CONSTRUCTION OF ZERO INFLATED POISSON MIXTURE DISTRIBUTIONS AND APPLICATION TO FERTILITY DATA

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Abstract

In applications involving count data, it is common to encounter the frequency of observed zeros significantly higher than predicted by the model based on the standard parametric family of discrete distributions. In such situations, Zero Inflated Poisson and Zero Inflated Negative Binomial distributions have been widely used in modeling the data, yet other models may be more appropriate in handling the data with excess zeros. Such situations normally result to misspecification of the statistical model leading to erroneous conclusions and bringing uncertainty into research and practice. Mixture models cover several distinct fields of the statistical science. Their broad acceptance as adequate models to describe diverse situations is evident from the plethora of their applications in the statistical literature. The univariate Poisson mixture distributions have been formed together with the structural properties of inflated power series distributions. However, the continuous mixture distributions with a Zero Inflated Poisson as the inflate model have neither been considered nor applied on fertility data. Therefore the problem was to form continuous Zero Inflated Poisson mixture distributions by using direct integration, in recursive formula, through expectation forms and by use of special functions. The mixture model formed had a Zero Inflated Poisson model as an inflate model and a prior distribution. Furthermore, the inflated mixture distributions obtained explicitly and by the method of moments had not been proved to be identical and application of ZIP mixture distribution on fertility data. This work concentrated on the construction of continuous ZIP mixture distributions together with their properties and the proofs of identities resulting from the continuous mixture distributions. According to this study, the method that resulted in a good number of mixture Poisson distributions compared to the other methods was that of obtaining recursive relations using integration by parts. This was a clear indication that there is no restriction on what kind of method to use for a particular given mixing distribution, that is, any method can be used whenever possible. However, some mixture distributions e.g the ZIP Lomax distribution, could not be constructed by direct integration. The ZIG distribution was then fitted to fertility data. Then ZIG model was chosen because it lies in the domain of $[0, \infty]$ and it is also used in modeling of rare and discrete events, which fits the characteristics of fertility data. This clearly showed that the Zero-Inflated mixture distributions are the most appropriate in modeling of count data. The models that were derived in this work could be used by actuaries in assessing the credit worthiness of an investor and in claims compensation. The demographers can also use these models to study different components in population.

CHAPTER ONE

INTRODUCTION

1.1 Introduction

A Poisson distribution is one of the most important counting distribution in insurance modeling. A mixed Poisson distribution is often used to model the number of losses or claims arising from a group of risks where the risk level among the group retains heterogeneity which can not be classified by underwriting criteria. However, it may be reasonable to assume that the risk level follows a probability distribution, and given the risk level the number of losses follows a Poisson distribution. Thus, the number of losses follows a mixed Poisson distribution. A mixed distribution is constructed when two probability distributions are mixed. Consider a probability distribution whose parameter is varying and also has a distribution. A ZIP mixture distribution is formed when a ZIP distribution with a parameter ρ is mixed with a prior distribution. Since this parameter is varying, it also has a distribution which is continuous, this gives rise to a ZIP mixture distribution. Greenwood and Yule, in [5], constructed a Negative Binomial distribution by mixing a Poisson distribution with its parameter. By advancing models on fertility, the study would build on the already existing theory while the study findings may shape the direction of policy formulation in the area of fertility determination Kenya. Kenya is currently grappling with the effects of overpopulation. If proper methods can be put in place and effected then, this will be a problem of the past. The DHS data are by nature over-dispersed hence the need of use of the Zero-Inflated Poisson mixture (ZIP) models to model the excess zeros. The ZIP models consist of two sub-models: the inflated model- which categorizes a woman as having or not having a child and the base model- which will determine the number of children a woman would have. The University of Madrid Carlos III (UCL), used a ZIP model with a negative binomial n assessing the impact of fertility decisions on infant mortality. Although the ZINB model had a good fit to the data, it had a low parsimony. This is so because the model had many parameters. The ZIP model will be the most suitable one because it is more parsimonious and has fewer parameters. In this work, the ZIP (Zero- Inflated Poisson) distribution was mixed with various continuous distributions to construct the ZIP mixture distributions. The odel was then fitted to fertility data. Then odel was chosen because it lies in the domain of $[0, \infty]$ and it is also used in modeling of rare and discrete events, which fits the characteristics of fertility data.

1.2 Statement of the problem

Mixture models have been used in various fields of statistics. In applications involving discrete data, it is common to encounter the frequency of observed zeros significantly higher than predicted by the model based on the standard parametric family of discrete distributions. In such situations, Zero-Inflated Poisson and Zero-Inflated Negative binomial distribution have been widely used in modeling the data, yet other models may be more appropriate in handling the data with excess zeros. The consequences of this is misspecifying the statistical model leading to erroneous conclusions and bringing uncertainty into research and practice. Therefore the problem was to identify by constructing other alternatives, to the models already present in the literature that may be more appropriate for modeling data with excess zeros together with their properties. Furthermore, the zero inflated mixture distributions obtained explicitly and by the method of moments have not been proved to be identical.

1.3 Objectives of the study

The main objective was to construct ZIP mixture distributions in explicit form, in recursive form, by use of special functions and in expectation form.

Specific Objectives

1. to construct the ZIP mixture models by direct integration and obtain their properties,
2. to construct and represent the ZIP mixture models in recursive form,,
3. to construct the ZIP mixture models by integration of expectation of r^{th} moment of the mixing distribution to prove the identities based on the results obtained by methods (1) and (3);
4. to derive the ZIP mixture by using generating function and Laplace Transforms of the mixing distributions and to represent them interms of special functions .
5. to fit the Zero Inflated Geometric distribution to fertility data.

1.4 Significance of the study

Mixed Poisson distributions have been used in a wide range of scientific fields for modeling nonhomogeneous populations. The study will also be useful to Actuaries in enriching their knowledge to develop and use statistical and financial models to make informed financial decisions, pricing, establishing the amount of liabilities, and setting capital requirements for uncertain future events.

This study will benefit and help future researchers in Traffic Accident Research as a guide in modeling accident data, with the focus in their studies and numerous others of similar kind, is on evaluating public policy on how successful was past (traffic) safety legislation in reducing the number of accidents. A good example on how applicable Mixed Poisson distributions are in actuarial data is given by Klugman, and others in their work, [7]. The driving habits of some automobile drivers were studied in a class of automobile insurance by counting the number of accidents per driver in a one-year time period. Poisson and Negative Binomial distributions were then fitted to the data and the two models compared using likelihood ratio test. The model that was selected as the best fitting was that of the Negative Binomial distribution which is a Mixed Poisson distribution with Gamma as the mixing distribution.

The study will be beneficial to those in the field of demography in understanding population history of areas and drivers of regional change. One of the applications is in the analysis of migration data. Researchers studying migration have widely used probability models with the primary purpose of modeling being simplification and to reduce a confusing mass of numbers to a few intelligible basic parameters, to make possible an approximate representation of reality without its complexity.

The model that will be developed can be adopted by the policy makers since it will be a parsimonious model that is, it has a high level of prediction performance.

1.5 Research Methodology

1.5.1 Introduction

The following mathematical tools were used in the construction of the continuous ZIP mixture distributions: recursive models, Laplace transforms, special functions and generating functions. A special function is a solution of elementary integrals, a Confluent Hypergeometric and Gauss Hypergeometric functions are examples of special functions.

1.5.2 Gamma function

Definition: A gamma function with parameter $\alpha > 0$ denoted by $\Gamma\alpha$ is defined as

$$\Gamma\alpha = \int_0^\infty e^{-t} t^{\alpha-1} dt$$

Properties

Property 1

$$\Gamma 1 = 1$$

Proof;

put$\alpha = 1$, then

$$\begin{aligned} \Gamma 1 &= \int_0^\infty e^{-t} dt \\ &= -e^{-t}|_0^\infty \\ &= 1 \end{aligned}$$

Property 2

$$\Gamma(\alpha + 1) = \alpha \Gamma \alpha$$

Proof;

$$\begin{aligned} \Gamma(\alpha + 1) &= \int_0^\infty e^{-t} t^{(\alpha+1)-1} dt \\ &= \int_0^\infty e^{-t} t^\alpha dt \end{aligned}$$

Using integration by parts

$$\text{let} u = t^\alpha \Rightarrow du = \alpha\, t^{\alpha-1}$$

and

$$dv = e^{-t}, \text{then} \Rightarrow v = -e^{-t}$$

$$\int_0^\infty e^{-t} t^\alpha dt =$$

$$\begin{aligned} &= -t^\alpha e^{-t}|_0^\infty + \int_0^\infty e^{-t} \alpha\, t^{\alpha-1} dt \\ &= \alpha \int_0^\infty e^{-t} t^{\alpha-1} dt \\ &= \alpha \Gamma \alpha \end{aligned}$$

Property 3

When $\alpha = n$ a positive integer, then

$$\Gamma(n+1) = n!$$

Proof;

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma n \\
&= n(n-1)\Gamma(n-1) \\
&= n(n-1)(n-2)\Gamma(n-2) \\
&= n(n-1)(n-2)\ldots 2.1(\Gamma 1) \\
&= n(n-1)(n-2)\ldots 2.(\Gamma 1) \\
&= n!
\end{aligned}$$

Property 4

$$\int_0^\infty e^{-\beta t} t^{\alpha-1} dt = \frac{\Gamma\alpha}{\beta^\alpha}$$

Proof;

Let

$$y = \beta t \,\text{then} \Rightarrow\ t = \frac{y}{\beta},\ dt = \frac{dy}{\beta}$$

$$\begin{aligned}
&= \int_0^\infty e^{-y}(\frac{y}{\beta})^{\alpha-1}\frac{dy}{\beta} \\
&= \frac{1}{\beta^\alpha}\int_0^\infty e^{-y}y^{\alpha-1}dy \\
&= \frac{\Gamma\alpha}{\beta^\alpha}
\end{aligned}$$

1.5.3 Laplace Transform

Laplace Transform is an important tool for constructing continuous probability distributions together with their corresponding properties.

Definition;

A function $K(s,\, t)$ of two independent variables s and t and that the integral

$$\int_a^b K(s,\, t) f(t) dt$$

is convergent is referred to as a kernel of transformation which is denoted by

$$L[f(t)] = \int_a^b K(s,\, t) f(t) dt$$

Laplace Transform is derived by letting $a = 0$ and $b = \infty$ together with a kernel of transformation. Laplace Transform when applied to a function, changes that function into a new function through integration. If $f(k)$ is a function defined for all $k > 0$, then its Laplace Transform is

$$L[f(k)] = \int_0^\infty e^{-st} f(k) dk$$

1.5.4 Hypergeometric Functions

Confluent Hypergeometric function

Kummer's confluent hypergeometric function denoted by ${}_1F_1(a;c;x)$ is defined as

$$\begin{aligned} {}_1F_1(a;c;x) &= 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)}\frac{x^n}{x!} \end{aligned}$$

This function has an integral representation given by

$${}_1F_1(a;c;x) = \frac{1}{B(a,\, c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{xt} dt$$

Gauss Hypergeometric function

Gauss Hypergeometric function denoted by ${}_2F_1(-\gamma; a; c; x)$ is given by

$$ {}_2F_1(-\gamma; a; c; x) = 1 + \frac{\gamma\, a}{c}\frac{x}{1!} + \frac{\gamma(\gamma+1)a(a+1)}{c(c+1)}\frac{x^2}{2!} + \dots $$

$$ = \sum_{n=0}^{\infty} \frac{\gamma(\gamma+1)(\gamma+2)\dots(\gamma+n-1)a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)}\frac{x^n}{n!} $$

with an integral representation given by

$$ {}_2F_1(-\gamma; a; c; x) = \frac{1}{B(a,\, c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-\gamma}dt $$

1.5.5 Recursive models

These models can be based on

1. integration by parts

$$ \int u dv = uv - \int v du $$

2. Panjer - Willmot model

$$ p_n \sum_{t=0}^{k} \alpha_t n^{(t)} = p_{n-1} \sum_{t=0}^{k} \beta_t (n-1)^{(t)} $$

for $n = 1, 2, 3, \dots$ implying that $n - 1 = 2, 3, \dots$ where

$$ n^{(t)} = n(n-1)(n-2)\dots(n-t+1) $$

and

$$ (n-1)^{(t)} = (n-1)(n-2)(n-3)\dots(n-t) $$

$$ n^{(0)} = (n-1)^{(0)} = 1 $$

The differential equation resulting from this is given by the following

$$\sum_{t=1}^{k} \alpha_t {s^t} G^{(t)}(s) + s \sum_{t=1}^{k} \beta_t {s^t} G^{(t)}(s) + (\alpha_0 - \beta\, s) G(s) = \alpha_0 p_0$$

1.6 Organization of the study

Let ZIP be a poisson distribution with an inflation parameter ρ . Then A . When a ZIP is mixed with a Gamma function with two parameters, the resulting distribution is a A .

In Chapter 3, the mixed ZIP distributions were constructed explicitly by using a gamma function. Exponential ,Gamma distribution with two parameters, Lindley and two parameter Generalized Lindley distributions were used as mixing distributions. The probability generating and moment generating functions of any distribution are very important in that they can be used in deriving various moments of the distribution.

Chapter 4 looked at the construction of mixture ZIP distributions by recursive method based on integration by parts.

In Chapter 5, the mixture ZIP distributions were formed by using two relationships. In the first instance, the Laplace Transform and the Generating functions of the mixing distribution were used to derive the mixed distribution. In the second scenario, given the r^{th} moment of the mixing distribution, we constructed the pdf of the mixed ZIP distribution through expectation. From this relationship, there were identities relating the distributions formed explicitly and those formed by r^{th} moment expectation, which were also proved.

Finally in Chapter 6, since some distributions could not be integrated directly or by recursive methods, therefore special functions were used: Confluent hypergeometric and the Gauss hypergeometric functions to get their pdf's.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Sometimes the number of zero's in a sample of data may not be exactly modeled by a Poisson distribution that is the data contains excess zero's which cannot be ignored. This gives rise to an "excess zero's" problem. In case they are ignored, then bias may be introduced into the analysis. The problem may be eliminated by using either a Zero inflated model or a hurdle model.

2.2 Inflated Models

A zero-inflated model is a statistical model based on a zero-inflated probability distribution and it has two kinds of zero's- true zero's and excess zero's. Foe example, in a fertility data, the true zeros are represented by those women who would have wished to have children but were unable due to biological reasons while the excess zero's represent the women who did not want to have children by choice. Gardner et.al in their paper, [3], suggested that using an inflation technique was adequate if the intention is to estimate the effect of the covariates. In April, 2009, the UCL team in their paper, [24], assessed the impacts of the fertility decisions of mothers on infant mortality. They used a Poisson regression to model the number of children. They fitted an inflated zero's model with negative binomial to the fertility decisions so as to eliminate the overdispersion of the Poisson model. They found out that there was a positive and statistically significant effect of infant mortality on the number of children a woman may decide to give birth to. University graduates had fewer children as compared to the junior college graduates since they get married at an older children. The study of proximate determinants of fertility enables the policymakers to come up with appropriate policies aimed at reducing the high levels of fertility hence hastening the achievement of Vision 2030 objectives, according to the work in [12]. Duddley and Poston in their study, [2], on estimation of fertility of United States women using count regression models, showed that the ZIP models performed better than the Poisson regression model with no inflated zeros thus making the ZIP models to be statistically appropriate for modeling of fertility. However, in their study, they used logistic regression to predict membership in the two latent groups. Inuwor, in his paper, [6], considered a model that takes into account zero observation. He assumed the Poisson distribution for the number of clusters migrating, and that the number of migrants in a cluster follows each of the members of the class of one-Inflated power series distributions namely: the binomial, the Poisson, the negative binomial, the geometric, the log-series, and the mis-recorded Poisson. Sarguta in her thesis,[19], constructed the Poisson mixture distributions by direct integration, recursive methods, use of special functions and in expectation forms. However, she found out that not all the mixed distributions could be constructed by using all the methods. For instance, the Poisson - Scaled Beta distribution could not be constructed by explicit method but by use of special functions. Tuwei in his dissertation, [23], constructed and derived the moments and maximum likelihood estimators of Zero- inflated power series distributions.

2.2.1 Zero Inflated Poisson Distribution

Poisson distribution is normally used when modeling count data. However, it is quite often that we encounter count data with a preponderance of zero's. In such a situation, the variance is likely to be greater (or less) than the mean thus disqualifying the use of the Poisson distribution. The overdispersion (or underdispersion) could be due to heterogeneity of the population or excess of zero's. For instance, in the fertility study of women, there are women who choose not to have children and some women may not have children due to biological reasons even though they would have wished to have children. In modeling such data, we introduce an inflation parameter, ρ, for excess zero's.

