are (a_1, a_2), then the subsets which contain these two unknown elements are A_6, A_7, A_8, A_9 with

$$A_6 \cap A_7 \cap A_8 \cap A_9 = \{a_1, a_2\},$$

$$A_6^c \cap A_7 \cap A_8 \cap A_9 = \{a_8\},$$

$$A_6 \cap A_7^c \cap A_8 \cap A_9 = \{a_5\},$$

$$A_6 \cap A_7 \cap A_8^c \cap A_9 = \{a_7\},$$

and

$$A_6 \cap A_7 \cap A_8 \cap A_9^c = \{a_{10}\}.$$

That is,

$$A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4} = \{a_1, a_2\}$$

and

$$A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}^c = \{a\}, \quad a \in S_{12}, 1 \leq i_4 \leq 4.$$

Thus, the system $\{A_1, A_2, \ldots, A_9; S_{12}\}$ is a 2-complete search design which detects an error in the search process.

Lemma 6.3:- Let $\{A_1, A_2, \ldots, A_m; S_n\}$ be a 2-complete search design. Further, let $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ be the subsets of the set S_n which contain the two unknown elements $(a_\ell, a_{\ell'})$ where $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, m\}$. Then, the 2-complete search design $\{A_1, A_2, \ldots, A_m; S_n\}$ will detect an error in the search process if and only if the intersection of any $(k-1)$ subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_{k-1}}$ is at most three elements with the two unknown elements $(a_\ell, a_{\ell'})$ being amongst the three.

Proof.

Suppose the system $\{A_1, A_2, \ldots, A_m; \; S_n\}$ is a 2-complete search design which detects an error in the search process. That is, for any set of indices $\{\iota_1, \iota_2, \ldots\ldots, \iota_k\} \subset \{1, 2, \ldots\ldots, m\}$

$$A_{\iota_1} \cap A_{\iota_2} \cap \cdots \cap A_{\iota_j} \cap \cdots\cdots A_{\iota_k} = \{a_\ell, a_{\ell'}\} \qquad (6.9)$$

and

$$A_{\iota_1} \cap A_{\iota_2} \cap \cdot \cap A^c_{\iota_j} \cap \cdot \cdot \cap A_{\iota_k} = \emptyset \text{ or } \{a_\iota\}$$
$$a_\iota \in S_n . \qquad (6.10)$$

Then, we are to prove that

$$A_{\iota_1} \cap A_{\iota_2} \cap \cdot \cap A_{\iota_{j-1}} \cap A_{\iota_{j+1}} \cap \cdot \cap A_{\iota_k} = \{a_\ell, a_{\ell'}\} \quad (6.11)$$

or

$$A_{\iota_1} \cap A_{\iota_2} \cap \cdot \cap A_{\iota_{j-1}} \cap A_{\iota_{j+1}} \cap \cdot \cap A_{\iota_k} = \{a_\ell, a_{\ell'}, a_{\ell''}\}$$
$$(6.12)$$

where, $\{a_\ell, a_{\ell'}, a_{\ell''}\} \in S_n$.

Now, suppose that

$$A_{\iota_1} \cap A_{\iota_2} \cap \cdot \cap A_{\iota_{j-1}} \cap A_{\iota_{j+1}} \cap \cdot \cap A_{\iota_k} \neq \{a_\ell, a_{\ell'}\}$$

and

$$A_{i_1} \cap A_{i_2} \cap \cdot\cdot \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdot\cdot \cap A_{i_k} \neq \{a_\ell, a_{\ell'}, a_{\ell''}\}.$$

