LIE SYMMETRY SOLUTIONS OF THE GENERALIZED BURGERS EQUATION

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ABSTRACT

Burgers equation: $u_t + uu_x = \lambda u_{xx}$ is a nonlinear partial differential equation which arises in model studies of turbulence and shock wave theory. In physical application of shock waves in fluids, coefficient λ , has the meaning of viscosity. For light fluids or gases the solution considers the inviscid limit as λ tends to zero. The solution of Burgers equation can be classified into two categories: Numerical solutions using both finite difference and finite elements approaches; the analytic solutions found by Cole and Hopf. In both cases the solutions have been valid for only $0 \leq \lambda \leq 1$. In this thesis, we have found a global solution to the Burgers equation with no restriction on λ i.e. $\lambda \in (-\infty, \infty)$. In pursuit of our objective, we have used, the Lie symmetry analysis. The method includes the development of infinitesimal transformations, generators, prolongations, and the invariant transformations of the Burgers equation. We have managed to determine all the Lie groups admitted by the Burgers equation, and used the symmetry transformations to establish all the solutions corresponding to each Lie group admitted by the equation. These solutions, which are appearing in literature for the first time are more explicit and more general than those previously obtained. This is a big contribution to the mathematical knowledge in the application of Burgers equation.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Lie group analysis is a mathematical theory that synthesizes symmetry of differential equations. This theory was originated by a great Norwegian mathematician of the nineteenth century known as Sophus Lie[13]. Lie pioneered the use of groups of transformations called Lie groups in the study of symmetry properties of differential equations with a view to their solutions. He discovered that the known *ad hoc* methods of integration of differential equations could easily be derived by his theory of continuous groups. He further, among other things, gave a classification of differential equations in terms of their symmetry groups, thereby identifying the set of equations which could be integrated or reduced to lower- order equations by group theoretic arguments. Lie's basic idea was to find all the Lie groups of a given partial differential equation (PDE) such that any solution of this PDE is transformed into another solution by the coordinate transformations of the respective Lie groups; i.e. all the groups with respect to which the set of solutions of the PDE is invariant. The solutions which results from this procedure are generally referred to as Lie symmetry solutions.

In this study we apply Lie symmetry analysis in the solution of the Burgers equation:

$$u_t + uu_x = \lambda u_{xx}. \qquad (1.1.0)$$

The Burgers equation mentioned above is one of the most difficult nonlinear PDE to be solved analytically. The equation appears in various physical applications. For example it models weak shock waves in compressible fluid dynamics. It is a one-space dimension version of Nevier Stoke's equations of fluid dynamics.

1.2 Statement of The Problem

The exact solutions of the Burgers equation:

$$u_t + uu_x = \lambda u_{xx}$$

can be classified in two groups as:

a)Numerical solutions

i) Finite Difference

ii) Finite Elements

b) Analytic solutions.

In numerical solutions, the values of the constant λ are restricted to $\lambda \in [0,1]$.

In fact when $\lambda < 0.001$, computation by means of the exact solution is not practical because of the slow convergence of the Fourier series Ames[2].The analytic solution has so far only been givev for $\lambda = 1$, Hopf and Cole [5] , Lamb[12], Raunch [22],and Gandarias [6].

In the physical application of shock waves in fluids, λ has the meaning of viscosity. Thus the solution considers the inviscid limit i.e., ($\lambda \to 0$) of the Burgers equation.

An attempt has also been made by Gandarias [6] to obtain potential symmetries for the Burgers equation and the corresponding local solution for only , $\lambda = 1$.

Popovych and Nataliya [21] obtained the infinitesimal symmetries for Burgers equation for only $\lambda = 1$. Both Mitchel and Griffins [15], and Roy [32] obtained stable numerical solution of the Burgers equation for $\lambda \in [0,1]$ and the numerical solution became unstable for the values of λ outside the interval $[0,1]$.

But it is not known what happens when $\lambda \to -\infty$ or when $\lambda \to \infty$.

To answer this question there is a need to find global solutions for $\lambda \in (-\infty, \infty)$.

Thus we have attempted in this study to solve the Burgers equation: $u_t + uu_x = \lambda u_{xx}$ for $\lambda \in (-\infty,\infty)$, analytically, using Lie symmetry analysis .

1.3 Objectives of The Study

The objective of this study was to find the global solution of the Burgers equation

$u_t + uu_x = \lambda u_{xx}$ for an arbitrary λ i.e. for $-\infty < \lambda < \infty$, using Lie symmetry analysis.