Consider the following data on women fertility

Number of children	0	1	2	3
Number of women	15	5	7	6
Probability	$\frac{15}{33}$	$\frac{5}{33}$	$\frac{7}{33}$	$\frac{6}{33}$

Assume that three more childless women are incorporated into the study, then the inflation parameter will be $\rho = \frac{3}{36}$. Therefore

$$P(Y=0) = \rho + (1-\rho)p_0 = \frac{3}{36} + \frac{33}{36}\left(\frac{15}{33}\right) = \frac{18}{36}$$

$$P(Y=1) = (1-\rho)p_1 = \frac{33}{36}\left(\frac{5}{33}\right) = \frac{5}{36}$$

$$P(Y=2) = (1-\rho)p_2 = \frac{33}{36}\left(\frac{7}{33}\right) = \frac{7}{36}$$

$$P(Y=3) = (1-\rho)p_3 = \frac{33}{36}\left(\frac{6}{33}\right) = \frac{6}{36}$$

2.2.2 Mixture Distributions

A mixed distribution is constructed when two probability distributions are mixed. Consider a probability distribution whose parameter is varying and also has a distribution. Then the integral or summation of these two distributions forms a mixed probability distribution. A ZIP continuous mixture distribution is formed by mixing a ZIP distribution with a continuous distribution. Let $f(y;\lambda)$ be a probability distribution function (pdf) or a probability mass function (pmf) of a random variable Y with parameter λ. If this parameter λ is varying, then it also becomes a random variable. Thus we have a conditional pdf or pmf $f(y|\lambda)$ and the unconditional or marginal distribution becomes

$$f(y) = \int_{-\infty}^{\infty} f(y|\lambda)g(\lambda)d\lambda$$

$$f(y) = \sum_{\lambda} f(y|\lambda)g(\lambda)$$

where $g(\lambda)$ is a pdf or pmf of λ and is called a mixing or prior distribution. For a Zero inflated Poisson mixture distribution, the distribution of y will be

$$\Pr(y = k) = \begin{cases} \int_0^\infty \left[\rho + (1-\rho)e^{-\lambda}g(\lambda)\right] d\lambda, & \text{k=0;} \\ \int_0^\infty \left[(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}g(\lambda)\right] d\lambda, & \text{k=1, 2, \ldots.} \end{cases}$$

Karlis and Xekalaki in their paper, [16] in their proposition 14 have given an alternative useful method which links the probability function of a mixed Poisson distribution to the moments of the mixing distribution. The pdf of a mixed distribution is related to the r^{th} moment of the mixing distribution. Willmot, in his paper, [25], used the relationship between generating function, $G_k(s)$ and the Laplace Transform $L_\lambda(s)$ of the mixing distribution to obtain $f(k)$. In his work,[26], he devised a method later, famously known as the Willmot Approach. Which is was used to obtain the recursive formulae for the Negative Binomial distribution, Generalized Pareto, Truncated distributions and many others, with The Poisson as the mixing distribution. Gupta et al in their paper, [4] obtained recursive forms of Poisson mixtures for the Gamma and Exponential distribution by using integration. Sankaran in his work, [17] obtained a recursive formula for Poisson - Inverse Gaussian using differential equation in pgf.

2.2.3 Summary

From all these, it was clear that no research had been done on the construction of the two case ZIP mixture distributions or even on the derivation of their recursive relations. In addition, the ZIP mixture distributions have not been used in modeling fertility data. This therefore formed the basis of the research in this work.

CHAPTER THREE

ZERO-INFLATED POISSON MIXTURES IN EXPLICIT FORMS

3.1 Introduction

In explicit construction of ZIP mixtures, a ZIP model is integrated simultaneously with the the distribution of the parameter λ. A zero-inflated model is a statistical model based on a zero-inflated probability distribution. It arises when probability mass at point zero exceeds the one allowed under the standard parametric family of discrete distributions. A probability distribution is said to be a mixture distribution if its distribution function $F(Y)$ can be written in the form

$$F(Y) = \int_{\Theta} F(Y|\lambda)dG(\lambda)$$

where $F(Y|\lambda)$ denotes the distribution function of the component densities considered to be indexed by a parameter λ with distribution function $G(\lambda)$, $\lambda\epsilon\Theta$.

The above definition can also be expressed in terms of probability density functions, thus

$$f(y) = \int_{\Theta} f(y|\lambda)g(\lambda)d\lambda$$

where $g(\lambda)$ is the mixing distribution.

A random variable Y follows a zero-inflated Mixed Poisson distribution with mixing distribution having probability density function g if its probability function is given by;

$$Prob(Y=k) = \begin{cases} \int_0^\infty \left[\rho + (1-\rho)e^{-\lambda}\right] g(\lambda)d\lambda, & \text{k=0;} \\ \int_0^\infty \left[(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}\right] g(\lambda)d\lambda, & \text{k=}1,2,\ldots. \end{cases}$$

3.1.1 The mean and variance of Zero-Inflated Mixed Poisson Distributions

Zero-inflation is a special case of over dispersion that contradicts the relationship between the mean and variance in a one-parameter exponential family. One way to address this is to use a two-parameter distribution so that the extra parameter permits a larger variance.

Theorem 3.1.1

The mean of a mixed distribution is equal to the mean of the underlying mixing distribution $g(\lambda)$, that is,

$$E(Y) = (1-\rho)E(\Lambda)$$

[11, page,16]

PROOF.

$$E(\Lambda) = \sum_{k=0}^{\infty} k\,p_k \qquad \text{(THREE.1)}$$

$$= \sum_{k=1}^{\infty} k\,p_k \qquad (\text{since for } k=0, \sum k, p_k = 0) \qquad \text{(THREE.2)}$$

$$= \sum_{k=1}^{\infty} k\,p_k \qquad \text{(THREE.3)}$$

$$= \sum_{k=1}^{\infty} k \int_0^\infty p_k(\lambda)\,dg(\lambda) \qquad \text{(THREE.4)}$$

$$= \int_0^\infty \sum_{k=1}^{\infty} k\,p_k(\lambda)dg(\lambda) \qquad \text{(THREE.5)}$$

By definition, $\sum_{k=1}^{\infty} k\, p_k = \lambda$, therefore,

$$\begin{aligned} E(Y) &= (1-\rho)\int_0^\infty \left[\sum_{k=1}^{\infty} k\, p_k\right] dg(\lambda) && \text{(THREE.6)} \\ &= (1-\rho)\int_0^\infty \lambda\, g(\lambda) d\lambda && \text{(THREE.7)} \\ &= (1-\rho)E(\Lambda). && \text{(THREE.8)} \end{aligned}$$

□

Theorem 3.1.2

[11, page,19] The variance of a mixed Poisson distribution is equal to the sum of the mean and variance of the underlying distribution, that is,

$$Var(Y) = (1-\rho)[E(\Lambda^2) + E(\Lambda)] - [(1-\rho)E(\Lambda)]^2$$

PROOF. Finding the second moment of the mixing distribution gives us

$$\begin{aligned} E(\Lambda^2) &= \sum_{k=0}^{\infty} k^2\, p_k && \text{(THREE.9)} \\ &= \sum_{k=1}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} && \text{(THREE.10)} \\ &= \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{(k-1)!} && \text{(THREE.11)} \\ &= \sum_{k=1}^{\infty} \left[\lambda^2 (k-1) e^{-\lambda} \frac{\lambda^{k-2}}{(k-1)!} + \lambda\, e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}\right] && \text{(THREE.12)} \\ &= \left[\lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda\, e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}\right] && \text{(THREE.13)} \\ &= [\lambda^2 e^{-\lambda} e^{\lambda} + \lambda\, e^{-\lambda} e^{\lambda}] && \text{(THREE.14)} \\ &= [\lambda^2 + \lambda]. && \text{(THREE.15)} \end{aligned}$$

The second moment of the mixed distribution is given by

$$\begin{aligned} E(Y^2) &= (1-\rho)\sum_{k=0}^{\infty} k^2 p_k & \text{(THREE.16)} \\ &= (1-\rho)\int_0^{\infty}\left[\sum_{k=0}^{\infty} k^2 p_k\right] dG(\lambda) & \text{(THREE.17)} \\ &= (1-\rho)\int_0^{\infty}[\lambda^2+\lambda]g(\lambda)d\lambda & \text{(THREE.18)} \\ &= (1-\rho)[E(\Lambda^2)+E(\Lambda)]. & \text{(THREE.19)} \end{aligned}$$

Therefore, the variance of the mixed distribution is

$$\begin{aligned} Var(Y) &= E(Y^2)-[E(Y)]^2 & \text{(THREE.20)} \\ &= (1-\rho)[E(\Lambda^2)+E(\Lambda)]-[(1-\rho)E(\Lambda)]^2. & \text{(THREE.21)} \end{aligned}$$

□

3.2 Mixing with Exponential Distribution

3.2.1 Construction

The pdf of Exponential distribution is

$$g(\lambda) = \mu\, e^{-\mu\lambda};\ \lambda > 0,\ \mu > 0$$

The Mixed Poisson distribution is obtained as follows:

$$\begin{aligned} Prob(Y=k) &= \begin{cases} \int_0^{\infty}\left[\rho+(1-\rho)e^{-\lambda}\right]\mu\, e^{-\mu\lambda}d\lambda, & \text{k=0;} \\ \int_0^{\infty}\left[(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}\right]\mu\, e^{-\mu\lambda}d\lambda, & \text{k=1, 2, \dots.} \end{cases} & \text{(THREE.22)} \\ &= \begin{cases} (1-\rho)\mu\int_0^{\infty}\lambda^{1-1}e^{-(1+\mu)\lambda}d\lambda, \text{ k=0;} \\ (1-\rho)\frac{\mu}{k!}\int_0^{\infty}e^{-(1+\mu)\lambda}\lambda^{k+1-1}d\lambda, \text{ k=1, 2, \dots.} \end{cases} & \text{(THREE.23)} \\ &= \begin{cases} (1-\rho)\frac{\mu}{(1+\mu)}, \text{ k=0;} \\ (1-\rho)\left(\frac{\mu}{1+\mu}\right)\left(\frac{1}{1+\mu}\right)^k, \text{ k=1, 2, \dots.} \end{cases} & \text{(THREE.24)} \end{aligned}$$

The resulting distribution is a ZIG with parameters $(\rho, (\frac{\mu}{1+\mu}))$.

3.2.2 Properties of the ZIG

From theorem 2.1.1, the mean of the mixed distribution will be given by

$$E(Y) = (1-\rho)E(\Lambda).$$

where

$$\begin{aligned} E(\Lambda) &= \int_0^\infty \lambda\mu\, e^{-\mu\lambda} d\lambda \\ &= \mu \int_0^\infty \lambda\, e^{-\mu\lambda} d\lambda. \end{aligned}$$

Using integration by parts,

Let

$$\begin{aligned} u &= \lambda \Rightarrow du = d\lambda \\ dv &= e^{-\mu\lambda} d\lambda \Rightarrow v = \int e^{-\mu\lambda} d\lambda = \frac{-1}{\mu} e^{-\mu\lambda}. \end{aligned}$$

Therefore

$$\begin{aligned} E(\Lambda) &= \mu\left[\frac{\lambda}{\mu} e^{-\mu\lambda}|_0^\infty + \frac{1}{\mu}\int_0^\infty e^{-\mu\lambda} d\lambda\right] && \text{(THREE.25)} \\ &= \mu\left[\frac{-\lambda}{\mu} e^{-\mu\lambda} - \frac{1}{\mu^2} e^{-\mu\lambda}\right]|_0^\infty && \text{(THREE.26)} \\ &= \frac{1}{\mu}. && \text{(THREE.27)} \end{aligned}$$

Hence

$$E(Y) = (1-\rho)\frac{1}{\mu}. \qquad \text{(THREE.28)}$$

From theorem 2.1.2, the variance of the Zero-Inflated Geometric distribution will be given by

$$Var(Y) = (1-\rho)[E(\Lambda^2) + E(\Lambda)] - [(1-\rho)E(\Lambda)]^2. \quad \text{(THREE.29)}$$

The second moment of the mixing distribution is given by

$$\begin{aligned} E(\Lambda^2) &= \int_0^{\infty} \lambda^2 \mu\, e^{-\mu\lambda} d\lambda \\ &= \mu \int_0^{\infty} \lambda^2\, e^{-\mu\lambda} d\lambda. \end{aligned}$$

Using integration by parts, Let

$$\begin{aligned} u &= \lambda^2 \Rightarrow du = 2\lambda\, d\lambda \\ dv &= e^{-\mu\lambda} d\lambda \Rightarrow v = \int e^{-\mu\lambda} d\lambda = \frac{-1}{\mu} e^{-\mu\lambda}. \end{aligned}$$

Therefore

$$\begin{aligned} E(\Lambda^2) &= \mu \left[\frac{-\lambda^2}{\mu} e^{-\mu\lambda}|_0^{\infty} - \frac{2}{\mu} \int_0^{\infty} \lambda\, e^{-\mu\lambda} d\lambda \right] \\ &= \frac{2}{\mu^2}. \end{aligned}$$

The variance of the mixed distribution is

$$\begin{aligned} Var(Y) &= (1-\rho)\left[\frac{2}{\mu^2} + \frac{1}{\mu}\right] - \left[(1-\rho)\frac{1}{\mu}\right]^2 & \text{(THREE.30)} \\ &= (1-\rho)\left(\frac{1+\rho+\mu}{\mu^2}\right). & \text{(THREE.31)} \end{aligned}$$

3.2.3 The Moment Generating Function of ZIG

The mgf of a discrete distribution is

$$M_Y(t) = E(e^{tY}).$$

For the ZIG

$$M_Y(t) = Pr(Y=0) + \sum_{k=1}^{\infty} e^{tk} Pr(Y=k) \tag{THREE.32}$$

$$= (1-\rho)\frac{\mu}{(1+\mu)} + (1-\rho)(\frac{\mu}{1+\mu}) \sum_{k=1}^{\infty} \left(\frac{1}{1+\mu}\right)^k (e^t)^k \tag{THREE.33}$$

$$= (1-\rho)(\frac{\mu}{1+\mu}) \left[\sum_{k=0}^{\infty} (\frac{e^t}{1+\mu})^k\right] \tag{THREE.34}$$

$$= (1-\rho)(\frac{\mu}{1+\mu})[1 + \frac{1}{\frac{e^t}{1+\mu}}] \tag{THREE.35}$$

$$= (1-\rho)\frac{\mu}{1+\mu-e^t}. \tag{THREE.36}$$

1. The r^{th} moment is obtained from the r^{th} derivative of $M_Y(t)$ w.r.t t and setting $t=0$

$$\mu_r^{'} = \frac{d^r M_Y(t)}{dt^r}|_{t=0}.$$

2. When $r=1$,

$$\mu_1^{'} = M_Y^{\prime}(t)|_{t=0}$$

$$M_Y^{\prime}(t) = \mu\, e^t (1-\rho)[1+\mu-e^t].$$

   Hence

$$\mu_1^{'} = \mu(1-\rho)[1+\mu-1]^{-2}$$
$$= \frac{(1-\rho)}{\mu}.$$

3. When $r=2$, $\mu_2^{''} = M_Y^{''}(t)|_{t=0}$

$$\mu_2^{''} = \mu(1-\rho)\left[\frac{e^t(1+\mu-e^t)^2 + 2e^{2t}(1+\mu-e^t)}{(1+\mu-e^t)^4}\right] \tag{THREE.37}$$

$$= \frac{(1-\rho)}{\mu}\left[1+\frac{2}{\mu}\right]. \tag{THREE.38}$$

4. The variance is then given by

$$Var(Y) = \mu''_2 - \mu'^2_1.$$

$$Var(Y) = \frac{(1-\rho)}{\mu}\left[1+\frac{2}{\mu}\right] - \left[\frac{(1-\rho)}{\mu}\right]^2 \quad \text{(THREE.39)}$$

$$= \frac{(1-\rho)}{\mu^2}[1+\mu+\rho]. \quad \text{(THREE.40)}$$

3.2.4 The Probability Generating Function of ZIG

The pgf of discrete random variable Y, is given by

$$\begin{aligned} G_Y(s) &= \sum_{k=0}^{\infty} Pr(Y=k)s^k \\ &= Pr(Y=0) + \sum_{k=1}^{\infty} Pr(Y=k)s^k \\ &= (1-\rho)\left(\frac{\mu}{1+\mu}\right) + (1-\rho)\left(\frac{\mu}{1+\mu}\right)\sum_{k=1}^{\infty}\left(\frac{s}{1+\mu}\right)^k \\ &= (1-\rho)\left[\frac{\mu}{1+\mu-s}\right]. \end{aligned}$$

The first and second differentials are given by

$$G_Y'(s) = 0 + (-1)(1-\rho)\mu(1+\mu-s)^{-2}(-1), \quad \text{(THREE.41)}$$

setting $s = 1$, we get

$$G_Y'(s=1) = (1-\rho)\frac{1}{\mu} \quad \text{(THREE.42)}$$

and

$$G_Y''(s) = (-2)(1-\rho)\mu(1+\mu-s)^{-3}(-1) \quad \text{(THREE.43)}$$

on setting $s = 1$

$$G_Y''(s=1) = (1-\rho)\frac{2}{\mu^2}. \quad \text{(THREE.44)}$$

Hence the mean and the variance is given by

$$E(Y) = (1-\rho)\frac{1}{\mu}, \tag{THREE.45}$$

and

$$\begin{aligned} Var(Y) &= G''_Y(1) + G'_Y(1) - [G'_Y(1)]^2 && \text{(THREE.46)} \\ &= (1-\rho)\frac{2}{\mu^2} + \frac{(1-\rho)}{\mu} - \left[\frac{(1-\rho)}{\mu}\right]^2 && \text{(THREE.47)} \\ &= (1-\rho)\left[\frac{1+\mu+\rho}{\mu^2}\right]. && \text{(THREE.48)} \end{aligned}$$

3.3 Mixing with a two parameter Gamma distribution

The pdf of Gamma distribution with two parameters is,

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda}\lambda^{\alpha-1};\ \lambda > 0,\ \beta > 0,\ \alpha > 0$$

3.3.1 Construction

The pdf of the mixed distribution is

$$Pr(Y=k) = \begin{cases} \int_0^\infty [\rho + (1-\rho)e^{-\lambda}\frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\lambda}\lambda^{\alpha-1}]d\lambda, & \text{k=0;} \\ \int_0^\infty [(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}\frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\lambda}\lambda^{\alpha-1}]d\lambda, & \text{k=1, 2, \dots.} \end{cases} \tag{THREE.49}$$

$$= \begin{cases} (1-\rho)\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(1+\beta)\lambda}\lambda^{\alpha-1}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(1+\beta)\lambda}\lambda^{\alpha-1+k}d\lambda, & \text{k=1, 2, \dots.} \end{cases} \tag{THREE.50}$$

$$= \begin{cases} (1-\rho)\frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{(1+\beta)^\alpha}, & \text{k=0;} \\ (1-\rho)\frac{\beta^\alpha}{(1+\beta)^{\alpha+k}}\frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)}, & \text{k=1, 2, \dots.} \end{cases} \tag{THREE.51}$$

$$= \begin{cases} (1-\rho)(\frac{\beta}{1+\beta})^\alpha, & \text{k=0;} \\ (1-\rho)\begin{pmatrix} \alpha+k-1 \\ k \end{pmatrix}\left(\frac{\beta}{1+\beta}\right)^\alpha\left(\frac{1}{1+\beta}\right)^k, & \text{k=1, 2, \dots.} \end{cases} \tag{THREE.52}$$

The mixed distribution is a ZINB distribution with parameters α, ρ and $\frac{\beta}{1+\beta}$.