That is, $A_{i_1} \cap A_{i_2} \cap \cdot\cdot \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \emptyset$ or a set consisting of one element a_j or a set consisting of two elements (a_{j_1}, a_{j_2}) where $a_{j_1}, a_{j_2} \notin \{a_\ell, a_{\ell'}\}$ or a set consisting of $a_{j_1'}, a_{j_2'}, \ldots, a_{j_r'}$ where $a_\ell, a_{\ell'} \notin \{a_{j_1'}, a_{j_2'}, \ldots, a_{j_r'}\}$ and $r > 2$ or a set consisting of a_ℓ and $a_{\ell'}$ and two or more other elements. But $A_{i_1} \cap \cdot\cdot \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdot\cdot \cap A_{i_k}$ cannot be the empty set or a set consisting of one element or a set consisting of two elements $\{a_{j_1}, a_{j_2}\}$ where $a_{j_1}, a_{j_2} \notin \{a_\ell, a_{\ell'}\}$ or a set consisting of $a_{j_1'}, a_{j_2'}, \ldots, a_{j_r'}$ where $a_\ell, a_{\ell'} \notin \{a_{j_1'}, a_{j_2'}, \ldots, a_{j_r'}\}$ since $A_{i_1} \cap \cdot\cdot \cap A_{i_j} \cap A_{i_{j+1}} \cdot\cdot \cap A_{i} = \{a_\ell, a_{\ell'}\}$; and so $a_\ell, a_{\ell'} \in A_{i_j}$ for $j = 1, 2, \ldots, k$. This leaves us with only one possibility that $A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cdots\cdots \cap A_{i_k}$ is a set consisting of $a_\ell, a_{\ell'}$ and two or more other elements. To investigate this possibility, we let a_j and $a_{j'}$ be the other elements in the set $A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots\cdots \cap A_{i_k}$. That is,

$$a_j, a_{j'} \in A_{i_1} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdot\cdot \cap A_{i_k}. \qquad (6.13)$$

Now, (6.9) and (6.13) imply that

$$a_j, a_{j'} \notin A_{i_j} .$$

That is ,

$$a_j, a_{j'} \in A^{c}_{i_j} ,$$

and so from (6.13)

$$a_j, a_{j'} \in A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A^{c}_{i_j} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k}$$

This contradicts (6.10). Thus $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots\cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}\}$ is not a set consisting of $a_\ell, a_{\ell'}$ and two or more other elements. We therefore, conclude that $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}\}$ or $\{a_\ell, a_{\ell'}, a_{\ell''}\}$.

Conversely, suppose

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}\}$$

or

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}, a_{\ell''}\}$$

where $\{a_\ell, a_{\ell'}, a_{\ell''}\} \in S_n$. Then, we are to prove that the 2-complete search design $\{A_1, A_2, \ldots, A_m; S_n\}$ detects an error in the search process. That is, we need to show that

$$A_{i_1} \cap A_{i_2} \cap \cdots\cdots \cap A_{i_j} \cap \cdots\cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}\}$$

and

$$A_{i_1} \cap A_{i_2} \cdots \cap A_{i_{j-1}} \cap A^c_{i_j} \cap A_{i_{j+1}} \cdots \cap A_{i_k} = \emptyset \text{ or } \{a_i\}$$
$$i \in S_n .$$

First we consider the case

$$A_{i_1} \cap A_{i_2} \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}\}. \tag{6.14}$$

From the fact that $\{A_1, A_2, \ldots, A_m; S_n\}$ is a 2-complete search design and the subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_j}, \ldots, A_{i_k}$ contain both the unknown elements $\{a_\ell, a_{\ell'}\}$, we see that

$$a_\ell, a_{\ell'} \in A_{i_j} \text{ and } a_\ell, a_{\ell'} \notin A^c_{i_j}. \tag{6.15}$$

From (6.14) and (6.15) we have

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_j} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}\}$$

and

$$A_{i_1} \cap A_{i_2} \cdots \cap A_{i_{j-1}} \cap A^c_{i_j} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \emptyset.$$

Thus, the 2-complete search design detects an error in the search process.

Next, we consider the case

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \{a_\ell, a_{\ell'}, a_{\ell''}\}. \tag{6.16}$$

Again, from the fact that $\{A_1, A_2, \ldots, A_m; S_n\}$ is a 2-complete search design and the subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ contain both the unknown elements $\{a_\ell, a_{\ell'}\}$, we see that $\{a_\ell, a_{\ell'}\} \in A_{i_j}$ and $\{a_\ell, a_{\ell'}\} \notin A^c_{i_j}$.