1.4 Significance of The Study

The results of this study provides an alternative method for solving the Burgers equation for $\lambda \in (-\infty,\infty)$ and other similar nonlinear partial differential equations. This is a significant contribution to the knowledge and further research.

1.5 Literature Review

The nonlinear algebraic theory of generalized solutions for large class of nonlinear PDE was originated by Rosinger[[23],[24]] who has since developed the theory further , culminating in the publication of four research monographs [[25],[26],[27],[28]]. In these monographs the algebraic theory, complete with applications in the study of nonlinear PDE, is well presented. Some of the major results obtained by Rosinger in this line of research include:

- The solution of the celebrated impossibility results of Schwatz [35] regarding the multiplication of distributions of Rosinger [30]
- The characterization of all possible nonlinear algebraic theories of generalized functions [29]
- The global solution of arbitrary nonlinear analytic PDE [27]

Lie symmetry groups for classical solutions of nonlinear PDE can be extended to symmetry groups for global generalized solutions. Nonlinear Lie group theory for global generalized solutions of nonlinear PDE was started by Rosinger [[27],[31]]. In collaboration with Michael Oberguggenberger [17] of Innsbruck, Austria, they have published a research monograph on the solution of continuous nonlinear PDEs through *order completion* Group Invariance of such global solutions have also been developed by ,Walus [38] and, Rudolph [33].

So far , some of the major results obtained by Rosinger and his collaborators are:

- The first nonlinear Lie Group theory of global generalized solutions of nonlinear PDEs
- Three solutions to Hilbert's fifth problem considered in its full generality
- The first solution of .Lewy problem of solvability of smooth PDEs.

The use of Lie Symmetry Analysis of differential equations in solving nonlinear PDEs was studied by Omolo-Ongati [19]. He particularly gave a stability approach to exact solutions of nonlinear PDEs provided by symmetry groups.

Ames, Lohner and aAdams [3] studied group properties the nonlinear wave equation

$u_{tt} = [f(x,u)u_x]_x$.

Torris and Valenti [37] studied the unperturbed nonlinear wave equation

$u_{tt} = [f(x,u)u_x]_x$. for $f. > 0, f_u \neq 0$. In their solution ,they assumed that $f. > 0, f_u \neq 0$.

But Omolo-Ongati [20] later showed that these assumptions made were unnecessary since the conditions present themselves naturally. He provided Lie symmetry solutions for approximate nonlinear hyperbolic equations.

Ibragimov [10] developed some invariant and symmetry solutions for heat equation and , for Burgers related equation but only for $\lambda = 1$. Hopf [7] studied Lie groups and their inversions as applied to PDEs. Stephani [36] made an attempt to solve the Burger's equation by first transforming the symmetry groups of the equivalent heat equation using Lie-Backund symmetry analysis. But in his method, the solution only consider the value of $\lambda \in [0,1]$ as in the numerical methods.

Lie symmetry solutions for nonlinear first order ordinary differential equations was developed for category of Abel's differential equations. Schwarz [34] gave an algorithm for computing infinitesimal symmetry for Abel's equation. Ibragimov [8] ,Ibragimov and Kolsrud [9] attempted to find some potential infinitesimal symmetries for an equation similar to the Burgers .No corresponding solutions were obtained.

The literature available shows that all the attempts, both numerical and analytic , to solve the Burgers equation have assumed the value of constant λ , between 0 and 1 i.e. $0 \le \lambda \le 1$.

In this study we have attempted to find exact solutions of the Burgers equation which is true for any real constant $\lambda : -\infty < \lambda < \infty$, using Lie symmetry analysis.

1.6 Research Methodology

The concept of Lie group theory has been used in solving the Burgers equation. These include : Lie groups of transformations; infinitesimal transformations; prolongations; infinitesimal generators and general applications of Lie groups to the solutions of differential equations. Finally, Lie invariant symmetry has been applied in obtaining exact solutions to the given Burgers equation.

CHAPTER 2

BASIC CONCEPTS

2.1 Lie Groups of Transformations

We first give the basic concept of a group.

Definition 2.1.1

A *group* G is a non empty set of mathematical elements with a composition ϕ defined between the elements satisfying the axioms:

(i) closure property:

$\forall x, y \in G, \phi(x, y) \in G..$

(ii) associative property:

$\forall x, y, z \in G, \phi(x, \phi(y, z)) = \phi(\phi(x, y), z) \in G.$

(iii) identity property:

$\exists!$ identity element $e \in G$ such that :

$\forall x \in G, \phi(e, x) = \phi(x, e) = x$

(iv) inverse property:

$\forall x \in G,$

$\exists!$ inverse element, $x^{-1} \in G, \phi(x^{-1}, x) = \phi(x, x^{-1}) = e$

Definition 2.1.2

Let

$x = (x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_m)$

lie in a region $D \subset R^m$.