3.3.2 Properties

The mean of the underlying mixing distribution is

$$E(\Lambda) = \int_0^\infty \lambda \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1} d\lambda.$$

Using integration by parts,

$$\begin{aligned} E(\Lambda) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left[-\frac{\lambda^\alpha}{\beta} e^{-\beta\lambda}|_0^\infty + \frac{\alpha}{\beta} \int_0^\infty e^{-\beta\lambda} \lambda^{\alpha-1} d\lambda \right] \\ &= \frac{\alpha}{\beta}. \end{aligned}$$

The mean of Y is

$$E(Y) = (1-\rho)\frac{\alpha}{\beta} \qquad \text{(THREE.53)}$$

The second moment of the mixing distribution is given by

$$E(\Lambda^2) = \int_0^\infty \lambda^2 \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1} d\lambda.$$

From integration by parts,

$$\begin{aligned} E(\Lambda^2) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left[\frac{-1}{\beta} e^{-\beta\lambda} \lambda^{\alpha+1}|_0^\infty + \frac{(\alpha+1)}{\beta} \int_0^\infty e^{-\beta\alpha} \lambda^\alpha d\lambda \right] \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left[0 + \frac{(\alpha+1)}{\beta} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} \right] \\ &= \frac{\alpha(\alpha+1)}{\beta^2}. \end{aligned}$$

Thus the variance of the mixed distribution is

$$\begin{aligned} Var(Y) &= (1-\rho)\left[\frac{\alpha(\alpha+1)}{\beta^2} + \frac{\alpha}{\beta}\right] - \left((1-\rho)\frac{\alpha}{\beta}\right)^2 && \text{(THREE.54)} \\ &= (1-\rho)\frac{\alpha(1+\beta+\alpha\rho)}{\beta^2}. && \text{(THREE.55)} \end{aligned}$$

3.3.3 The Moment Generating Function of ZINB

For the ZINB,

$$\begin{aligned} M_Y(t) &= Pr(Y=0) + \sum_{k=1}^{\infty} e^{tk} Pr(Y=k) \\ &= (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left[1+\sum_{k=1}^{\infty}(-1)^k \left(\begin{array}{c} -\alpha \\ k \end{array}\right)\left(\frac{e^t}{1+\beta}\right)^k\right] \\ &= (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left[1-(\frac{e^t}{1+\beta})\right]^{-\alpha}. \end{aligned}$$

1. When $r=1$

$$\begin{aligned} \mu_1' &= (1-\rho)\alpha\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left[1-(\frac{1}{1+\beta})e^t\right]^{(-\alpha-1)}\left(\frac{e^t}{1+\beta}\right) \\ &= (1-\rho)\frac{\alpha}{\beta}. \end{aligned}$$

2. When $r=2$

$$\begin{aligned} \mu_2'' &= (1-\rho)\alpha\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left[\frac{(\frac{e^t}{1+\beta})(1-\frac{e^t}{1+\beta})^{\alpha+1}+(\frac{1}{1+\beta})^2 e^{2t}(\alpha+1)(0-\frac{e^t}{1+\beta})^{\alpha}}{(1-\frac{1}{1+\beta}e^t)^{2\alpha+2}}\right] \\ &= (1-\rho)\alpha\left[\frac{(\frac{1}{1+\beta})(\frac{\beta}{1+\beta})+(\alpha+1)(\frac{1}{1+\beta})^2}{(\frac{\beta}{1+\beta})^2}\right] \\ &= (1-\rho)\frac{\alpha(1+\beta+\alpha)}{\beta^2}. \end{aligned}$$

3. The variance will be

$$Var(Y) = (1-\rho)\frac{\alpha(1+\beta+\alpha)}{\beta^2} - \left[(1-\rho)\frac{\alpha}{\beta}\right]^2 \quad \text{(THREE.56)}$$

$$= (1-\rho)\frac{\alpha(1+\beta+\alpha\rho)}{\beta^2}. \quad \text{(THREE.57)}$$

3.3.4 The Probability Generating Function of ZINB

The pgf of the resultant distribution is then derived as follows;

$$\begin{aligned} G_Y(S) &= (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha} + (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}\sum_{k=1}^{\infty}(-1)^k\binom{-\alpha}{k}\left(\frac{s}{1+\beta}\right)^k \\ &= (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left[1+\left(1-\frac{s}{1+\beta}\right)^{-\alpha}-1\right] \\ &= (1-\rho)\frac{\beta^{\alpha}}{[1+\beta-s]^{\alpha}}. \end{aligned}$$

Then

$$G'_Y(s) = (1-\rho)\alpha\beta^{\alpha}[1+\beta-s]^{-\alpha-1},$$

and

$$G''_Y(s) = (1-\rho)\alpha(\alpha+1)\beta^{\alpha}[1+\beta-s]^{-\alpha-2}$$

. Hence, the mean and variance then become

$$E(Y) = G'_Y(1) = (1-\rho)\frac{\alpha}{\beta} \qquad \text{(THREE.58)}$$

and

$$\begin{aligned} Var(Y) &= G''_Y(1) + G'_Y(1) - [G'_Y(1)]^2 \\ &= (1-\rho)\frac{\alpha(\alpha+1)}{\beta^2} + (1-\rho)\frac{\alpha}{\beta} - \left[(1-\rho)\frac{\alpha}{\beta}\right]^2 \\ &= (1-\rho)\frac{\alpha}{\beta^2}[1+\beta+\alpha\rho]. \end{aligned}$$

3.4 Mixing with Lindley Distribution

The pdf for Lindley distribution as given by Sankaran in [18] is,

$$g(\lambda) = \frac{\theta^2}{1+\theta}(\lambda+1)e^{-\theta\lambda};\ \lambda > 0,\ \theta > 0$$

3.4.1 Construction

The mixed distribution is

$$Pr(Y=k) = \begin{cases} \int_0^\infty \rho + (1-\rho)e^{-\lambda}\frac{\theta^2}{\theta+1}(\lambda+1)e^{-\lambda\theta}d\lambda, & \text{k=0;} \\ \int_0^\infty (1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}\frac{\theta^2}{\theta+1}(\lambda+1)e^{-\lambda\theta}d\lambda, & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(THREE.59)}$$

$$= \begin{cases} (1-\rho)\frac{\theta^2}{\theta+1}\int_0^\infty (\lambda+1)e^{-(1+\theta)\lambda}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\theta^2}{\theta+1}\int_0^\infty \lambda^k(\lambda+1)e^{-(1+\theta)\lambda}d\lambda, & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(THREE.60)}$$

$$= \begin{cases} (1-\rho)\frac{\theta^2}{\theta+1}\left[\frac{1}{(1+\theta)^2}+\frac{1}{1+\theta}\right], & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\theta^2}{\theta+1}\left[\frac{\Gamma(k+2)}{(1+\theta)^{k+2}}+\frac{\Gamma(k+1)}{(1+\theta)^{k+1}}\right], & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(THREE.61)}$$

Hence a ZIP Lindley with parameters ρ, $\lambda > 0$, $\theta > 0$, will be given by

$$P(Y=k) = \begin{cases} (1-\rho)\frac{\theta^2}{\theta+1}\left[\frac{1}{(1+\theta)^2}+\frac{1}{1+\theta}\right], & \text{k=0;} \\ (1-\rho)\theta^2\left[\frac{2+k+\theta}{(1+\theta)^{k+3}}\right], & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(THREE.62)}$$

Properties

The mean of the mixing distribution is given by

$$E(\Lambda) = \int_0^\infty \lambda\frac{\theta^2}{1+\theta}(\lambda+1)e^{-\theta\lambda}d\lambda \quad \text{(THREE.63)}$$

$$= \frac{\theta^2}{1+\theta}\left[\int_0^\infty \lambda^2 e^{-\theta\lambda}d\lambda + \int_0^\infty \lambda\, e^{-\theta\lambda}d\lambda\right]. \quad \text{(THREE.64)}$$

By using integration by parts equation 3.63 becomes;

$$\begin{aligned} E(\Lambda) &= \frac{\theta^2}{1+\theta}\left[-\frac{\lambda^2}{\theta}e^{-\lambda\theta}|_0^\infty + \frac{2}{\theta}\int_0^\infty \lambda\, e^{-\lambda\theta}d\lambda + \int_0^\infty \lambda\, e^{-\lambda\theta}d\lambda\right] \\ &= \frac{\theta^2}{1+\theta}\left[\frac{2}{\theta}\left(0+\frac{1}{\theta}\int_0^\infty e^{-\lambda\theta}d\lambda\right)+0+\frac{1}{\theta}\int_0^\infty e^{-\lambda\theta}d\lambda\right] \\ &= \frac{\theta^2}{1+\theta}\left[\frac{2}{\theta}\left(0-\frac{1}{\theta^2}(0-1)\right)+\left(0+\frac{1}{\theta^2}\right)\right] \end{aligned}$$

which when simplified gives

$$= \frac{\theta^2}{1+\theta}\left[\frac{2}{\theta^3}+\frac{1}{\theta^2}\right]. \quad \text{(THREE.65)}$$

Thus, the mean of the mixed distribution is

$$E(Y) = (1-\rho)\left(\frac{2+\theta}{\theta(1+\theta)}\right).$$

The second moment of the underlying mixing distribution is

$$\begin{aligned} E(\Lambda^2) &= \int_0^\infty \lambda^2 \frac{\theta^2}{1+\theta}(\lambda+1)e^{-\theta\lambda}d\lambda & \text{(THREE.66)} \\ &= \frac{\theta^2}{1+\theta}\left[\int_0^\infty \lambda^3 e^{-\theta\lambda}d\lambda + \int_0^\infty \lambda^2 e^{-\theta\lambda}d\lambda\right]. & \text{(THREE.67)} \end{aligned}$$

By using integration by parts, equation 3.69 reduces to

$$\begin{aligned} E(\Lambda^2) &= \frac{\theta^2}{1+\theta}\left[0+\frac{3}{\theta}\left(0+\frac{2}{\theta}\left[0+\frac{1}{\theta}\int_0^\infty e^{-\lambda\theta}d\lambda\right]\right)+\left(\frac{2}{\theta}\left[0-\frac{1}{\theta^2}e^{-\lambda\theta}|_0^\infty\right]\right)\right] \\ &= \frac{\theta^2}{1+\theta}\left[\frac{6}{\theta^2}\left(0-\frac{1}{\theta^2}e^{-\lambda\theta}|_0^\infty\right)+\left[\frac{2}{\theta}\left(0-\frac{1}{\theta^2}(0-1)\right)\right]\right] \\ &= \frac{\theta^2}{1+\theta}\left[\frac{6}{\theta^2}\left(\frac{1}{\theta^2}\right)+\frac{2}{\theta^3}\right] \\ &= \frac{\theta^2}{1+\theta}\left[\frac{6}{\theta^4}+\frac{2}{\theta^3}\right]. \end{aligned}$$

The variance of the mixture distribution will then be given by

$$\begin{aligned} Var(Y) &= (1-\rho)\left[\frac{\theta^2}{1+\theta}\left(\frac{6}{\theta^4}+\frac{2}{\theta^3}\right)+\frac{\theta^2}{1+\theta}\left(\frac{2}{\theta^3}+\frac{1}{\theta^2}\right)\right]-\left[(1-\rho)\frac{\theta^2}{1+\theta}\left(\frac{2}{\theta^3}+\frac{1}{\theta^2}\right)\right]^2. \\ &= (1-\rho)\frac{\theta^2}{1+\theta}\left[\frac{6}{\theta^4}+\frac{4}{\theta^3}+\frac{1}{\theta^2}\right]-\left[(1-\rho)\frac{\theta^2}{1+\theta}\left(\frac{2}{\theta^3}+\frac{1}{\theta^2}\right)\right]^2. \end{aligned}$$

3.5 Mixing with Generalized Lindley distribution

A Generalized Lindley distribution as given by Mahmoudi et al in [13] is,

$$g(\lambda) = \frac{\theta^2(\theta\lambda)^{\alpha-1}(\alpha+\lambda)e^{-\theta\lambda}}{(\theta+1)\Gamma(\alpha+1)}; \lambda > 0,\ \theta > 0,\ \alpha > 0$$

3.5.1 Construction

The mixed distribution is

$$
\begin{aligned}
Pr(Y=k) &= \begin{cases} \rho+(1-\rho)\int_0^\infty e^{-\lambda}\frac{\theta^2(\theta\lambda)^{\alpha-1}(\alpha+\lambda)e^{-\theta\lambda}}{(\theta+1)\Gamma(\alpha+1)}d\lambda, & \text{k=0;} \\ (1-\rho)\int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!}\frac{\theta^2(\theta\lambda)^{\alpha-1}(\alpha+\lambda)e^{-\theta\lambda}}{(\theta+1)\Gamma(\alpha+1)}d\lambda, & \text{k=1, 2, \ldots.} \end{cases} \\
&= \begin{cases} \rho+(1-\rho)\frac{\theta^{\alpha+1}}{(\theta+1)(\Gamma(\alpha+1))}\left[\int_0^\infty \alpha\, e^{-(1+\theta)\alpha}\lambda^{\alpha-1}d\lambda+\int_0^\infty e^{-(1+\theta)\lambda}\lambda^{\alpha+1-1}d\lambda\right], & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\theta^{\alpha+1}}{(\theta+1)(\Gamma(\alpha+1))}\left[\int_0^\infty \alpha\, e^{-(1+\theta)\lambda}\lambda^{(\alpha+k-1)}d\lambda+\int_0^\infty \lambda^{\alpha+k+1-1}e^{-(1+\theta)\lambda}d\lambda\right], & \text{k=1,} \end{cases} \\
&= \begin{cases} \rho+(1-\rho)\frac{\theta^{\alpha+1}}{(\theta+1)(\Gamma(\alpha+1))}\left[\frac{\alpha\Gamma(\alpha)}{(1+\theta)^\alpha}+\frac{\Gamma(\alpha+1)}{(1+\theta)^{\alpha+1}}\right], & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\theta^{\alpha+1}}{(\theta+1)(\Gamma(\alpha+1))}\left[\frac{\alpha\Gamma(\alpha+k)}{(1+\theta)^{\alpha+k}}+\frac{\Gamma(\alpha+k+1)}{(1+\theta)^{\alpha+k+1}}\right], & \text{k=1, 2, \ldots.} \end{cases}
\end{aligned}
$$

Thus the resultant distribution , a Zero-Inflated Generalized Poisson Lindley distribution with two parameters $\lambda > 0$, $\theta > 0$, $\alpha > 0$, is given by

$$
P(Y=k) = \begin{cases} \rho+(1-\rho)\frac{\theta^{\alpha+1}}{(\theta+1)^\alpha}\left[\frac{1}{(1+\theta)^\alpha}+\frac{1}{(1+\theta)^{\alpha+1}}\right] & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\theta^{\alpha+1}}{(\theta+1)^{\alpha+k+1}}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+1)}\left[\alpha+\frac{\alpha+k}{1+\theta}\right] & \text{k=1, 2, \ldots.} \end{cases} \qquad \text{(THREE.68)}
$$