From (6.16) we have

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_j} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \{a_{\ell'}, a_{\ell''}\}$$

and

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_j} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k} = \{a_{\ell''}\}.$$

Thus, the 2-complete search design $\{A_1, A_2, \ldots, A_m; S_n\}$ detects an error in the search process. Hence the proof.

Example 6.8: Consider Example 4.1 in Section 4.1 of Chapter 4. The subsets $A_1, A_2, \ldots, A_9$ are:

$$A_1 = \{a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\},$$

$$A_2 = \{a_2, a_3, a_4, a_8, a_9, a_{10}, a_{11}, a_{12}\},$$

$$A_3 = \{a_2, a_3, a_4, a_5, a_6, a_7, a_{11}, a_{12}\},$$

$$A_4 = \{a_1, a_3, a_4, a_6, a_7, a_8, a_9, a_{12}\},$$

$$A_5 = \{a_1, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{11}\},$$

$$A_6 = \{a_1, a_2, a_4, a_5, a_7, a_9, a_{10}, a_{12}\},$$

$$A_7 = \{a_1, a_2, a_4, a_6, a_7, a_8, a_{10}, a_{11}\},$$

$$A_8 = \{a_1, a_2, a_3, a_5, a_6, a_8, a_{10}, a_{12}\},$$

$$A_9 = \{a_1, a_2, a_3, a_5, a_7, a_8, a_9, a_{11}\}.$$

As displayed in the example, every pair of elements (a_i, a_j), $i \neq j = 1, 2, \ldots, 12$, can be detected by at at most four subsets. The intersection of any three subsets can easily be verified to contain atmost three elements. Hence the system $\{A_1, A_2, \ldots\ldots, A_9; S_{12}\}$ is a 2-complete search design which detects an error in the search process.

6.4 DETERMINING TWO UNKNOWN ELEMENTS IN THE PRESENCE OF NOISE USING PARTITION SEARCH DESIGNS.

We recall that a partition search design consists of two stages, namely:

(i) Determining subsets $A_1, A_2 \ldots\ldots\ldots, A_m$ of the set $S_n = \{a_1, a_2, \ldots\ldots, a_n\}$ such that for any two distinct elements $(u,v) \in S_n$ there exists two subsets A_{i_1} and A_{i_2} of S_n such that $u \in A_{i_1}$ and $v \in A_{i_2}$ and $A_{i_1} \cap A_{i_2} = \emptyset$.

(ii) Identifying the two unknown elements (u,v) from the sets A_{i_1} and A_{i_2} separately using a separating system.

Now, suppose that we search for these two unknown elements in the presence of noise; that is, it is possible for an error to occur. If the error occurs in stage one, that is, a subset A_i is declared to contain the unknown element while it does not, then this error would be detected in the second stage while searching for the unknown element u or v in a subset in which it does not belong. If the error occurs in the second stage; that is the observed values of the functions $f_1, f_2, \ldots\ldots, f_m$ at the unknown element may be in error, then this error can be detected without being corrected by applying single-error detecting systems, described in Section 6.2.1 of this Chapter; or the error can be detected and corrected by applying error-correcting systems described in Section 6.2.3 again of this Chapter.

Example 6.9:- Consider the partition search design of Example 4.2 in Section 4.1 of Chapter 4. The subsets $A_1, A_2, A_3, A_4, A_5, A_6$ are:

$$A_1 = \{a_1, a_2, a_3, a_4\}, \qquad A_2 = \{a_5, a_6, a_7, a_8\},$$

$$A_3 = \{a_1, a_2, a_7, a_8\}, \qquad A_4 = \{a_3, a_4, a_5, a_6\},$$

$$A_5 = \{a_1, a_3, a_5, a_7\}, \qquad A_6 = \{a_2, a_4, a_6, a_8\}.$$

Let $\{a_1, a_2\}$ be the unknown pair of elements. Then the subsets A_5 and A_6 will detect the pair since $a_1 \in A_5$ $a_2 \in A_6$ and $A_5 \cap A_6 = \emptyset$. Suppose, a subset say A_4 is erroneously, found to contain an unknown element say a_2, then this error will be detected without being corrected in the second stage, where the identity of the unknown element is determined. In this case, we will be trying to identify the unknown element from a set in which it does not belong.