Consider the set of transformations;

$$x^* = X(x,\varepsilon) \tag{2.1.1}$$

defined for each $x \in D$ depending on real parameter ε where $\varepsilon \in S \subset R$.

Suppose

$\phi(\varepsilon,\delta)$

defines a composition law of parameters ε , $\delta \in S$ then (2.1.1) forms a *group of transformations* on D if

(i) for each

$\varepsilon \in S,\ x^* \in D$

(ii) S with ϕ forms a group G.

(iii) $x^* = x$ when $\varepsilon = e$ i.e. $X(x,e) = x$

(iv) if $x^* = X(x,\varepsilon)$ and $x^{**} = X(x^*,\delta)$ then $x^{**} = X(x,\phi(\varepsilon,\delta))$

that is the group transformation from x to x^* via ε , followed by x^* to x^{**} via, δ is equivalent to a single transformation from x to x^{**} via $\phi(\varepsilon,\delta)$.

We say , a group of transformation which depends on a single real parameter ε defines , one-parameter (ε) Lie group of transformations if in addition

(v) ε is a continuous parameter i.e. $\varepsilon \in S$ is an interval in R.

(vi) X is infinitely differentiable with respect to x in D and $\varepsilon \in S$.

(vii) $\phi(\varepsilon,\delta)$ is C^∞ continuous .

Example 2.1.1

Consider a one-parameter (ε) group of transformations

$x^* = X(x,\varepsilon) = x + \varepsilon$

Checking for the properties of a one-parameter (ε) Lie group of transformations, we have

$$x^* = X(x,\varepsilon) = X(x,0) = x\,.$$

$$x^{**} = X(x^*,\delta) = x^* + \delta = x + (\varepsilon + \delta) = x + \phi(\varepsilon,\delta)$$

Clearly

$x^* = X(x,\varepsilon)$ defines a simple group on G.

Hence $x^* = X(x,\varepsilon)$ is a Lie group of transformations.

Example 2.1.2

For the two-dimensional group of transformations

$$x^* = X(x,y;\varepsilon) = \left(x + \varepsilon, \frac{xy}{x+\varepsilon}\right),$$

it is evident that

$$x^* = x + \varepsilon\,,$$

and

$$y^* = \frac{xy}{x+\varepsilon}$$

We therefore arrive at

$$x^{**} = X(x^*,\delta) = x^* + \delta = x + (\varepsilon + \delta) = x + \phi(\varepsilon,\delta),$$

$$y^{**} = \frac{x^*\,y^*}{x^* + \delta} = xy(x + \phi(\varepsilon,\delta))$$

and

$$X(x,y;0) = (x,y)..$$

Hence the transformation

$$X(x,y;\varepsilon) = \left(x + \varepsilon, \frac{xy}{x+\varepsilon}\right)$$

forms a Lie group of transformations.

2.2 Determination of Infinitesimal Transformations

Let

$x^* = X(x,\varepsilon)$

be a one-parameter (ε) Lie group of transformations with identity $\varepsilon = 0$ and law of composition ϕ.

Application of Taylor expansion about $\varepsilon = 0$ gives

$$x^* = X(x,\varepsilon) = X(x,0)x + \varepsilon\left[\frac{\partial X(x,\varepsilon)}{\partial \varepsilon}\right]_{\varepsilon=0} + \frac{\varepsilon^2}{2!}\left[\frac{\partial X(x,\varepsilon)}{\partial \varepsilon}\right]_{\varepsilon=0} + \ldots\ldots \tag{2.2.1}$$

If we let

$\xi(x) = \left[\frac{\partial X(x,\varepsilon)}{\partial \varepsilon}\right]_{\varepsilon=0}$ then (2.2.1) becomes

$$x^* = x + \varepsilon\xi(x) + {}^\circ(\varepsilon^2) \tag{2.2.2}$$

The transformations

$x^* = x + \varepsilon\xi(x)$ in (2.2.2) is known as the *infinitesimal transformation* of the one-parameter Lie group of transformations (2.1.1).

The components of $\xi(x)$ are called the *infinitesimals* of (2.1.1)

The symmetry of a group G is a Lie group of transformations which maps solutions into solutions, Stephani [36].