Properties

The mean of the Generalized Lindley distribution is given by

$$
\begin{aligned}
E(\Lambda) &= \int_0^\infty \frac{\lambda(\theta\lambda)^{\alpha-1}(\alpha+\lambda)e^{-\theta\lambda}}{(1+\theta)\Gamma(\alpha+1)}d\lambda \\
&= \frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\alpha\int_0^\infty \lambda^\alpha e^{-\theta\lambda}d\lambda+\int_0^\infty \lambda^{\alpha+1}e^{-\theta\lambda}d\lambda\right] \\
&= \frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+1)}{\theta^{\alpha+1}}+\frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}}\right] \\
&= \frac{1}{1+\theta}\left[\alpha+\frac{\alpha+1}{\theta}\right].
\end{aligned}
$$

The mean of the mixed ZIP distribution will be

$$
E(Y) = \frac{(1-\rho)}{1+\theta}\left[\alpha+\frac{\alpha+1}{\theta}\right].
$$

The second moment of the mixing distribution will be

$$\begin{aligned}
E(\Lambda^2) &= \int_0^\infty \frac{\lambda^2(\theta\lambda)^{\alpha-1}(\alpha+\lambda)e^{-\theta\lambda}}{(1+\theta)\Gamma(\alpha+1)} d\lambda \\
&= \frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)} \left[\alpha \int_0^\infty \lambda^{\alpha+1} e^{-\theta\lambda} d\lambda + \int_0^\infty \lambda^{\alpha+3-1} e^{-\theta\lambda} d\lambda\right] \\
&= \frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)} \left[\frac{\alpha\Gamma(\alpha+2)}{\theta^{\alpha+2}} + \frac{\Gamma(\alpha+3)}{\theta^{\alpha+3}}\right] \\
&= \frac{1}{\alpha(1+\theta)} \left[\frac{\alpha+1}{\theta} + \frac{(\alpha+2)(\alpha+1)}{\theta^2}\right].
\end{aligned}$$

Hence the variance of Y will be

$$\begin{aligned}
Var(Y) &= (1-\rho)[Var(\Lambda) + E(Y)] \\
&= \frac{(1-\rho)}{1+\theta}\left[\frac{\alpha(\alpha+1)}{\theta} + \frac{(\alpha+1)(\alpha+2)}{\theta^2} + \alpha + \frac{(\alpha+1)}{\theta}\right] - \left(\frac{(1-\rho)}{1+\theta}\left[\alpha + \frac{\alpha+1}{\theta}\right]\right)^2 \\
&= \frac{(1-\rho)}{1+\theta}\left[\left(\frac{\alpha(\alpha+1)}{1+\theta} + \frac{(\alpha+1)}{\theta} + \frac{(\alpha+2)(\alpha+1)}{\theta^2}\right) - \left(\frac{(1-\rho)}{1+\theta}\left[\alpha + \frac{(\alpha+1)}{\theta}\right]^2\right)\right].
\end{aligned}$$

3.6 Summary

Most ZIP mixture distributions could be constructed explicitly by using direct integration irrespective of the form of the mixing distribution. Whether the mixing distribution is a one parameter distribution or a has several parameters, belongs to the $[0,1]$ domain or has strictly positive observations, a resultant ZIP mixture distribution can be constructed.

CHAPTER FOUR

ZERO-INFLATED POISSON MIXTURES IN RECURSIVE FORMS

4.1 Introduction

A main difficulty with the use of Mixed Poisson distribution is that their probability mass function $f(x)$ is difficult to evaluate according to Albert in his paper, [1]. One way of of solving this problem was to express the ZIP mixed distributions in terms of recursive relations. A number of methods for deriving such recursive relations had been developed, starting with the works of Katz in [9], Panjer in [10], Wilmot in [26], etc.

4.2 Recursive Models based on Integration

4.2.1 Mixing with Rectangular Distribution

If the mixing distribution is $U(a, b)$, then the recursive formula, for the Zero Inflated Poisson- Rectangular distribution is given by the following: The mixing distribution is

$$g(\lambda) = \frac{1}{b-a}, \; a \leq \lambda \leq b$$

Therefore, the mixed distribution will be given by

$$Prob(Y=k) = \begin{cases} \int_a^b [\rho + (1-\rho)e^{-\lambda}]\frac{1}{b-a}d\lambda, & \text{k=0;} \\ \int_0^\infty [(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}]\frac{1}{b-a}d\lambda, & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(FOUR.1)}$$

$$= \begin{cases} \frac{(1-\rho)}{b-a}[\int_0^b e^{-\lambda} - \int_0^a e^{-\lambda}]d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!(b-a)}[\int_0^b e^{-\lambda}\lambda^k d\lambda - \int_0^a e^{-\lambda}\lambda^k d\lambda], & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(FOUR.2)}$$

$$= \begin{cases} \frac{(1-\rho)}{b-a}\left[e^{-a} - e^{-b}\right], & \text{k=0;} \\ \frac{(1-\rho)}{k!(b-a)}\left[\Gamma_b(k+1) - \Gamma_a(k+1)\right], & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(FOUR.3)}$$

Consider the following integration function,

$$I_k = \int_0^b e^{-\lambda}\lambda^k d\lambda,$$

by using integration by parts, the integration function becomes

$$\begin{aligned} \int_0^b e^{-\lambda}\lambda^k d\lambda &= -\lambda^k e^{-\lambda}|_0^b + k\int_0^b e^{-\lambda}\lambda^{k-1}d\lambda \\ &= -[b^k e^{-b} - 0] + k\Gamma_b(k). \end{aligned}$$

Thus

$$\begin{aligned} \Gamma_b(k+1) &= -b^k e^{-b} + k\Gamma_b(k) \\ &= -b^k e^{-b} - kb^{k-1}e^{-b} + k(k-1)\Gamma_b(k-1) \\ &= -b^k e^{-b} - kb^{k-1}e^{-b} + k(k-1)[-b^{k-2}e^{-b} + (k-2)\Gamma_b(k-2)] \\ &= -e^{-b}[b^k + kb^{k-1} + k(k-1)b^{k-2} + k(k-1)(k-2)b^{k-3} + \cdots \\ &\quad +k(k-1)(k-2)(k-3)\cdots[k-(k-1)]b^{k-k}]. \end{aligned}$$

Therefore

$$\frac{(1-\rho)}{k!(b-a)}\Gamma_b(k+1) = (1-\rho)\frac{-e^{-b}}{(b-a)}\left[\frac{b^k}{k!} + \frac{b^{k-1}}{(k-1)!} + \frac{b^{k-2}}{(k-2)!} + \cdots + \frac{b}{1!} + \frac{1}{0!}\right].$$

Similarly

$$\frac{(1-\rho)}{k!(b-a)}\Gamma_a(k+1) = (1-\rho)\frac{-e^{-a}}{(b-a)}\left[\frac{a^k}{k!} + \frac{a^{k-1}}{(k-1)!} + \frac{a^{k-2}}{(k-2)!} + \cdots + \frac{a}{1!} + \frac{1}{0!}\right].$$

The mixture distribution when $Y = k$ then becomes

$$\begin{aligned} Pr(Y=k) &= \frac{(1-\rho)}{(b-a)}\left[(\frac{e^{-a}a^k - e^{-b}b^k}{k!}) + (\frac{e^{-a}a^{k-1} - e^{-b}b^{k-1}}{(k-1)!}) + \right. \\ &\quad \left. \cdots + \frac{e^{-a}a - e^{-b}b}{1!} + (e^{-a} - e^{-b})\right] \end{aligned}$$

and when $Y = k + 1$, is

$$Pr(Y = k+1) = \frac{(1-\rho)}{(b-a)}\left[\frac{e^{-a}a^{k+1} - e^{-b}b^{k+1}}{(k+1)!}\right] + Pr(Y = k). \quad \text{(FOUR.4)}$$

The recursive formula becomes

$$Pr(Y = k+1) = \begin{cases} \frac{(1-\rho)}{b-a}[e^{-a} - e^{-b}], & \text{k=0;} \quad \text{(FOUR.5)} \\ \frac{(1-\rho)}{(b-a)}\{\frac{e^{-a}a^{k+1} - e^{-b}b^{k+1}}{(k+1)!}\} + Pr(Y = k), & \text{k=1, 2, \ldots.} \quad \text{(FOUR.6)} \end{cases}$$

4.2.2 Mixing with Poisson-Inverse Gaussian Distribution

If the Inverse Gaussian mixing distribution is given by

$$g(\lambda) = (\frac{\phi}{2\pi\lambda^3})^{\frac{1}{2}} \exp\{\frac{-\phi(\lambda-\mu)^2}{2\mu^2\lambda}\}, \ \lambda > 0, \ \mu > 0, \ \phi > 0$$

then the recursive formula for Zero Inflated Poisson-Inverse Gaussian distribution becomes

$$\begin{aligned} Pr(Y=k) &= \begin{cases} \int_0^\infty [\rho + (1-\rho)e^{-\lambda}](\frac{\phi}{2\pi\lambda^3})^{\frac{1}{2}} \exp\{\frac{-\phi(\lambda-\mu)^2}{2\mu^2\lambda}\}d\lambda, & \text{k=0;} \\ \int_0^\infty [(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}](\frac{\phi}{2\pi\lambda^3})^{\frac{1}{2}} \exp\{\frac{-\phi(\lambda-\mu)^2}{2\mu^2\lambda}\}d\lambda, & \text{k=1, 2, \ldots.} \end{cases} \\ &= \begin{cases} (1-\rho)(\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{\frac{-3}{2}} e^{-\lambda(1+\frac{\phi}{2\mu^2}) - \frac{\phi}{2\lambda}} d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{k-\frac{3}{2}} e^{-\lambda(1+\frac{\phi}{2\mu^2}) - \frac{\phi}{2\lambda}} d\lambda, & \text{k=1, 2, \ldots.} \end{cases} \end{aligned}$$

Let the indicator function be given by the following,

$$I_k = \int_0^\infty \lambda^{k-\frac{3}{2}} e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda.$$

Using integration by parts, let

$$u = e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}}$$

and

$$dv = \lambda^{k-\frac{3}{2}} d\lambda,$$

then

$$du = \left[-(1+\frac{\phi}{2\mu^2}) + \frac{\phi}{2\lambda^2}\right] e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda$$

and

$$v = \frac{\lambda^{k+1-\frac{3}{2}}}{k+1-\frac{3}{2}}.$$

This implies that

$$\begin{aligned} I_k &= -\int_0^\infty \frac{\lambda^{k+1-\frac{3}{2}}}{k+1-\frac{3}{2}} \left[-(1+\frac{\phi}{2\mu^2}) + \frac{\phi}{2\lambda^2}\right] e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda \\ &= \left(1+\frac{\phi}{2\mu^2}\right) \frac{1}{(k+1-\frac{3}{2})} \int_0^\infty \lambda^{k+1-\frac{3}{2}} e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda \\ &\quad - \frac{\phi}{2(k+1-\frac{3}{2})} \int_0^\infty \lambda^{k-1-\frac{3}{2}} e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda \\ &= \left(1+\frac{\phi}{2\mu^2}\right) \frac{1}{(k+1-\frac{3}{2})} I_{k+1} - \frac{\phi}{2(k+1-\frac{3}{2})} I_{k-1}. \end{aligned}$$

This implies that

$$k!Pr(Y=k) = \left(1+\frac{\phi}{2\mu^2}\right) \frac{(k+1)!}{(k+1-\frac{3}{2})} Pr(Y=k+1) - \frac{\phi(k-1)!}{2(k+1-\frac{3}{2})} Pr(Y=k-1)$$

which can be further simplified to

$$kPr(Y=k) = (1+\frac{\phi}{2\mu^2}) \frac{k(k+1)}{(k+1-\frac{3}{2})} Pr(Y=k+1) - \frac{\phi}{2k-1} Pr(Y=k-1).$$

Therefore, the recursive formula is

$$(1-\rho)\left[(1+\frac{\phi}{2\mu^2})\frac{k(k+1)}{(k+1-\frac{3}{2})}Pr(Y=k+1)\right]$$

$$= (1-\rho)\left[kPr(Y=k)+\frac{\phi}{2k-1}Pr(Y=k-1)\right],\ k=1,\ 2,\ 3,\cdots \qquad \text{(FOUR.7)}$$

with

$$Pr(Y=-1)=0.$$

4.2.3 Mixing with an Exponential Distribution

An exponential distribution with one parameter is given by

$$g(\lambda)=\mu\, e^{-\mu\lambda}\ \lambda>0,\ \mu>0.$$

The recursive formula for the Zero Inflated Geometric distribution is derived as follows

$$Pr(Y=k) \quad = \quad \begin{cases} (1-\rho)\mu\int_0^\infty e^{-(1+\mu)\lambda}d\lambda, & \text{k=0;} \\ (1-\rho)\frac{\mu}{k!}\int_0^\infty e^{-(1+\mu)\lambda}\lambda^k d\lambda & \text{k=1, 2,}\ldots. \end{cases}$$

Let the indicator function in k be represented by

$$I_k=\frac{k!Pr(Y=k)}{(1-\rho)}\approx\int_0^\infty \mu\, e^{-(1+\mu)\lambda}\lambda^k d\lambda,$$

then using integration by parts, let

$$u=e^{-(1+\mu)\lambda}$$

and

$$dv=\lambda^k d\lambda,$$

then

$$du=-(1+\mu)e^{-(1+\mu)\lambda}d\lambda.$$

Then

$$\begin{aligned} I_k &= \frac{1}{k+1}\int_0^\infty (1+\mu)e^{-(1+\mu)\lambda}\lambda^{k+1}d\lambda \\ &= \left(\frac{1+\mu}{k+1}\right) I_{k+1}. \end{aligned}$$

$$I_{k+1} = \left(\frac{k+1}{1+\mu}\right) I_k.$$

It follows that

$$\frac{(k+1)!Pr(Y=k+1)}{(1-\rho)\mu} = \left(\frac{k+1}{1+\mu}\right)\frac{k!}{(1-\rho)\mu}Pr(Y=k).$$

Therefore,

$$\begin{aligned} Pr(Y=k+1) &= \left(\frac{k+1}{1+\mu}\right)\frac{k!}{(k+1)!}Pr(Y=k) \\ &= \left(\frac{1}{1+\mu}\right)Pr(Y=k). \end{aligned}$$

The recursive formula will then be given by

$$Pr(Y=k+1) = \begin{cases} \rho+(1-\rho)\frac{\mu}{(1+\mu)}, & \text{k=0;} \\ \left(\frac{1}{1+\mu}\right)Pr(Y=k), & \text{k=1, 2, \dots.} \end{cases}$$

4.2.4 ZINB distribution

A ZINB distribution is a mixture of a Zero inflated Poisson distribution and a two parameter Gamma distribution. If the pdf of a Gamma distribution with two parameters is given by

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma\alpha}e^{-\beta\lambda}\lambda^{\alpha-1},\ \lambda > 0.\alpha > 0,\ \beta > 0$$

then the recursive formula for ZINB is derived as follows;

$$Pr(Y=k) = \begin{cases} (1-\rho)\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(1+\beta)\lambda}\lambda^{\alpha-1}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(1+\beta)\lambda}\lambda^{\alpha-1+k}d\lambda, & \text{k=1, 2, \dots.} \end{cases}$$

Suppose that

$$\frac{k!}{(1-\rho)}\frac{\Gamma\alpha}{\beta^{\alpha}}P(Y=k)=I_k\approx\int_0^{\infty}e^{-(1+\beta)\lambda}\lambda^{\alpha-1+k}d\lambda$$

Using integration by parts

$$\begin{aligned} I_k &= \int_0^{\infty}\frac{1+\beta}{\alpha+k}e^{-(1+\beta)\lambda}\lambda^{\alpha+k}d\lambda \\ &= \left(\frac{1+\beta}{\alpha+k}\right)I_{k+1}, \end{aligned}$$

such that

$$I_{k+1}=\left(\frac{\alpha+k}{1+\beta}\right)I_k.$$

It then follows that

$$\frac{(k+1)!}{(1-\rho)}\frac{\Gamma\alpha}{\beta^{\alpha}}Pr(Y=k+1)=\frac{(\alpha+k)}{(1+\beta)}\frac{k!}{(1-\rho)}\frac{\Gamma\alpha}{\beta^{\alpha}}Pr(Y=k).$$

Hence

$$Pr(Y=k+1)=\left(\frac{\alpha+k}{(1+\beta)(k+1)}\right)Pr(Y=k).$$

The recursive formula is

$$Pr(Y=k+1) = \begin{cases} (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}, & \text{k=0;} \\ \left(\frac{\alpha+k}{(k+1)(1+\beta)}\right)Pr(Y=k), & \text{k=1, 2, \ldots.} \end{cases}$$

4.2.5 Mixing with Inverse - Gamma Distribution

The pdf of an Inverse - Gamma distribution is

$$g(\lambda)=\frac{\beta^{\alpha}}{\Gamma\alpha}\frac{e^{-\frac{\beta}{\alpha}}}{\lambda^{\alpha+1}},\ \lambda>0,\ \alpha>0,\ \beta>0$$