An error in the second stage, say an error made in identifying the unknown a_2 from A_6 will be detected without being corrected by applying single-error detecting system. That is, if we apply single-error detecting system $\{f_1, f_2, \ldots., f_6\}$, then the intersection of the subsets $A_1, A_2, \ldots\ldots\ldots, A_{\sigma}$. where $A_i = f_i^{-1}(f_i(a_2))$ will be either $\{a_2\}$ or Ø. It will be $\{a_2\}$ if no error is made and Ø if an error is made in identifying, the unknown element, a_2 from A_{σ}.

CHAPTER 7

CONCLUDING REMARKS

In this thesis the problem of search for one unknown and two unknown elements from a set S_n consisting of n distinguishable elements has been studied. The study has dealt with search models which assume noiseless conditions and those which take noise into account.

Starting with the case of one unknown element in the set S_n, binary and non-binary separating systems which detect the unknown element have been studied. Properties of these separating systems have also been given in the thesis.

It has been shown in the study that some geometrical structures like Projective geometries and Euclidean geometries are separating systems and therefore can be used to separate the elements of the set S_n. The duration of the search process for detecting one unknown element using some of these geometrical structures has been obtained.

For detecting two unknown elements from the finite set S_n, two designs have been constructed. These designs are 2-complete search design and the partition search design. The 2-complete search design is based on the property that the intersection of a given number of subsets of S_n which contain the two unknown elements consists of the two unknown elements. On the other hand, a partition search design divides the set S_n into two parts with each part containing one unknown element. The two unknown elements are then identified separately from each part.

Two different methods of constructing 2-complete search design have been discussed in the thesis. The two methods which are both based on properties of balanced incomplete block designs can be described briefly as follows:

(i) The elements of the set S_n are identified with the blocks and the functions $f_1, f_2, \ldots\ldots, f_m$ are identified with the objects of a BIB design with some specific properties. These properties are given in the thesis.

(ii) The elements of the set S_n are identified with the objects and the functions $f_1, f_2, \ldots, f_m$ are identified with the blocks of a BIB design after deleting a number of blocks. A simple formula for computing the number of blocks to be deleted is given in the thesis.

Methods of constructing partition search designs have also been discussed in the thesis. Some of the methods discussed are the halving and the $\frac{1}{x}$ - procedures. It is shown in the thesis that the $\frac{1}{3}$ - procedure, which partitions the set S_n into three disjoint parts, provides the best results.

Probabilities of termination of the search process after N steps, and duration of the search process for detecting two unknown elements have been derived for both the 2- complete search design and the partition search design. Comparing the number of elements n' of the set S_n and the expected duration of the search process, it was observed that partition search design is very economical for large values of n.

Lastly, the study has dealt with the detection of one unknown element and of two unknown elements from a finite set S_n in the presence of noise. The study has attempted to obtain designs which would detect an error without correcting it or detect the error and correct it. To achieve this, systematic strategies of choosing the functions $f_1, f_2, \ldots, f_m$ in the case of separating systems and of choosing the subsets $A_1, A_2, \ldots, A_m$ in the case of 2-complete search and partition search designs has been proposed. In this strategy all the functions $f_1, f_2, \ldots, f_m$ and all the subsets $A_1, A_2, \ldots, A_m$ were systematically chosen in the determination of one unknown element and two unknown elements respectively. We note here that it is not possible to detect without correcting , or detect and correct an error, if only a few functions or a few subsets are chosen at random to determine one unknown element or two unknown elements.

Search models studied in this thesis have a variety of practical applications. A list of these applications is given in Chapter 1 of the thesis. We conclude by listing some problems which require further investigations:

(i) Construction of strategies which determine one unknown element from a finite set S_n in the presence of noise with probability $1 - \varepsilon$.

(ii) Construction of economical partition search designs which determine more than two unknown elements.

(iii) The relationship between combinatorial search models and probabilistic search models.

(iv) Construction of random search models based on finite plane Projective and Euclidean geometries which give sharper bounds to the expected duration of the search process.

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