That is, image $\bar{y}(\bar{x})$ of any solution $y(x)$ of a differential equation is again a solution of the differential equation.

Example 2.2.1

Consider one-parameter (ε) Lie group of transformations

$x^* = x + \varepsilon x$

and

$$y^* = y + 2\varepsilon y + \varepsilon^2 y \quad : \qquad -1 < \varepsilon < \infty$$

We see that

$$X(x, y; \varepsilon) = (x^*, y^*) = \left((1+\varepsilon)x, (1+\varepsilon)^2 y\right)$$

and from (2.2.2) that the corresponding infinitesimal is given by

$$\xi(x, y) = \left[\frac{\partial X(x, y; \varepsilon)}{\partial \varepsilon}\right]_{\varepsilon=0}$$

$$= \left[\frac{\partial X(x^*, y^*)}{\partial \varepsilon}\right]_{\varepsilon=0} = \left[(x, 2(1+\varepsilon)y)\right]_{\varepsilon=0} = (x, 2y)$$

Lemma 2.2.1

$$X(x; \varepsilon + \Delta\varepsilon) = X(X(x; \varepsilon); \phi(\varepsilon^{-1}, \varepsilon + \Delta\varepsilon)) \qquad (2.2.3a).$$

Proof

$$\begin{aligned} X(X(x); \phi(\varepsilon^{-1}, \varepsilon + \Delta\varepsilon)) &= X(x; \phi(\varepsilon, \phi(\varepsilon^{-1}, \varepsilon + \Delta\varepsilon)) \\ &= X(x; \phi(\phi(\varepsilon, \varepsilon^{-1}), \varepsilon + \Delta\varepsilon)) \\ &= X(x; (\phi(0, \varepsilon + \Delta\varepsilon)) \\ &= X(x; \varepsilon + \Delta\varepsilon) \end{aligned}$$

Theorem 2.2.1 [Lie's first fundamental theorem]

There exists a parameterization $\tau(\varepsilon)$ such that the Lie group of transformations (2.1.1) is equivalent to the solution of the initial value problem (IVP) for the first order differential equations

$$\frac{dx^*}{d\tau} = \xi(x^*) \qquad (2.2.3b).$$

with initial conditions $x^* = x$, when $\tau = 0$ (2.2.3c).

In particular $\tau(\varepsilon) = \int_0^{\varepsilon} \alpha(\varepsilon') d\varepsilon'$ (2.2.3d).

where $\alpha(\varepsilon) = \frac{\partial\phi(\varepsilon,\delta)}{\partial\delta}\Big|_{(\varepsilon,\delta)=(\varepsilon^{-1},\varepsilon)}$ (2.2.3e).

and $\alpha(0) = 1$. (2.2.3f).

[ε^{-1} denotes the inverse of ε]

Proof.

First we show that (2.1.1) leads to (2.2.3b), (2.2.3c), (2.2.3d), (2.2.3e). Expand the left-hand side of (2.2.3a) in a power series in $\Delta\varepsilon$ about $\Delta\varepsilon$ =0 so that

$$X(x;\varepsilon+\Delta\varepsilon) = x^* + \frac{\partial X(x;\varepsilon)}{\partial\varepsilon}\Delta\varepsilon + o((\Delta\varepsilon)^2) \quad (2.2.3h).$$

where x^* is given by (2.1.1) .Then expanding $\phi(\varepsilon^{-1},\varepsilon+\Delta\varepsilon)$ in a power series in $\Delta\varepsilon$ about $\Delta\varepsilon$ =0 we have

$$\phi(\varepsilon^{-1},\varepsilon+\Delta\varepsilon) = \phi(\varepsilon^{-1},\varepsilon) + \alpha(\varepsilon)\Delta\varepsilon + o((\Delta\varepsilon)^2)$$

$$= \alpha(\varepsilon)\Delta\varepsilon + o((\Delta\varepsilon)^2) \quad (2.2.3i).$$

where $\alpha(\varepsilon)$ is defined by (2.2.3e). Consequently , after the right-hand side of (2.2.3a) in a power series in $\Delta\varepsilon$ about $\Delta\varepsilon$ =0, we obtain

$$X(x;\varepsilon+\Delta\varepsilon) = X(x^*;\phi(\varepsilon^{-1},\varepsilon+\Delta\varepsilon))$$

$$= X(x^*;\alpha(\varepsilon)\Delta\varepsilon + o((\Delta\varepsilon)^2))$$

$$= X(x^*;0) + \alpha(\varepsilon)\Delta\varepsilon\frac{\partial X}{\partial\delta}(x^*;\delta)\Big|_{\delta=0} + o((\Delta\varepsilon)^2))$$

$$= x^* + \alpha(\varepsilon)\xi(x^*)\Delta\varepsilon + o((\Delta\varepsilon)^2)) . \quad (2.2.3j).$$