The resultant distribution is

$$Pr(Y=k) = \begin{cases} \rho + (1-\rho)\int_0^\infty e^{-\lambda}\frac{\beta^\alpha}{\Gamma\alpha}\frac{e^{-\frac{\beta}{\alpha}}}{\lambda^{\alpha+1}}d\lambda, & \text{k=0;} \\ (1-\rho)\int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!}\frac{\beta^\alpha}{\Gamma\alpha}\frac{e^{-\frac{\beta}{\alpha}}}{\lambda^{\alpha+1}}d\lambda, & \text{k=1, 2, \dots.} \end{cases}$$

which can be rewritten as

$$Pr(Y=k) = \begin{cases} (1-\rho)\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{-(\lambda+\frac{\beta}{\lambda})}\lambda^{-\alpha-1}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{-(\lambda+\frac{\beta}{\lambda})}\lambda^{k-\alpha-1}d\lambda, & \text{k=1, 2, \dots.} \end{cases}$$

Assume that the integration function is

$$\frac{\Gamma\alpha}{\beta^\alpha}\frac{k!}{(1-\rho)} = I_k = \int_0^\infty e^{-(\lambda+\frac{\beta}{\lambda})}\lambda^{k-\alpha-1}d\lambda$$

Using integration by parts, let

$$u = e^{-(\lambda+\frac{\beta}{\lambda})}$$

and

$$dv = \lambda^{k-\alpha-1}d\lambda,$$

then

$$du = -(1-\frac{\beta}{\lambda^2})e^{-(\lambda+\frac{\beta}{\lambda})}d\lambda$$

Then, let

$$\begin{aligned} I_k &= \frac{1}{(k-\alpha)}\int_0^\infty \lambda^{k-\alpha}e^{-(\lambda+\frac{\beta}{\lambda})}d\lambda - \frac{\beta}{(k-\alpha)}\int_0^\infty \lambda^{k-\alpha-2}e^{-(\lambda+\frac{\beta}{\lambda})}d\lambda \\ &= \left(\frac{1}{k-\alpha}\right)I_{k+1} - \left(\frac{\beta}{k-\alpha}\right)I_{k-1}. \end{aligned}$$

The integration function in $k+1$ will be

$$I_{k+1} = (k-\alpha)I_k + \beta I_{k-1}.$$

which implies that

$$\frac{\Gamma\alpha}{\beta^\alpha}\frac{(k+1)!}{(1-\rho)}Pr(Y=k+1) = \frac{(k-\alpha)k!}{(1-\rho)}\frac{\Gamma\alpha}{\beta^\alpha}Pr(Y=k) + \beta\frac{(k-1)!}{(1-\rho)}\frac{\Gamma\alpha}{\beta^\alpha}Pr(Y=k-1),$$

and so,

$$k(k+1)Pr(Y=k+1) = k(k-\alpha)Pr(Y=k) + \beta\, Pr(Y=k-1).$$

which is the recursive formula with $Pr(Y=-1)=0$.

4.2.6 Mixing with Poisson - Beta distribution

The Poisson - Beta distribution is

$$g(\lambda) = \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)},\ 0 \le \lambda \le 1,\ \alpha,\ \beta > 0$$

The recursive formula is derived as follows;

$$\begin{aligned} Pr(Y=k) &= \begin{cases} (1-\rho)\int_0^1 e^{-\lambda}\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\int_0^1 e^{-\lambda}\lambda^k\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}d\lambda, & \text{k=}1,2,\ldots. \end{cases} \\ &= \begin{cases} \frac{(1-\rho)}{B(\alpha,\beta)}\int_0^1 e^{-\lambda}\lambda^{\alpha-1}(1-\lambda)^{\beta-1}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!B(\alpha,\beta)}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta-1}d\lambda, & \text{k=}1,2,\ldots. \end{cases} \end{aligned}$$

Suppose that

$$I_k(\alpha,\beta) = \frac{k!B(\alpha,\beta)}{(1-\rho)}Pr(Y=k) \approx \int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta-1}d\lambda.$$

Let $u = e^{-\lambda}\lambda^{\alpha+k-1}$ and $dv = (1-\lambda)^{\beta-1}d\lambda$, then

$$du = (-e^{-\lambda}\lambda^{\alpha+k-1} + (\alpha+k-1)e^{-\lambda}\lambda^{\alpha+k-2})d\lambda$$

and

$$v = -\frac{(1-\lambda)^{\beta}}{\beta}.$$

Therefore,

$$\begin{aligned}
I_k(\alpha,\beta) &= -\frac{1}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta}d\lambda + \frac{(\alpha+k-1)}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-2}(1-\lambda)^{\beta}d\lambda \\
&= -\frac{1}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)(1-\lambda)^{\beta-1}d\lambda + \\
&\quad \frac{(\alpha+k-1)}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-2}(1-\lambda)(1-\lambda)^{\beta-1}d\lambda \\
&= -\frac{1}{\beta}\{\int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta-1}d\lambda - \int_0^1 e^{-\lambda}\lambda^{\alpha+k}(1-\lambda)^{\beta-1}d\lambda\} \\
&\quad +\{-\frac{\alpha+k-1}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-2}(1-\lambda)^{\beta-1}d\lambda - \int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta-1}d\lambda\} \\
&= -\frac{1}{\beta}\{I_k(\alpha,\,\beta) - I_{k+1}(\alpha,\,\beta)\} + \frac{(\alpha+k-1)}{\beta}\{I_{k-1}(\alpha,\,\beta) - I_k(\alpha,\,\beta)\}.
\end{aligned}$$

and hence

$$\begin{aligned}
I_{k+1}(\alpha,\,\beta) &= \beta\, I_k(\alpha,\,\beta) + I_k(\alpha,\,\beta) + (\alpha+k-1)I_x(\alpha,\,\beta) - (\alpha+k-1)I_{x-1}(\alpha,\,\beta) \\
&= (\alpha+\beta+k)I_k(\alpha,\,\beta) - (\alpha+k-1)I_{k-1}(\alpha,\,\beta).
\end{aligned}$$

This implies that

$$(k+1)!Pr(Y=k+1) \;=\; (\alpha+\beta+k)k!Pr(Y=k) - (\alpha+k-1)(k-1)!Pr(Y=k-1),$$

which can be further simplified as

$$k(k+1)Pr(Y=k+1) = (\alpha+\beta+k)kPr(Y=k) - (\alpha+k-1)Pr(Y=k-1),$$

with

$$Pr(Y=-1) = 0.$$

4.2.7 Mixing with Inverted - Beta distribution

The mixing distribution is

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{B(\alpha,\,\beta)(1+\lambda)^{\alpha+\beta}},\ \lambda > 0,\ \alpha > 0,\ \beta > 0,$$

with the mixed distribution as

$$Pr(Y=k) = \begin{cases} (1-\rho)\int_0^\infty e^{-\lambda}\frac{\lambda^{\alpha-1}}{B(\alpha,\beta)(1+\lambda)^{\alpha+\beta}}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\int_0^\infty e^{-\lambda}\lambda^k\frac{\lambda^{\alpha-1}}{B(\alpha,\beta)(1+\lambda)^{\alpha+\beta}}d\lambda, & \text{k=}1,2,\ldots. \end{cases} \quad \text{(FOUR.8)}$$

Let,

$$\frac{k!B(\alpha,\beta)}{(1-\rho)}Pr(Y=k) = \int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-1}}{(1+\lambda)^{\alpha+\beta}}d\lambda = I_k$$

Let $u = e^{-\lambda}\lambda^{\alpha+k-1}$ and $dv = \frac{d\lambda}{(1+\lambda)^{\alpha+\beta}}$, then

$$du = -e^{-\lambda}\lambda^{\alpha+k-1} + e^{-\lambda}(\alpha+k-1)\lambda^{\alpha+k-2}$$

and

$$v = -\frac{(1+\lambda)^{-(\alpha+\beta-1)}}{(\alpha+\beta-1)}.$$

then the integral function in k becomes

$$\begin{aligned} I_k &= \frac{1}{(\alpha+\beta-1)}\int_0^\infty (1+\lambda)^{-(\alpha+\beta-1)}\{e^{-\lambda}(\alpha+k-1)\lambda^{\alpha+k-2} - e^{-\lambda}\lambda^{\alpha+k-1}\}d\lambda \\ &= \frac{1}{(\alpha+\beta-1)}\{\int_0^\infty \frac{e^{-\lambda}(\alpha+k-1)\lambda^{\alpha+k-2}(1+\lambda)}{(1+\lambda)(1+\lambda)^{-(\alpha+\beta-1)}}d\lambda - \int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-1}(1+\lambda)}{(1+\lambda)(1+\lambda)^{-(\alpha+\beta-1)}}d\lambda\} \\ &= \frac{(\alpha+k-1)}{(\alpha+\beta-1)}\{\int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-2}}{(1+\lambda)^{-(\alpha+\beta)}}d\lambda + \int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-1}}{(1+\lambda)^{-(\alpha+\beta)}}d\lambda\} - \\ &\quad \frac{1}{(\alpha+\beta-1)}\{\int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-1}}{(1+\lambda)^{-(\alpha+\beta)}}d\lambda + \int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k}}{(1+\lambda)^{-(\alpha+\beta)}}d\lambda\} \\ &= \frac{(\alpha+k-1)}{(\alpha+\beta-1)}\{I_{k-1}+I_k\} - \frac{1}{(\alpha+\beta-1)}\{I_k+I_{k+1}\}. \end{aligned}$$

Therefore

$$(\alpha+\beta-1)I_k = (\alpha+k-1)(I_{k-1}+I_k) - (I_k+I_{k+1}),$$

which implies that

$$I_{k+1} = (\alpha+k-1)I_k + (k-\beta-1)I_{k-1}.$$

Thus

$$\frac{(k+1)!B(\alpha,\beta)}{(1-\rho)}Pr(Y=k+1)=\frac{(k-\beta-1)k!B(\alpha,\beta)}{(1-\rho)}Pr(Y=k)$$
$$+\frac{(\alpha+k-1)(k-1)!B(\alpha,}{(1-\rho)}$$

which when simplified gives the recursive formula of a Zero Inflated Poisson- Inverted Beta distribution as

$$k(k+1)Pr(Y=k+1)=k(k-\beta-1)Pr(Y=k)+(\alpha+k-1)Pr(Y=k-1),$$

with $Pr(Y=-1)=0$.

4.3 Summary

Irrespective of the domain of the mixing distribution, the recursive formulae could be developed of the mixing and the Zero Inflated Poisson distribution. However, recursive formulae for mixing distributions in the $[0,\infty]$ domain could not be derived.

CHAPTER FIVE

ZERO-INFLATED POISSON MIXTURES IN EXPECTATION FORMS

5.1 Introduction

In this chapter the derivation of the relationship between a ZIP mixed Poisson pmf, $f(y)$, and the Laplace of the mixing distribution was established. Then the relationship between the pgf of y and the Laplace Transform of the mixing distribution wa also established. The Exponential mixing distribution and Gamma mixing distributions with one and two parameters were used as special cases. The relationship between the pdf of a ZIP distribution and the r^{th} moment of the mixing distribution was also derived. Several mixing distributions were used to find the resultant ZIP mixed distributions. When these distributions were equated to distributions that were derived explicitly, there were identities, which were also proved.

5.1.1 Relationship between a ZIP mixture and Laplace of mixing distributions

For a ZIP mixed distribution, whose pmf is as given

$$Pr(Y=k) = \begin{cases} \int_0^\infty \left[\rho + (1-\rho)e^{-\lambda}\right] g(\lambda)d\lambda, & \text{k=0;} \\ \int_0^\infty \left[(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}\right] g(\lambda)d\lambda, & \text{k=1, 2, \dots.} \end{cases}$$
$$= \begin{cases} (1-\rho)E[e^{-\lambda}], & \text{k=0;} \\ (1-\rho)E[\frac{e^{-\lambda}\lambda^k}{k!}], & \text{k=1, 2, \dots.} \end{cases}$$

The Laplace Transform of the mixing distribution is defined as

$$L_\lambda(s) = (1-\rho)E[e^{-\lambda s}], \qquad \text{(FIVE.1)}$$

On differentiating successively with respect to s

$$L'_\lambda(s) = (1-\rho)E[-\lambda e^{-\lambda s}] \qquad \text{(FIVE.2)}$$
$$L''_\lambda(s) = (1-\rho)E[\lambda^2 e^{-\lambda s}], \qquad \text{(FIVE.3)}$$

In general,

$$L^k_\lambda(s) = (1-\rho)(-1)^k E[\lambda^k e^{-\lambda s}],$$

When $s = 1$, then

$$L^k_\lambda(1) = (1-\rho)(-1)^k E[\lambda^k e^{-\lambda}],$$

Therefore, the mixed ZIP distribution can be written as

$$Pr(Y=k) = \begin{cases} (1-\rho)E[e^{-\lambda}], & \text{k=0;} \\ \frac{(1-\rho)}{k!}E[e^{-\lambda}\lambda^k], & \text{k=1, 2, \dots.} \end{cases} \qquad \text{(FIVE.4)}$$
$$= \begin{cases} (1-\rho)E[e^{-\lambda}], & \text{k=0;} \\ \frac{(1-\rho)}{k!}(-1)^k L^k_\lambda(1), & \text{k=1, 2, \dots.} \end{cases} \qquad \text{(FIVE.5)}$$

which is the ZIP mixture distribution expressed in terms of the Laplace transform of the mixing distribution.

5.1.2 Relationship between pgf of Y and Laplace Transform of mixing distribution

The probability generating function of the ZIP mixed distribution is given by

$$
\begin{aligned}
G_k(s) &= \begin{cases} (1-\rho)\sum_{k=0}^{\infty}\int_0^{\infty}\left[e^{-\lambda}g(\lambda)d\lambda\right]s^k, & \text{k=0;} \\ (1-\rho)\sum_{k=0}^{\infty}\int_0^{\infty}\left[\frac{e^{-\lambda}\lambda^k}{k!}g(\lambda)d\lambda\right]s^k, & \text{k=}1,2,\ldots. \end{cases} \\
&= \begin{cases} (1-\rho)\int_0^{\infty}\left[\sum_{k=0}^{\infty}e^{-\lambda}s^k\right]g(\lambda)d\lambda, & \text{k=0;} \\ (1-\rho)\int_0^{\infty}\left[\sum_{k=0}^{\infty}\frac{e^{-\lambda}(\lambda s)^k}{k!}\right]g(\lambda)d\lambda, & \text{k=}1,2,\ldots. \end{cases} \\
&= \begin{cases} (1-\rho)L_\lambda(1-s), & \text{k=0;} \\ (1-\rho)L_\lambda(1-s), & \text{k=}1,2,\ldots. \end{cases}
\end{aligned}
$$

where $L_\lambda(1-s)$ is the Laplace Transform of the distribution of lambda.

5.1.3 Relationship between mixed ZIP distribution and Moments of the mixing distribution

Karlis and Xekalaki in [8], in Proposition 14 gave an alternative formula linking the probability function of a mixed Poisson distribution to the moments of the mixing distribution. Consider the following ZIP mixed distribution, which can be written in terms of the r^{th} moment of the mixing distribution as follows

$$
Pr(Y=k) = \begin{cases} \int_0^{\infty}[\rho+(1-\rho)e^{\lambda}]g(\lambda)d\lambda, & \text{k=0;} \\ \int_0^{\infty}[(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}]g(\lambda)d\lambda, & \text{k=}1,2,\ldots. \end{cases} \quad \text{(FIVE.6)}
$$

$$
= \begin{cases} \int_0^{\infty}[\rho+(1-\rho)\sum_{j=0}^{\infty}\frac{(-\lambda)^j}{j!}]g(\lambda)d\lambda, & \text{k=0;} \\ \int_0^{\infty}[(1-\rho)\sum_{j=0}^{\infty}\frac{(-\lambda)^j}{j!}\frac{\lambda^k}{k!}]g(\lambda)d\lambda, & \text{k=}1,2,\ldots. \end{cases} \quad \text{(FIVE.7)}
$$

$$
= \begin{cases} (1-\rho)\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\int_0^{\infty}\lambda^j g(\lambda)d\lambda, & \text{k=0;} \\ (1-\rho)\sum_{j=0}^{\infty}\frac{(-1^j}{j!k!}\int_0^{\infty}\lambda^{j+k}g(\lambda)d\lambda, & \text{k=}1,2,\ldots. \end{cases} \quad \text{(FIVE.8)}
$$

Let $j = x$, then $f(k)$ can be written as

$$Pr(Y = k) = \begin{cases} (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!} E(\Lambda^x), & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!k!} E(\Lambda^{x+k}), & \text{k=1, 2, \ldots.} \end{cases} \qquad \text{(FIVE.9)}$$

where $E(\Lambda^r)$ is the r^{th} moment of the mixing distribution.