Equating (2.2.3h) and (2.2.3j) we see that

$x^* = X(x;\varepsilon)$ satisfies the initial value problem for the system of differential equations

$$\frac{dx^*}{d\varepsilon} = \alpha(\varepsilon)\xi(x^*) \quad (2.2.3k).$$

with $x^* = x$, at $\varepsilon = 0$. (2.2.3l).

From (2.2.2) it follows it follows that $\alpha(0) = 1$. The parameterization $\tau(\varepsilon) = \int_0^{\varepsilon} \alpha(\varepsilon')d\varepsilon'$ leads to (2.2.3b,c).

Since $\frac{\partial \xi(x)}{\partial x_i}$, $i = 1.2.3........,n$ is continuous , it follows from the existence and uniqueness theorem for an (IVP) for a system of first order differential equations , that the solution of (2.2.3b,c), and hence (2.2.3k,l) , exists and is unique. This solution must be (2.1.1),which completes the proof.

From the above theorem , without loss of generality , we assume that a one-parameter (ε) Lie group of transformations is parameterized such that its laws of composition

$\phi(\varepsilon, \delta) = \varepsilon + \delta$

and

$\varepsilon^{-1} = \varepsilon$, where $\varepsilon = 0$ is the neutral element. That is the one-parameter Lie group of transformations (2.1.1) now becomes;

$$\frac{dx^*}{d\varepsilon} = \xi(x^*)$$

with initial conditions $x^* = x$, at $\varepsilon = 0$ (2.2.4)

where $\xi(x)$ is the infinitesimal of (2.1.1)

Example 2.2.2

Considering the groups of translations

$x^* = x + \varepsilon$ (2.2.4a)

$y^* = y,$ (2.2.4b)

the law of composition here is $\phi(\varepsilon, \delta) = \varepsilon + \delta$, and $\varepsilon^{-1} = -\varepsilon$.
Applying lemma (2.2.1) and theorem (2.2.1) we obtain

$$\frac{\partial \phi(\varepsilon,\delta)}{\partial \delta} = 1 \text{ and hence } \alpha(\varepsilon) \equiv 1 .$$

Let $X = (x, y; \varepsilon)$.

Then the group (2.2.4a,b) is $X(x, y; \varepsilon) = (x + \varepsilon, y)$.

Thus $\frac{\partial X(x,y;\varepsilon)}{\partial \varepsilon} = (1,0)$. Hence $\xi(x,y) = \frac{\partial X(x,y;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = (1,0)$

Consequently (2.2.3k,l) become

$$\frac{dx^*}{d\varepsilon} = 1 \;,\; \frac{dy^*}{d\varepsilon} = 0 \tag{2.2.4c}$$

with $x^* = x,\ y^* = y,\ at\ \ \varepsilon = 0$ (2.2.4d)

The solution of the (IVP) (2.2.4c,d) is seen to be (2.2.4a,b).

Definition 2.2.1

The infinitesimal generator of the one-parameter Lie group of transformations (2.1.1) is the operator

$$V = V(x) = \xi(x)\nabla$$

$$= \xi(x)\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \ldots\ldots\ldots, \frac{\partial}{\partial x_n}\right)$$

$$= \sum_i^n \xi_i(x)\frac{\partial}{\partial x_i}$$

where (2.2.5)

∇ is the gradient operator defined by where

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \ldots\ldots\ldots, \frac{\partial}{\partial x_n}\right)$$

with $\xi(x) = (\xi_1(x), \xi_2(x), \ldots\ldots\ldots\ldots\ldots\ldots, \xi_n(x))$.

Theorem 2.2.2

The one-parameter Lie group of transformations (2.1.1) is equivalent to

$$x^* = e^{\varepsilon v} x = X(x,\varepsilon) = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} x \qquad (2.2.6)$$

where

$v = v(x)$

is defined by (2.2.5) and

$v^m = v v^{m-1}$, $m = 1,2,3,\ldots\ldots\ldots$ with $v^0 x = x$

The transformation (2.2.6) above is called *Lie series*

Proof.

Let

$$V = V(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i} \qquad (2.2.6a)$$

and

$$V(x^*) = \sum_{i=1}^{n} \xi_i(x^*) \frac{\partial}{\partial x^*_i} \qquad (2.2.6b)$$

where

$$x^* = V(x;\varepsilon) \qquad (2.2.6c)$$

is the Lie group of transformations (2.1.1). From Taylor's theorem , expanding (2.2.6c) about $\varepsilon = 0$, we get

$$x^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \left(\frac{\partial^k V(x;\varepsilon)}{\partial \varepsilon^k} \Big|_{\varepsilon=0} \right) = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \left(\frac{d^k x^*}{d\varepsilon^k} \Big|_{\varepsilon=0} \right) \qquad (2.2.6d)$$

For any differential function $F(x)$,

$$\frac{d}{d\varepsilon} F(x^*) = \sum_{i=0}^{n} \left(\frac{\partial^k F(x^*)}{\partial x_i^*} \frac{dx_i^*}{d\varepsilon} \right)$$

$$= \sum_{i=0}^{n} \xi_i(x^*) \left(\frac{\partial^k F(x^*)}{\partial x_i^*} \right) = V(x^*)F(x^*) \qquad (2.2.6e)$$

Hence it follows that $\frac{dx^*}{d\varepsilon} = V(x^*)x^*,$ (2.2.6f)

$$\frac{d^2x^*}{d\varepsilon^2} = \frac{d}{d\varepsilon}\left(\frac{dx^*}{d\varepsilon}\right) = V(x^*)V(x^*)x^* = V^2(x^*)x^*, \qquad (2.2.6g)$$

Consequently

$$\frac{d^k x^*}{d\varepsilon^k}\bigg|_{\varepsilon=0} = V^k(x)x = V^k x, \quad k = 1,2,3,\dots\dots\dots \qquad (2.2.6h)$$

which leads to (2.2.6).

Example 2.2.3

For the infinitesimal generator

$$V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

we see that its corresponding Lie series may be obtained.

By theorem 2.2.2 we have

$$x^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} x \quad , \quad y^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} y$$

The infinitesimal for (x^*, y^*) is

$$\xi(x) = (\xi_1(x,y), \xi_2(x,y))$$

so that

$$\xi_1(x,y) = y, \xi_2(x,y) = -x$$

We need to find $v^k x$ and $v^k y$; k=1,2,3,........

$$V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

$$v^1x = y, v^2x = v(v^1x) = v^1y = -x, v^3x = -y, v^4x = x,\ v^5x = y, \ldots\ldots\ldots\ldots$$

$$v^1y = -x, v^2y = -y, v^3y = x, v^4y = y, v^5y = -x, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It is evident that

$$v^{4n}x = x, v^{4n-1}x = -y, v^{4n-2}x = -x, v^{4n-3}x = y\ , n = 1,2,3, \ldots\ldots\ldots \tag{2.2.6i}$$

$$v^{4n}y = y, v^{4n-1}y = x,\ v^{4n-2}y = -y, v^{4n-3}y = -x\ , n = 1,2,3, \ldots\ldots\ldots \tag{2.2.6j}$$

Alternating and recurrent series for the above equations (2.2.6i) ,(2.2.6j) yield

$$x^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} x$$
$$= \left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \frac{\varepsilon^6}{6!} + \ldots\ldots\ldots\right)x + \left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \frac{\varepsilon^7}{7!} + \ldots\ldots\right)y$$

$$y^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} y$$
$$= \left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \frac{\varepsilon^6}{6!} + \ldots\ldots\ldots\right)y - \left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \frac{\varepsilon^7}{7!} + \ldots\ldots\right)x$$

Thus

$$x^* = x\cos\varepsilon + y\sin\varepsilon$$

$$y^* = y\cos\varepsilon - x\sin\varepsilon$$

is the corresponding explicit one-parameter Lie group of transformations.

Example 2.2.4

Consider the infinitesimal generator,

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

We see that its corresponding Lie series are of the form:

$$x^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} x \ , \quad y^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} y$$

From theorem (2.2.2) we therefore have,

$$v^1 x = -y , v^2 x = v(v^1 x) = v^1 y = -x , v^3 x = y , v^4 x = x , \ v^5 x = -y , \ldots\ldots\ldots\ldots$$

$$v^1 y = x , v^2 y = -y , v^3 y = -x , v^4 y = y , v^5 y = x , \ldots\ldots\ldots\ldots\ldots\ldots$$