5.2 Special cases

5.2.1 Exponential distribution

For an exponential distribution, the r^{th} moment is

$$\begin{aligned} E(\Lambda^r) &= \int_0^{\infty} \lambda^r \mu\, e^{-\mu\lambda} d\lambda \\ &= \mu \int_0^{\infty} \lambda^r e^{-\mu\lambda} d\lambda. \end{aligned}$$

Let $y = \mu\lambda$, $\Longrightarrow \lambda = \frac{y}{\mu}$, and $d\lambda = \frac{dy}{\mu}$, then the r^{th} moment is

$$\begin{aligned} E(\Lambda^r) &= \mu \int_0^{\infty} \left(\frac{y}{\mu}\right)^r e^{-y} \frac{dy}{\mu} \\ &= \frac{1}{\mu^r} \int_0^{\infty} y^{r+1-1} e^{-y} dy \\ &= \frac{1}{\mu^r} \Gamma(r+1) \\ &= \frac{r!}{\mu^r}. \end{aligned}$$

Then equation 5.6, will be given by

$$Pr(Y = k) = \begin{cases} (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!} \frac{x!}{\mu^x}, & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty} \frac{(-1^x)}{x!k!} \frac{(x+k)!}{\mu^{x+k}}, & \text{k=1, 2, \ldots.} \end{cases} \qquad \text{(FIVE.10)}$$

We have the following identities from equation 4.7

1. $(1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!} \frac{x!}{\mu^x} = (1-\rho)\left(\frac{\mu}{1+\mu}\right)$.

2. $(1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!k!} \frac{(x+k)!}{\mu^{x+k}} = (1-\rho)(\frac{\mu}{1+\mu})\left(\frac{1}{1+\mu}\right)^k$.

PROOF. For the first identity, showing that the LHS of the equation is equal to the RHS of the equation,

$$\begin{aligned}\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{x!}{\mu^x} &= \sum_{x=0}^{\infty}\binom{-1}{x}\left(\frac{1}{\mu}\right)^x \\ &= \left(1+\frac{1}{\mu}\right)^{-1} \\ &= \frac{\mu}{1+\mu}.\end{aligned}$$

For the second identity,

$$\begin{aligned}\frac{\Gamma(k+1)}{k!\mu^k}\sum_{x=0}^{\infty}(-1)^x\frac{(x+k)!}{x!\Gamma(k+1)\mu^x} &= \frac{\Gamma(k+1)}{k!\mu^k}\sum_{x=0}^{\infty}(-1)^x\binom{(x+k+1-1)}{x}\frac{1}{\mu^x} \\ &= \frac{\Gamma(k+1)}{k!\mu^k}\sum_{x=0}^{\infty}(-1)^x\binom{-(k+1)}{x}\left(\frac{1}{\mu}\right)^{k+1} \\ &= \frac{\Gamma(k+1)}{k!\mu^k}\left(1+\frac{1}{\mu}\right)^{-(k+1)} \\ &= \frac{k!}{k!\mu^k}\left(\frac{\mu}{1+\mu}\right)^{k+1} \\ &= \left(\frac{\mu}{1+\mu}\right)\left(\frac{1}{1+\mu}\right)^k.\end{aligned}$$

□

5.2.2 Gamma distribution with two parameters

The r^{th} moment is

$$\begin{aligned}E(\Lambda^r) &= \int_0^{\infty}\lambda^r\frac{\beta^\alpha}{\Gamma\alpha}e^{-\beta\lambda}\lambda^{\alpha-1}d\lambda \\ &= \frac{\beta^\alpha}{\Gamma\alpha}\int_0^{\infty}e^{-\beta\lambda}\lambda^{\alpha+r-1}d\lambda \\ &= \frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma(\alpha+r)}{\beta^{\alpha+r}}.\end{aligned}$$

The mixed distribution becomes

$$Pr(Y=k) \quad = \quad \begin{cases} (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!}\frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma(\alpha+x)}{\beta^{\alpha+x}}, & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!k!}\frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma(\alpha+x+k)}{\beta^{\alpha+x+k}}, & \text{k=}1,2,\ldots, \end{cases} \qquad \text{(FIVE.11)}$$

with the following identities

1. $(1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!}\frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma(\alpha+x)}{\beta^{\alpha+x}} = (1-\rho)\left(\frac{\beta}{1+\beta}\right)^\alpha.$

2. $(1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!k!}\frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma(\alpha+x+k)}{\beta^{\alpha+x+k}} = (1-\rho)\begin{pmatrix}\alpha+k-1\\ k\end{pmatrix}\frac{\beta^\alpha}{(1+\beta)^{\alpha+k}}.$

PROOF. For the first identity,

$$\begin{aligned}
\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma\alpha\beta^x} &= \sum_{x=0}^{\infty}(-1)^x\begin{pmatrix}\alpha+x-1\\ x\end{pmatrix}\left(\frac{1}{\beta}\right)^x \\
&= \sum_{x=0}^{\infty}\begin{pmatrix}-\alpha\\ x\end{pmatrix}\left(\frac{1}{\beta}\right)^\alpha \\
&= \left(1+\frac{1}{\beta}\right)^{-\alpha} \\
&= \left(\frac{\beta}{1+\beta}\right)^\alpha.
\end{aligned}$$

For the second identity,

$$\begin{aligned}
\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x+k)}{x!k!\Gamma\alpha\beta^{x+k}} &= \frac{\Gamma(\alpha+k)}{\beta^k k!\Gamma\alpha}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x+k)}{x!\Gamma(\alpha+k)}\frac{1}{\beta^{x+k}} \\
&= \frac{\Gamma(\alpha+k)}{k!\Gamma\alpha}\frac{1}{\beta^k}\sum_{x=0}^{\infty}(-1)^x\begin{pmatrix}\alpha+x+k-1\\ x\end{pmatrix}\left(\frac{1}{\beta}\right)^x \\
&= \frac{\Gamma(\alpha+k)}{k!\Gamma\alpha}\frac{1}{\beta^k}\sum_{x=0}^{\infty}\begin{pmatrix}-(\alpha+k)\\ x\end{pmatrix}\left(\frac{1}{\beta}\right)^{\alpha+k} \\
&= \frac{\Gamma(\alpha+k)}{k!\Gamma\alpha}\frac{1}{\beta^k}\left(1+\frac{1}{\beta}\right)^{-(\alpha+k)} \\
&= \begin{pmatrix}\alpha+k-1\\ k\end{pmatrix}\frac{\beta^\alpha\beta^k}{(1+\beta)^{\alpha+k}}\frac{1}{\beta^k} \\
&= \begin{pmatrix}\alpha+k-1\\ k\end{pmatrix}\left(\frac{\beta}{1+\beta}\right)^\alpha\left(\frac{1}{1+\beta}\right)^k.
\end{aligned}$$

□

5.2.3 Mixing with Generalized Lindley distribution

The distribution of a two parameter Generalized Lindley distribution is of the form

$$g(\lambda) = \frac{\theta^2(\theta\lambda)^{\alpha-1}(\alpha+\lambda)e^{-\theta\lambda}}{(1+\theta)\Gamma(\alpha+1)}, \alpha,\ \lambda,\ \theta > 0.$$

The r^{th} moment will be given by

$$\begin{aligned}
E(\Lambda^r) &= \int_0^\infty \frac{\lambda^r\theta^2(\theta\lambda)^{\alpha-1}(\alpha+\lambda)e^{-\theta\lambda}}{(1+\theta)\Gamma(\alpha+1)}d\lambda \\
&= \frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\int_0^\infty \alpha\lambda^{\alpha+r-1}e^{-\theta\lambda}d\lambda + \int_0^\infty \lambda^{\alpha+r+1-1}e^{-\theta\lambda}d\lambda\right] \\
&= \frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+r)}{\theta^{\alpha+r}} + \frac{\Gamma(\alpha+r+1)}{\theta^{\alpha+r+1}}\right].
\end{aligned}$$

The ZIP mixed distribution is

$$Pr(Y=k) = \begin{cases} \rho + (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!}\frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+x)}{\theta^{\alpha+x}} + \frac{\Gamma(\alpha+x+1)}{\theta^{\alpha+x+1}}\right], & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!k!}\frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+x+k)}{\theta^{\alpha+x+k}} + \frac{\Gamma(\alpha+x+k+1)}{\theta^{\alpha+x+k+1}}\right], & \text{k=1, 2, \ldots.} \end{cases}$$

Identities

1.

$$(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+x)}{\theta^{\alpha+x}} + \frac{\Gamma(\alpha+x+1)}{\theta^{\alpha+x+1}}\right]$$

$$= (1-\rho)\frac{\theta^{\alpha+1}}{(1+\theta)}\left[\frac{1}{(1+\theta)^{\alpha}} + \frac{1}{(1+\theta)^{\alpha+1}}\right].$$

2.

$$(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!k!}\frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+x+k)}{\theta^{\alpha+x+k}} + \frac{\Gamma(\alpha+x+k+1)}{\theta^{\alpha+x+k+1}}\right]$$

$$= \frac{(1-\rho)}{k!}\frac{\theta^{\alpha+1}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+k)}{(1+\theta)^{\alpha+k}} + \frac{\Gamma(\alpha+k+1)}{(1+\theta)^{\alpha+k+1}}\right].$$

PROOF.

$$\frac{\alpha}{\theta^{\alpha}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma(\alpha+1)}\frac{1}{\theta^x} + \frac{1}{\theta^{\alpha+1}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma(\alpha+1)}\frac{1}{\theta^x}$$

$$= \frac{\alpha}{\theta^{\alpha}\alpha}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma\alpha}\frac{1}{\theta^x} + \frac{1}{\theta^{\alpha+1}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma(\alpha+1)}\frac{1}{\theta^x}$$

$$
\begin{aligned}
&= \frac{1}{\theta^{\alpha}}\sum_{x=0}^{\infty}(-1)^{x}\binom{x+\alpha-1}{x}\left(\frac{1}{\theta}\right)^{x}+\frac{1}{\theta^{\alpha+1}}\sum_{x=0}^{\infty}(-1)^{x}\binom{x+\alpha+1-1}{x}\left(\frac{1}{\theta}\right)^{x}\\
&= \frac{1}{\theta^{\alpha}}\sum_{x=0}^{\infty}\binom{-\alpha}{x}\left(\frac{1}{\theta}\right)^{\alpha}+\frac{1}{\theta^{\alpha+1}}\sum_{x=0}^{\infty}\binom{-(\alpha+1)}{x}\left(\frac{1}{\theta}\right)^{\alpha+1}\\
&= \frac{1}{\theta^{\alpha}}\left(1+\frac{1}{\theta}\right)^{-\alpha}+\frac{1}{\theta^{\alpha+1}}(1+\frac{1}{\theta})^{-(\alpha+1)}\\
&= \frac{1}{\theta^{\alpha}}\left(\frac{\theta}{1+\theta}\right)^{\alpha}+\frac{1}{\theta^{\alpha+1}}\left[\frac{\theta}{1+\theta}\right]^{\alpha+1}\\
&= \frac{1}{(1+\theta)^{\alpha}}+\frac{1}{(1+\theta)^{\alpha+1}},
\end{aligned}
$$

which completes the proof for the first identity.

For the second part,

$$
\begin{aligned}
&\frac{\alpha}{\theta^{k+\alpha}}\sum_{x=0}^{\infty}(-1)^{x}\frac{\Gamma(x+\alpha+k)}{x!\theta^{x}}+\frac{1}{\theta^{\alpha+k+1}}\sum_{x=0}^{\infty}(-1)^{x}\frac{\Gamma(x+k+\alpha+1)}{x!\theta^{x}}\\
&= \frac{\alpha\Gamma(k+\alpha)}{\theta^{k+\alpha}}\sum_{x=0}^{\infty}(-1)^{x}\frac{\Gamma(x+\alpha+k)}{x!\Gamma(k+\alpha)}\frac{1}{\theta^{x}}+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\sum_{x=0}^{\infty}(-1)^{x}\frac{\Gamma(x+k+\alpha+1)}{x!\Gamma(k+\alpha+1)}\frac{1}{\theta^{x}}\\
&= \frac{\alpha\Gamma(\alpha+k)}{\theta^{k+\alpha}}\sum_{x=0}^{\infty}(-1)^{x}\binom{x+k+\alpha-1}{x}\left(\frac{1}{\theta}\right)^{x}\\
&\quad+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\sum_{x=0}^{\infty}(-1)^{x}\binom{x+k+\alpha+1-1}{x}\left(\frac{1}{\theta}\right)^{x}\\
&= \frac{\alpha\Gamma(\alpha+k)}{\theta^{k+\alpha}}\sum_{x=0}^{\infty}\binom{-(k+\alpha)}{x}\left(\frac{1}{\theta}\right)^{k+\alpha}+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\sum_{x=0}^{\infty}\binom{-(k+\alpha+1)}{x}\left(\frac{1}{\theta}\right)^{k+\alpha+1}.
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha\Gamma(\alpha+k)}{\theta^{k+\alpha}}\left(1+\frac{1}{\theta}\right)^{-(k+\alpha)}+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\left(1+\frac{1}{\theta}\right)^{-(k+\alpha+1)}\\
&= \frac{\alpha\Gamma(\alpha+k)}{\theta^{k+\alpha}}\left(\frac{\theta}{1+\theta}\right)^{k+\alpha}+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\left(\frac{\theta}{1+\theta}\right)^{k+\alpha+1}\\
&= \frac{\alpha\Gamma(\alpha+k)}{(1+\theta)^{k+\alpha}}+\frac{\Gamma(k+\alpha+1)}{(1+\theta)^{\alpha+k+1}}.
\end{aligned}
$$

□

5.2.4 Mixing with Lindley Distribution

A Lindley distribution is of the form

$$g(\lambda) = \frac{\theta^2}{1+\theta}(\lambda+1)e^{-\theta\lambda},\ \lambda > 0,\ \theta > 0,$$

with the r^{th} moment as

$$\begin{aligned} E(\Lambda^r) &= \int_0^\infty \lambda^r \frac{\theta^2}{1+\theta}(\lambda+1)e^{-\theta\lambda}d\lambda \\ &= \frac{\theta^2}{1+\theta}\int_0^\infty \left[\lambda^{r+1+1-1}e^{-\theta\lambda} + \lambda^{r+1-1}e^{-\theta\lambda}\right] d\lambda \\ &= \frac{\theta^2}{1+\theta}\left[\frac{\Gamma(r+2)}{\theta^{r+2}} + \frac{\Gamma(r+1)}{\theta^{r+1}}\right]. \end{aligned}$$

Therefore $f(k)$ can be written in terms of the r^{th} moment as

$$Pr(Y=k) = \begin{cases} (1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(x+2)}{\theta^{x+2}} + \frac{\Gamma(x+1)}{\theta^{x+1}}\right], & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty}\frac{(-1^x}{x!k!}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(x+k+2)}{\theta^{x+k+2}} + \frac{\Gamma(x+k+1)}{\theta^{x+k+1}}\right], & \text{k=1, 2, \dots.} \end{cases}$$

which should be the same as equation (3.6.1).