It is evident that

$$v^{4n} x = x , v^{4n-1} x = y , v^{4n-2} x = -x , v^{4n-3} x = -y \ , n = 1,2,3,\ldots\ldots\ldots\ldots \quad (2.2.6k)$$

$$v^{4n} y = y , v^{4n-1} y = -x , \ v^{4n-2} y = -y , v^{4n-3} y = x \ , n = 1,2,3,\ldots\ldots\ldots\ldots \quad (2.2.6l)$$

Alternating and recurrent series for the above equations (2.2.6k) ,(2.2.6l) yield

$$x^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} x$$

$$= \left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \frac{\varepsilon^6}{6!} + \ldots\ldots\right) x - \left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \frac{\varepsilon^7}{7!} + \ldots\ldots\right) y$$

$$y^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} y$$

$$= \left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \frac{\varepsilon^6}{6!} + \ldots\ldots\right) x + \left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \frac{\varepsilon^7}{7!} + \ldots\ldots\right) y$$

Thus

$$x^* = x \cos\varepsilon - y \sin\varepsilon \ , \quad y^* = y \cos\varepsilon + x \sin\varepsilon$$

Example 2.2.5

The infinitesimal generator

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} ,$$

readily yields the series : $x^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} x$, $y^* = \sum_{k=0}^{\infty} \frac{\varepsilon^k v^k}{k!} y$,

where

$$v^1x = x, v^2x = v(v^1x) = v^1y = x, v^3x = x, v^4x = x, \; v^5x = x, \ldots\ldots\ldots\ldots (2.2.6m)$$

$$v^1y = y, v^2y = y, v^3y = y, v^4y = y, v^5y = y, \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.2.6ln)$$

It is clear from equations (2.2.6m) , (2.2.6n) that

$$x^* = e^{\varepsilon}x$$

and

$$y^* = e^{\varepsilon}y \; .$$

Hence the corresponding explicit one-parameter Lie group of transformations are determined.

2.3 Extended Transformations (Prolongations)

To be able to apply a point transformation

$$\left.\begin{aligned} x^* &= X(x, y; \varepsilon) \\ y^* &= Y(x, y; \varepsilon) \end{aligned}\right\} \qquad (2.3.1)$$

to the differential equation,

$$H\left(x, y, y', y'', y''', \ldots\ldots, y^{(n)}\right) = 0, \; y' = \frac{dy}{dx}, \ldots\ldots, y^{(n)} = \frac{d^n y}{dx^n} \qquad (2.3.2)$$

we must know how to transform the derivatives $y^{(n)}$, that is, how to *extend*

(or prolong) the point transformation to the derivatives. The task here is extending on the transformation (2.3.1) acting on

(x, y) to the $(x, y, y_1, y_2, y_3 \ldots. y_n)$ space , with the property of preserving the contact conditions relating the differentials

$dx, dy, dy_1, dy_2, \ldots\ldots\ldots. dy_m..$ i.e.

$$dy = y_1dx, \; dy_1 = y_2dx, \;\; dy_2 = y_3dx, \;\; \ldots\ldots\ldots\ldots \;\; , \; dy_m = y_{m+1}dx, \qquad (2.3.3)$$

From (2.3.1) the transformed derivatives are defined by

$$dy^* = y^*_1\, dx^*,\ dy^*_1 = y^*_2\, dx^*,\ \ dy^*_2 = y^*_3\, dx^*, \dots\ \ dy^*_m = y^*_{m+1}\, dx^*, \qquad .(2.3.4)$$

Using (2.1.0) and (2.1.1) it can be shown in particular, that

$$y_1^* = \frac{dy^*}{dx^*} = Y_1(x, y, y_1; \varepsilon) = \frac{\dfrac{\partial Y(x,y;\varepsilon)}{\partial x} + y_1 \dfrac{\partial Y(x,y;\varepsilon)}{\partial y}}{\dfrac{\partial X(x,y;\varepsilon)}{\partial x} + y_1 \dfrac{\partial X(x,y;\varepsilon)}{\partial y}} \qquad (2.3.5)$$

$$y_2^* = \frac{dy_1^*}{dx^*} = Y_2(x, y, y_1, y_2; \varepsilon)$$

$$= \frac{\dfrac{\partial Y_1(x,y,y_1;\varepsilon)}{\partial x} + y_1 \dfrac{\partial Y_1(x,y,y_1;\varepsilon)}{\partial y} + y_2 \dfrac{\partial Y_1(x,y,y_1;\varepsilon)}{\partial y_1}}{\dfrac{\partial X(x,y;\varepsilon)}{\partial x} + y_1 \dfrac{\partial X(x,y;\varepsilon)}{\partial y}} \qquad (2.3.6)$$