This implies that there are have two identities, i.e

1.
$$(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(x+2)}{\theta^{x+2}} + \frac{\Gamma(x+1)}{\theta^{x+1}}\right]$$
$$= (1-\rho)\frac{\theta^2}{1+\theta}\left(\frac{1}{(1+\theta)^2} + \frac{1}{1+\theta}\right). \qquad \text{(FIVE.12)}$$

2. $(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!k!}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(x+k+2)}{\theta^{x+k+2}} + \frac{\Gamma(x+k+1)}{\theta^{x+k+1}}\right] = (1-\rho)\frac{\theta^2}{1+\theta}\left[\frac{2+k+\theta}{(1+\theta)^{k+2}}\right].$

Proof. For the first identity,

$$\frac{1}{\theta^2}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+2)}{x!}\frac{1}{\theta^x} + \frac{1}{\theta}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+1)}{x!}\frac{1}{\theta^x}$$

$$
\begin{aligned}
&= \frac{\Gamma 2}{\theta^2}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+2)}{x!\Gamma 2\theta^x}+\frac{1}{\theta}\sum_{x=0}^{\infty}(-1)^x\left(\frac{1}{\theta}\right)^x \\
&= \frac{1\Gamma 1}{\theta^2}\left(1+\frac{1}{\theta}\right)^{-2}+\frac{1}{\theta}\left(1+\frac{1}{\theta}\right)^{-1} \\
&= \frac{1}{(1+\theta)^2}+\frac{1}{1+\theta},
\end{aligned}
$$

while for the second identity,

$$
\begin{aligned}
&\frac{\Gamma(k+2)}{\theta^{k+2}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+k+2)}{\Gamma(k+2)x!k!}\frac{1}{\theta^x}+\frac{\Gamma(k+1)}{\theta^{k+1}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+k+1)}{\Gamma(k+1)x!}\frac{1}{\theta^x} \\
&= = \frac{\Gamma(k+2)}{k!\theta^{k+2}}\sum_{x=0}^{\infty}(-1)^x\binom{x+k+2-1}{x}\left(\frac{1}{\theta}\right)^x+\frac{\Gamma(k+1)}{k!\theta^{k+1}}\sum_{x=0}^{\infty}(-1)^x\binom{x+k+1-1}{x}\left(\frac{1}{\theta}\right)^x \\
&= \frac{\Gamma(k+2)}{k!\theta^{k+2}}\sum_{x=0}^{\infty}\binom{-(k+2)}{x}\left(\frac{1}{\theta}\right)^{k+2}+\frac{\Gamma(k+1)}{k!\theta^{k+1}}\sum_{x=0}^{\infty}\binom{-(k+1)}{x}\left(\frac{1}{\theta}\right)^{k+1} \\
&= \frac{\Gamma(k+2)}{k!\theta^{k+2}}\left(1+\frac{1}{\theta}\right)^{-(k+2)}+\frac{\Gamma(k+1)}{k!\theta^{k+1}}\left(1+\frac{1}{\theta}\right)^{-(k+1)} \\
&= \frac{(k+1)k!}{k!\theta^{k+2}}\left(\frac{\theta}{1+\theta}\right)^{k+2}+\frac{\Gamma(k+1)}{k!\theta^{k+1}}\left(\frac{\theta}{1+\theta}\right)^{k+1} \\
&= \frac{2+k+\theta}{(1+\theta)^{k+2}}.
\end{aligned}
$$

□

5.3 Exponential Distribution

The Laplace Transform of a ZIG distribution is

$$
\begin{aligned}
L(s) &= (1-\rho)\mu\int_0^{\infty}e^{-(s+\mu)\lambda}d\lambda \\
&= (1-\rho)\frac{\mu}{s+\mu}[-e^{(s+\mu)\lambda}|_0^{\infty}] \\
&= (1-\rho)\frac{\mu}{s+\mu}
\end{aligned}
$$

On getting the first three derivatives of the Laplace Transform and then generalizing for the k^{th} derivation,

$$\begin{aligned} L'(s) &= (1-\rho)[\frac{-\mu}{(s+\mu)^2}] \\ L''(s) &= (1-\rho)\frac{2\mu}{(s+\mu)^3} \\ L'''(s) &= (1-\rho)\left[\frac{-6\mu}{(s+\mu)^4}\right] \end{aligned}$$

Thus the k^{th} derivative will be

$$L^k(s) = \begin{cases} (1-\rho)[\frac{(-1)^k k!\mu}{(s+\mu)^{k+1}}], & \text{k=0;} \\ (1-\rho)[\frac{(-1)^k k!\mu}{(s+\mu)^{k+1}}], & \text{k=1, 2, \dots.} \end{cases} \quad \text{(FIVE.13)}$$

When $s = 1$, then for a generalized two cases, we have

$$L^k(1) = \begin{cases} (1-\rho)[\frac{(-1)^k k!\mu}{(1+\mu)^{k+1}}], & \text{k=0;} \\ (1-\rho)[\frac{(-1)^k k!\mu}{(1+\mu)^{k+1}}], & \text{k=1, 2, \dots.} \end{cases}$$

Now,

$$Pr(Y=k) = \begin{cases} (1-\rho)[\frac{(-1)^k k!\mu}{(1+\mu)^{k+1}}\frac{(-1)^k}{k!}], & \text{k=0;} \\ (1-\rho)[\frac{(-1)^k k!\mu}{(1+\mu)^{k+1}}\frac{(-1)^k}{k!}], & \text{k=1, 2, \dots.} \end{cases} \quad \text{(FIVE.14)}$$

$$= \begin{cases} \rho + (1-\rho)\frac{\mu}{(1+\mu)}, & \text{k=0;} \\ (1-\rho)(\frac{\mu}{(1+\mu)})(\frac{1}{1+\mu})^k, & \text{k=1, 2, \dots.} \end{cases} \quad \text{(FIVE.15)}$$

Using equation (5.5),

$$G_\lambda(s) = \begin{cases} (1-\rho)\frac{\mu}{\mu+1-s}, & \text{k=0;} \\ (1-\rho)\frac{\mu}{(1+\mu-s)}, & \text{k=1, 2, \dots.} \end{cases}$$

$$= \begin{cases} (1-\rho)\frac{\mu}{(1+\mu)[\frac{1+\mu-s}{1+\mu}]}, & \text{k=0;} \\ (1-\rho)\frac{\mu}{(1+\mu)[\frac{1+\mu-s}{1+\mu}]}, & \text{k=1, 2, \dots.} \end{cases}$$

$$= \begin{cases} (1-\rho)\frac{\mu}{(1+\mu)}\left[1-\frac{s}{1+\mu}\right]^{-1}, & \text{k=0;} \\ (1-\rho)\frac{\mu}{(1+\mu)}\left[1-\frac{s}{1+\mu}\right]^{-1}, & \text{k=1, 2, \dots.} \end{cases}$$

$$= \begin{cases} (1-\rho)\frac{\mu}{(1+\mu)}\sum_{k=0}^{\infty}\left[\frac{s}{1+\mu}\right]^{k}, & \text{k=0;} \\ (1-\rho)\frac{\mu}{(1+\mu)}\sum_{k=0}^{\infty}\left[\frac{s}{1+\mu}\right]^{k}, & \text{k=1, 2, \dots.} \end{cases}$$

$$= \begin{cases} (1-\rho)\frac{\mu}{(1+\mu)}\sum_{k=0}^{\infty}\left[\frac{s}{1+\mu}\right]^{k}, & \text{k=0;} \\ (1-\rho)\frac{\mu}{(1+\mu)}\left(\frac{s}{1+\mu}\right)\sum_{k=0}^{\infty}\left[\frac{s}{1+\mu}\right]^{k}, & \text{k=1, 2, \dots.} \end{cases}$$

$$= \begin{cases} (1-\rho)\frac{\mu}{(1+\mu)}, & \text{k=0;} \\ (1-\rho)\frac{\mu}{(1+\mu)}\left(\frac{1}{1+\mu}\right)^{k}, & \text{k=1, 2, \dots.} \end{cases}$$

which is the same as equation 5.7.

5.4 Gamma with one parameter

The distribution of Gamma is given by

$$g(\lambda) = \frac{e^{-\lambda}\lambda^{\alpha-1}}{\Gamma\alpha}, \ \lambda > 0, \ \alpha > 0,$$

with a Laplace Transform of the form

$$\begin{aligned} L(s) &= \frac{(1-\rho)}{\Gamma\alpha}\int_0^\infty e^{-(1+s)\lambda}\lambda^{\alpha-1})d\lambda, \\ &= (1-\rho)\frac{1}{(1+s)^\alpha} \end{aligned}$$

On successively differentiating the Laplace Transform, we have

$$\begin{aligned} L'(s) &= (1-\rho)[\frac{-\alpha}{(1+s)^{\alpha+1}}] \\ L''(s) &= (1-\rho)[\frac{-\alpha}{(1+s)^{\alpha+1}}] \\ L'''(s) &= (1-\rho)[\frac{-\alpha(\alpha+1)(\alpha+2)}{(1+s)^{\alpha+3}}] \end{aligned}$$

On generalizing,

$$L^k(s) = (1-\rho)\left[\frac{(-1)^k(\alpha+k-1)!}{(\alpha-1)!}\frac{1}{(1+s)^{\alpha+k}}\right], \text{ k=0, 1, 2, \ldots.}$$

When $s = 1$

$$L^k(1) = \begin{cases} (1-\rho)\left[\frac{1}{2^{\alpha+k}}\right], & \text{k=0;} \\ (1-\rho)\left[\frac{(-1)^k(\alpha+k-1)!}{(\alpha-1)!}\frac{1}{2^{\alpha+k}}\right], & \text{k=1, 2, \ldots.} \end{cases}$$

Using equation 5.2., the ZIP mixture distribution can be rewritten as

$$Pr(Y=k) = \begin{cases} (1-\rho)\left[\frac{(-1)^k(\alpha+k-1)!}{(\alpha-1)!}\frac{1}{2^{\alpha+k}}\frac{(-1)^k}{k!}\right], & \text{k=0;} \\ (1-\rho)\left[\frac{(-1)^k(\alpha+k-1)!}{(\alpha-1)!}\frac{1}{2^{\alpha+k}}\frac{(-1)^k}{k!}\right], & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(FIVE.16)}$$

$$= \begin{cases} (1-\rho)\left[\frac{1}{2^{\alpha+k}}\right], & \text{k=0;} \\ (1-\rho)\left[\frac{(\alpha+k-1)!}{(\alpha-1)!k!}\frac{1}{2^{\alpha+k}}\right], & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(FIVE.17)}$$

$$= \begin{cases} \rho + (1-\rho)\frac{1}{2^{\alpha}}, & \text{k=0;} \\ (1-\rho)\begin{pmatrix} \alpha+k-1 \\ k \end{pmatrix}\left(\frac{1}{2}\right)^{\alpha}\left(\frac{1}{2}\right)^{k}, & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(FIVE.18)}$$

From equation 5.5, the pgf of Y can be written as

$$
\begin{aligned}
G_k(s) &= \begin{cases} (1-\rho)\frac{1}{(1+1-s)^\alpha}, & \text{k=0;} \\ (1-\rho)\frac{1}{(1+1-s)^\alpha}, & \text{k=1, 2, \dots.} \end{cases} \\
&= \begin{cases} (1-\rho)\frac{1}{2^\alpha(1-\frac{s}{2})^\alpha}, & \text{k=0;} \\ (1-\rho)\frac{1}{2^\alpha(1-\frac{s}{2})^\alpha}, & \text{k=1, 2, \dots.} \end{cases} \\
&= \begin{cases} (1-\rho)\frac{1}{2^\alpha}\sum_{k=0}^\infty \begin{pmatrix} -\alpha \\ k \end{pmatrix} \left(\frac{s}{2}\right)^k, & \text{k=0;} \\ (1-\rho)\frac{1}{2^\alpha}\sum_{k=0}^\infty \begin{pmatrix} -\alpha \\ k \end{pmatrix} \left(\frac{s}{2}\right)^k, & \text{k=1, 2, \dots.} \end{cases} \\
&= \begin{cases} (1-\rho)\frac{1}{2^\alpha}, & \text{k=0;} \\ (1-\rho)\frac{1}{2^\alpha}\sum_{k=0}^\infty (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} \left(\frac{1}{2}\right)^k, & \text{k=1, 2, \dots.} \end{cases} \\
&= \begin{cases} (1-\rho)\frac{1}{2^\alpha}, & \text{k=0;} \\ (1-\rho)\frac{1}{2^{\alpha+k}}\sum_{k=0}^\infty (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} \left(\frac{1}{2}\right)^k, & \text{k=1, 2, \dots.} \end{cases} \\
&= \begin{cases} (1-\rho)\frac{1}{2^\alpha}, & \text{k=0;} \\ (1-\rho) \begin{pmatrix} \alpha+k-1 \\ k \end{pmatrix} \left(\frac{1}{2}\right)^\alpha \left(\frac{1}{2}\right)^k, & \text{k=1, 2, \dots.} \end{cases}
\end{aligned}
$$

which is the same as equation 5.18.

5.5 Gamma with two parameters

The Gamma distribution is of the form

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta\lambda}\lambda^{\alpha-1},\ \lambda > 0,\ \alpha > 0,\ \beta > 0$$

The Laplace Transform of the mixing distribution is

$$\begin{aligned} L(s) &= (1-\rho)\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{-(s+\beta)\lambda}\lambda^{\alpha-1})d\lambda, \\ &= (1-\rho)\frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma\alpha}{(s+\beta)^\alpha} \\ &= (1-\rho)\left(\frac{\beta}{s+\beta}\right)^\alpha \end{aligned}$$

On finding the first three derivatives,

$$\begin{aligned} L'(s) &= (1-\rho)\left[\beta^\alpha\frac{-\alpha}{(s+\beta)^{\alpha+1}}\right] \\ L''(s) &= (1-\rho)\beta^\alpha\frac{\alpha(\alpha+1)}{(s+\beta)^{\alpha+2}} \\ L'''(s) &= (1-\rho)\beta^\alpha\left[\frac{-\alpha(\alpha+1)(\alpha+2)}{(s+\beta)^{\alpha+3}}\right], \text{ k=1, 2,} \ldots \end{aligned}$$

On generalizing to the k^{th}, derivative

$$L^k(s) = \begin{cases} (1-\rho)\beta^\alpha\left[\frac{1}{(s+\beta)^\alpha}\right], & \text{k=0;} \\ (1-\rho)\beta^\alpha\left[\frac{(-1)^k(\alpha+k-1)!}{(\alpha-1)!}\frac{1}{(s+\beta)^{\alpha+k}}\right], & \text{k=1, 2,} \ldots \end{cases}$$

On replacing s with 1, then

$$L^k(1) = \begin{cases} (1-\rho)\beta^\alpha\left[\frac{1}{(1+\beta)^{\alpha+k}}\right], & \text{k=0;} \\ (1-\rho)\beta^\alpha\left[\frac{(-1)^k(\alpha+k-1)!}{(\alpha-1)!}\frac{1}{(1+\beta)^{\alpha+k}}\right], & \text{k=1, 2,} \ldots \end{cases}$$

From equation (5.2), the mixed distribution can then be expressed as

$$Pr(Y=k) = \begin{cases} (1-\rho)\frac{\beta^\alpha}{(1+\beta)^\alpha}, & \text{k=0;} \\ (1-\rho)\beta^\alpha\left[\frac{(-1)^k(\alpha+k-1)!}{k!(\alpha-1)!}\frac{(-1)^k}{(1+\beta)^{\alpha+k}}\right], & \text{k=1, 2,} \ldots \end{cases} \quad \text{(FIVE.19)}$$

$$= \begin{cases} (1-\rho)\frac{\beta^\alpha}{(1+\beta)^\alpha}, & \text{k=0;} \\ (1-\rho)\begin{pmatrix} \alpha+k-1 \\ k \end{pmatrix}\left(\frac{\beta}{1+\beta}\right)^\alpha\left(\frac{1}{1+\beta}\right)^k, & \text{k=1, 2,} \ldots \end{cases} \quad \text{(FIVE.20)}$$

Using the Laplace Transform of the Gamma distribution;

$$
\begin{aligned}
G_k(s) &= \begin{cases} (1-\rho)\left(\frac{\beta}{\beta+1-s}\right)^{\alpha}, & \text{k=0;} \\ (1-\rho)\left(\frac{\beta}{\beta+1-s}\right)^{\alpha}, & \text{k=1, 2, ...} \end{cases} \\
&= \begin{cases} (1-\rho)\left(\frac{\beta}{(1+\beta)[1-\frac{s}{1+\beta}]}\right)^{\alpha}, & \text{k=0;} \\ (1-\rho)\left(\frac{\beta}{(1+\beta)[1-\frac{s}{1+\beta}]}\right)^{\alpha}, & \text{k=1, 2, ...} \end{cases} \\
&= \begin{cases} (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left[1-\frac{s}{1+\beta}\right]^{-\alpha}, & \text{k=0;} \\ (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left[1-\frac{s}{1+\beta}\right]^{-\alpha}, & \text{k=1, 2, ...} \end{cases} \\
&= \begin{cases} (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}\sum_{k=0}^{\infty}\begin{pmatrix} -\alpha \\ k \end{pmatrix}\left(\frac{s}{1+\beta}\right)^{k}, & \text{k=0;} \\ (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}\sum_{k=1}^{\infty}(-1)^k\begin{pmatrix} \alpha \\ k \end{pmatrix}\left(\frac{1}{1+\beta}\right)^{k}, & \text{k=1, 2, ...} \end{cases} \\
&= \begin{cases} (1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}, & \text{k=0;} \\ (1-\rho)\begin{pmatrix} \alpha+k-1 \\ k \end{pmatrix}\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left(\frac{1}{1+\beta}\right)^{k}, & \text{k=1, 2, ...} \end{cases}
\end{aligned}
$$

which is a ZINB with parameters α, ρ, and$\frac{\beta}{1+\beta}$ which is the same as equation 5.20.

5.6 Summary

The ZIP mixture distributions were represented interms of the expectation of the r^{th} moment of the underlying distribution. These distributions were found to be identical to those constructed explicitly and identities proved thereof. Even though, the Laplace transform is also a special function, it had to be used in this chapter because the identities were to be proved.

CHAPTER SIX

ZERO-INFLATED POISSON MIXTURES IN TERMS OF SPECIAL FUNCTIONS

6.1 Introduction

This chapter entails expressing of Zero Inflated Poisson mixture distributions in terms of Hypergeometric functions, that is; the Confluent and Gauss Hypergeometric functions with their construction.

6.2 Mixing with Inverted Beta distribution

The Scaled Beta distribution is

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{(1+\lambda)^{\alpha+\beta}B(\alpha,\,\beta)},\ 0<\lambda<\infty,\ \alpha,\ \beta>0$$

The ZIP mixed distribution is constructed as follows

$$Pr(Y=k) = \begin{cases} \int_0^\infty \rho + (1-\rho)e^{-\lambda}\frac{\lambda^{\alpha-1}}{(1+\lambda)^{\alpha+\beta}B(\alpha,\beta)}d\lambda, & \text{k=0;} \\ \int_0^\infty (1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}\frac{\lambda^{\alpha-1}}{(1+\lambda)^{\alpha+\beta}B(\alpha,\beta)}d\lambda, & \text{k=1, 2, \dots.} \end{cases} \tag{SIX.1}$$

$$= \begin{cases} \frac{(1-\rho)}{B(\alpha,\beta)}\int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha-1}}{(1+\lambda)^{\alpha+\beta}}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!B(\alpha,\beta)}\int_0^\infty \frac{e^{-\lambda}\lambda^{k+\alpha-1}}{(1+\lambda)^{\alpha+\beta}}d\lambda, & \text{k=1, 2, \dots.} \end{cases} \tag{SIX.2}$$

Consider a Confluent Hypergeometric distribution of the second kind, given by

$$\Psi(a,\, c,\, 1) = \frac{1}{\Gamma a}\int_0^\infty \frac{e^{-t}t^{a-1}}{(1+t)^{a-c+1}}dt$$

$$\Psi(k+\alpha,\, c,\, 1) = \frac{1}{\Gamma(k+\alpha)}\int_0^\infty \frac{e^{-t}t^{k+\alpha-1}}{(1+t)^{k+\alpha-c+1}}dt \tag{SIX.3}$$

Comparing equation (6.2) and equation (6.3)

$$\begin{aligned} \alpha+\beta &= k+\alpha-c+1 \\ \beta &= k-c+1 \\ c &= k-\beta+1 \end{aligned}$$

This implies that

$$\begin{aligned} \Psi(k+\alpha,\, k-\beta+1,\, 1) &= \frac{1}{\Gamma(k+\alpha)}\int_0^\infty \frac{e^{-\lambda}\lambda^{k+\alpha-1}}{(1+t)^{k+\alpha-k+\beta-1+1}}dt \\ &= \frac{1}{\Gamma(k+\alpha)}\int_0^\infty \frac{e^{-\lambda}\lambda^{k+\alpha-1}}{(1+t)^{\alpha+\beta}}dt \end{aligned}$$

Hence equation(6.2) can be written as

$$f(k) = \begin{cases} (1-\rho)B(\alpha,\,\beta)\frac{1}{\Gamma\alpha}\int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha-1}}{(1+\lambda)^{\alpha+\beta}}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!B(\alpha,\beta)}\frac{1}{\Gamma(k+\alpha)}\int_0^\infty \frac{e^{-\lambda}\lambda^{k+\alpha-1}}{(1+\lambda)^{\alpha+\beta}}d\lambda, & \text{k=1, 2, \dots.} \end{cases}$$

$$= \begin{cases} (1-\rho)B(\alpha,\,\beta)\Psi(\alpha,\, k-\beta+1,\, 1), & \text{k=0;} \\ \frac{(1-\rho)}{k!B(\alpha,\beta)}\Psi(k+\alpha,\, 1-\beta,\, 1), & \text{k=1, 2, \dots.} \end{cases}$$

which is a confluent hypergeometric function.

6.3 Mixing with Lomax Distribution

The pdf of the Lomax distribution is

$$g(\lambda) = \frac{\alpha\beta^{\alpha}}{(\lambda+\beta)^{\alpha+1}},\ \alpha > 0,\ \beta > 0,\ \lambda > 0$$

The ZIP mixture distribution is derived as follows

$$Pr(Y=k) = \begin{cases} (1-\rho)\alpha\beta^{\alpha}\int_0^{\infty}\frac{e^{-\lambda}}{(\lambda+\beta)^{\alpha+1}}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)\alpha\beta^{\alpha}}{k!}\int_0^{\infty}\frac{e^{-\lambda}\lambda^k}{(\lambda+\beta)^{\alpha+1}}d\lambda, & \text{k}=1,2,\ldots. \end{cases} \quad \text{(SIX.4)}$$

$$= \begin{cases} (1-\rho)\alpha\beta^{\alpha}\int_0^{\infty}\frac{e^{-\lambda}}{[\beta(\frac{\lambda+\beta}{\beta})]^{\alpha+1}}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)\alpha\beta^{\alpha}}{k!}\int_0^{\infty}\frac{e^{-\lambda}\lambda^k}{[\beta(\frac{\lambda+\beta}{\beta})]^{\alpha+1}}d\lambda, & \text{k}=1,2,\ldots. \end{cases} \quad \text{(SIX.5)}$$

$$= \begin{cases} (1-\rho)\alpha\beta^{\alpha}\int_0^{\infty}\frac{e^{-\lambda}}{\beta^{\alpha+1}[1+\frac{\lambda}{\beta}]^{\alpha+1}}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)\alpha\beta^{\alpha}}{k!}\int_0^{\infty}\frac{e^{-\lambda}\lambda^k}{\beta^{\alpha+1}[1+\frac{\lambda}{\beta}]^{\alpha+1}}d\lambda, & \text{k}=1,2,\ldots. \end{cases} \quad \text{(SIX.6)}$$

Let $\frac{\lambda}{\beta} = t$ then $\lambda = \beta\, t$ and $d\lambda = \beta\, dt$, then equation (6.6) can be written as

$$Pr(Y=k) = \begin{cases} (1-\rho)\alpha\beta^{\alpha}\int_0^{\infty}\frac{e^{-\beta\, t}}{\beta^{\alpha+1}[1+t]^{\alpha+1}}\beta\, dt, & \text{k=0;} \\ \frac{(1-\rho)\alpha\beta^{\alpha}}{k!}\int_0^{\infty}\frac{e^{-\beta\, t}(\beta\, t)^k}{\beta^{\alpha+1}[1+t]^{\alpha+1}}\beta\, dt, & \text{k}=1,2,\ldots. \end{cases} \quad \text{(SIX.7)}$$

$$= \begin{cases} (1-\rho)\alpha\int_0^{\infty}\frac{e^{-\beta\, t}}{(1+t)^{\alpha+1}}dt, & \text{k=0;} \\ \frac{(1-\rho)\alpha\beta^{k}}{k!}\int_0^{\infty}\frac{e^{-\beta\, t}t^{k+1-1}}{(1+t)^{\alpha+1}}dt, & \text{k}=1,2,\ldots. \end{cases} \quad \text{(SIX.8)}$$

Comparing equation (6.8) with equation (6.3), where

$$\begin{aligned} \alpha+1 &= k+\alpha-c+1 \\ k &= c \end{aligned}$$

equation (6.8) then becomes

$$Pr(Y=k)=\begin{cases}(1-\rho)\alpha\Psi(1,\,0,\,\beta), & \text{k=0;}\\ (1-\rho)\alpha\beta^{k}\Psi(k+1,\,k,\,\beta), & \text{k=1,2,....}\end{cases}$$

which is a confluent hypergeometric distribution.

6.4 Mixing with Scaled Beta Distribution

The mixing distribution is

$$g(\lambda)=\frac{\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}}{B(\alpha,\,\beta)\mu^{\alpha+\beta-1}},0\leq\lambda<\mu$$

The mixed distribution is

$$Pr(Y=k)=\begin{cases}(1-\rho)\int_0^{\mu}\frac{e^{-\lambda}\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}}{B(\alpha,\beta)\mu^{\alpha+\beta-1}}d\lambda, & \text{k=0;}\\ (1-\rho)\int_0^{\mu}\frac{e^{-\lambda}\lambda^{k+\alpha-1}(\mu-\lambda)^{\beta-1}}{k!B(\alpha,\beta)\mu^{\alpha+\beta-1}}d\lambda, & \text{k=1,2,....}\end{cases}$$

The pgf of k is

$$G_k(s)=\int_0^{\mu}\frac{e^{-\lambda(s-1)}\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}}{B(\alpha,\,\beta)\mu^{\alpha+\beta-1}}d\lambda$$

Let $\lambda=\mu\,t$ then $d\lambda=\mu\,dt$ therefore the pgf becomes

$$\begin{aligned}G_k(s) &= \int_0^{\mu}\frac{e^{-\mu(s-1)t)}(\mu\,t)^{\alpha-1}(\mu-\mu\,t)^{\beta-1}}{B(\alpha,\,\beta)\mu^{\alpha+\beta-1}}\mu\,dt\\ &= \int_0^{1}\frac{e^{\mu(1-s)t)}t^{\alpha-1}(\mu-t)^{\beta-1}}{B(\alpha,\,\beta)}dt\\ &= {}_1F_1\{\alpha,,\alpha+\beta,\,\mu(s-1)\}\end{aligned}$$

which is a Confluent Hypergeometric function.

6.5 Summary

Some ZIP mixture distributions could not be constructed by using the three methods earlier discussed instead, they were represented by using the Confluent hypergeometric functions of first and second kind.

CHAPTER SEVEN

APPLICATION TO FERTILITY DATA

7.1 Introduction

The terminology "fertility" refers to the number of children a woman of childbearing ages from 15 to 49 years is able to give birth to in her lifetime. Fertility is a one of the main components of a population and it contributes to the changes in size, structure, and composition of the population in any country. In this chapter, a Zero Inflated Geometric distribution was fitted to the data. The ZIG distribution was found to be the most suitable because it belongs to the distributions in the density domain of $[0, \infty]$. The number of children is a discrete and count variable therefore it assumes only positive values. The table shows the descriptive measures of the data

7.2 Estimation of parameters

The data was extracted from the Kenya Demographic Health Survey 2014 database. It is a record on the number of children in each household. When descriptive analysis was carried out the data, it was clear that it is highly skewed, which necessitated the use of a zero inflated distribution. Moreover, there was evidence of overdispersion.

There were 16967 households with a portion of 70% having no children. The highest number of children was seven.

The estimators were estimated using the method of moments as follows; The Zero Inflated distribution has the probability density function given by

$$Prob(Y = k) = \begin{cases} (1-\rho)\frac{\mu}{(1+\mu)}, & \text{k=0;} \\ (1-\rho)\left(\frac{\mu}{1+\mu}\right)\left(\frac{1}{1+\mu}\right)^k, & \text{k=1, 2, \ldots.} \end{cases} \quad \text{(SEVEN.1)}$$

For a ZIG distribution, the first and second central moments are given by

$$\mu_1' = (1-\rho)\frac{1}{\mu}, \text{ and } \mu_2' = (1-\rho)\left[\frac{2}{\mu^2} + \frac{1}{\mu}\right].$$

while the sample moments are

$$m_1' = \frac{\sum_{i=1}^n y_i}{n} \text{ and } m_2' = \frac{\sum_{i=1}^n y_i^2}{n}.$$

Further, the first central moment is equivalent to the first sample moment. This means that $\mu_1 = m_1'$, which implies that

$$(1-\rho)\frac{1}{\mu} = \bar{y} \quad \text{(SEVEN.2)}$$

and

$$\frac{\sum_{i=1}^n y_i^2}{n} = (1-\rho)\left[\frac{2}{\mu^2} + \frac{1}{\mu}\right] \quad \text{(SEVEN.3)}$$

On replacing equation 7.2 into equation 7.3 yielded

$$\sum_{i=1}^n y_i^2 = (1-\rho)\frac{2n}{\mu^2} + n\bar{y},$$

which was further simplified to get the estimator of ρ as

$$\hat{\rho} = 1 - \frac{\hat{\mu}^2}{2}\left[\frac{\sum_{i=1}^n y_i^2}{n} - \bar{y}\right]. \quad \text{(SEVEN.4)}$$

while that of μ was given by

$$\hat{\mu} = \frac{2}{\left[\frac{\sum_{i=1}^{n} y_i^2}{n\bar{y}} + 1\right]}. \qquad \text{(SEVEN.5)}$$

From the data, the mean number of children, $\bar{y} = 0.707796$. This is due to the fact that there are so many households with no children. On substituting this value in equation 7.5, yielded a value of 0.69622 as the estimate of $\hat{\mu}$ while $\hat{\rho} = 0.850303$. Therefore the fitted model was

$$Prob(Y = k) = \begin{cases} 0.06144, & \text{k=0;} \\ 0.06144(0.5896)^k, & \text{k=}1, 2, \ldots, 7 \end{cases} \qquad \text{(SEVEN.6)}$$

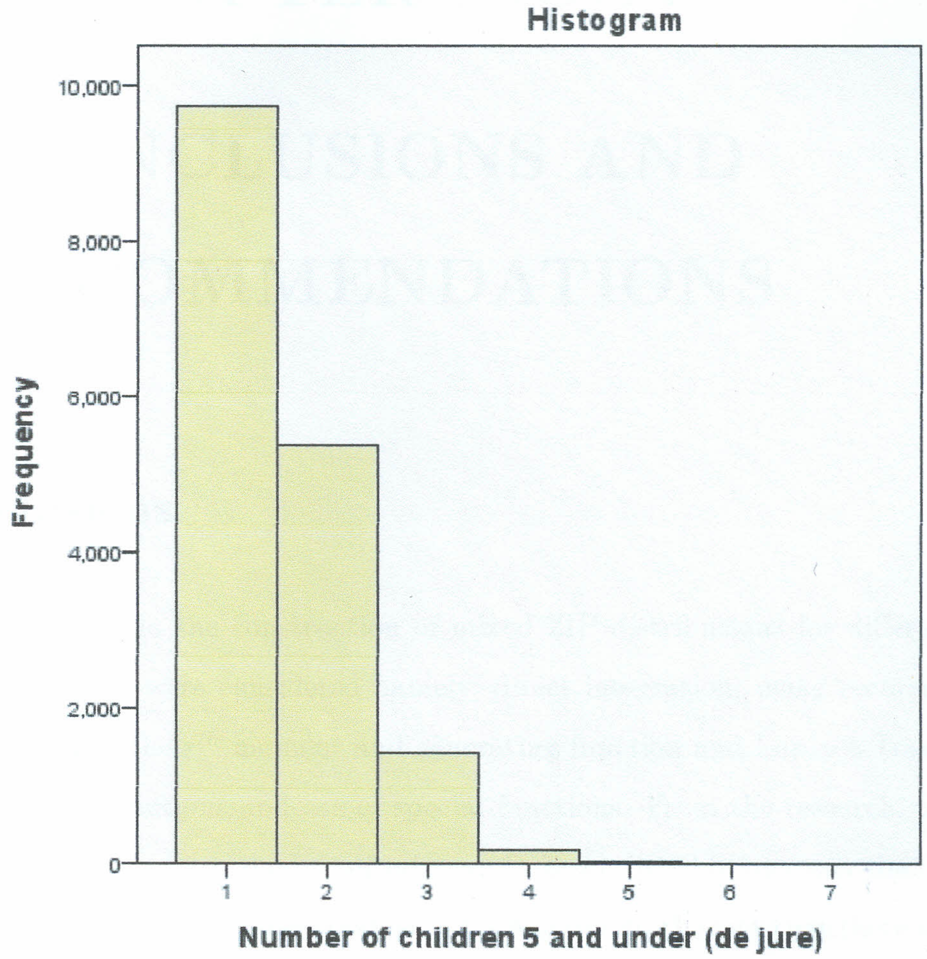
7.3 Summary

The shape of the graph clearly shows that the mixed models constructed are appropriate for modeling fertility data because of the diminishing tail.

Statistics

Number of children 5 and under (de jure)

N	Valid	36430
	Missing	0
Mean		.71
Median		.00
Std. Deviation		.908
Variance		.825
Skewness		1.189
Std. Error of Skewness		.013


Histogram
10,000
8,000
6,000
4,000
2,000
0
Frequency
1
2
3
4
5
6
7
Number of children 5 and under (de jure)

CHAPTER EIGHT

CONCLUSIONS AND RECOMMENDATIONS

8.1 Conclusions

Four methods were used in the construction of mixed ZIP distributions for different mixing distributions that were considered namely; direct integration; using recursive formulae; expectation of the r^{th} moment and generating function and Laplace Transform of the mixing distributions and use of special functions. From the research, the method that resulted in a good number of mixture Poisson distributions was that of recursive representation of the mixture models. This was due to the fact that there are no conditions that need to be considered and hence it is straightforward. Some ZIP mixture distributions were obtained by using more than one method. For instance, a ZIG, could be obtained explicitly or by use of a pgf of a one parameter exponential distribution. However, some mixture distributions could not be constructed by direct integration, for example the ZIP Lomax distribution could only be expressed by use of special functions. A Zero Inflated Geometric function parameters were also estimated and applied on fertility data which was obtained from the KNDS 2014 survey.

8.2 Recommendations

When a distribution is derived, it is paramount that it is tested on a set of data and tests-of-goodness of fit carried out. This is only possible when there is a way in which the maximum likelihood estimators can be tested. During the application on the fertility data, parameter estimates were estimated by the method of moments. Other methods like the maximum likelihood may also be used and applications done on sets of real data for instance on a claims data. Since all the available continuous distributions were not exhausted, therefore more work can be done by considering the distributions which we did not use, by using the methods of construction already used and also other methods can be studied or researched on. Mixed Poisson distributions exhibit several interesting properties as given by, (read) [8]. More research can be done on the properties of the mixed ZIP distributions constructed in this work. The research concentrated on the construction of ZIP continuous mixing distributions, even though further research on discrete or countable mixtures where the parameter of the base model follow discrete. It would be interesting to know the parameters as well as the structure of the resultant mixture distributions. There is therefore need to examine the following for each continuous mixture ZIP distributions obtained:

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