Theorem 2.3.1

The Lie group of transformations (2.3.5) and (2.3.6) extend to $n-th$ extension, $n > 2$, which is the following one-parameter Lie group of transformation acting on $(x, y, y_1, y_2, y_3.....y_n)$ -space:

$$
\left.\begin{aligned}
x^* &= X(x, y; \varepsilon) \\
y^* &= Y(x, y; \varepsilon) \\
y_1{}^* &= Y_1(x, y, y_1; \varepsilon) \\
y_2{}^* &= Y_2(x, y, y_1, y_2; \varepsilon) \\
&\cdot \\
&\cdot \\
y_n{}^* &= Y_n(x, y, y_1, y_2, y_3, \ldots\ldots y_n; \varepsilon) \\
\text{with} & \\
y_n{}^* &= Y_n(x, y, y_1, y_2, y_3, \ldots\ldots y_n; \varepsilon) \\
&= \frac{\dfrac{\partial Y_{n-1}}{\partial x} + y_1 \dfrac{\partial Y_{n-1}}{\partial y} + y_2 \dfrac{\partial Y_{n-1}}{\partial y_1} + \ldots.. + y_n \dfrac{\partial Y_{n-1}}{\partial y_{n-1}}}{\dfrac{\partial X(x, y; \varepsilon)}{\partial x} + y_1 \dfrac{\partial X(x, y; \varepsilon)}{\partial y}}
\end{aligned}\right\} \qquad (2.3.7)
$$

For proof ,see Olver[18]

Example 2.3.1

Given the scaling group

$$x^* = X(x, y; \varepsilon) = e^{\varepsilon} x$$

$$y^* = Y(x, y; \varepsilon) = e^{2\varepsilon} y$$

then it's first extension $y_1{}^*$ is given by

$$y_1{}^* = \frac{dy^*}{dx^*} = Y_1(x, y, y_1; \varepsilon) = \frac{\dfrac{\partial Y(x, y; \varepsilon)}{\partial x} + y_1 \dfrac{\partial Y(x, y; \varepsilon)}{\partial y}}{\dfrac{\partial X(x, y; \varepsilon)}{\partial x} + y_1 \dfrac{\partial X(x, y; \varepsilon)}{\partial y}} = e^{\varepsilon} y_1$$

Its second extension $y_2{}^*$ is given by

$$y_2{}^* = \frac{dy_1{}^*}{dx^*} = Y_2(x, y, y_1, y_2; \varepsilon)$$

$$= \frac{\dfrac{\partial Y_1(x, y, y_1; \varepsilon)}{\partial x} + y_1 \dfrac{\partial Y_1(x, y, y_1; \varepsilon)}{\partial y} + y_2 \dfrac{\partial Y_1(x, y, y_1; \varepsilon)}{\partial y_1}}{\dfrac{\partial X(x, y; \varepsilon)}{\partial x} + y_1 \dfrac{\partial X(x, y; \varepsilon)}{\partial y}} = y_2$$

and by (2.3.7) the $k-th$ extension becomes

$$y_k{}^* = Y_k(x, y, y_1, y_2, y_3, ...y_i; \varepsilon) = e^{2-k} y_k; i = 1,2,3............k$$

By definition of the Lie group of transformations, the $k-th$ extension or prolongation of example 2.3.1 above, is also a Lie group of transformations. Thus the study of extended Lie group of transformations reduce to that of infinitesimal transformations. There we need to determine explicit formula for developing extended infinitesimal transformations and corresponding infinitesimal generators.

Consider , the one -parameter Lie group of transformations

$$\left.\begin{aligned} x^* &= X(x, y; \varepsilon) = x + \varepsilon\xi(x, y) + o(\varepsilon^2) \\ y^* &= Y(x, y; \varepsilon) = y + \varepsilon\eta(x, y) + o(\varepsilon^2) \end{aligned}\right\} \qquad (2.3.8)$$

with infinitesimal

$$\xi_*(x, y) = [\xi(x, y), \eta(x, y)] \qquad (2.3.9)$$

and corresponding infinitesimal generator

$$V = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \qquad (2.3.10).$$

The $k-th$ extension of (2.3.8) is given by

$$
\left.\begin{aligned}
x^* &= X(x,y;\varepsilon) = x + \varepsilon\xi(x,y) + o(\varepsilon^2) \\
y^* &= Y(x,y;\varepsilon) = y + \varepsilon\eta(x,y) + o(\varepsilon^2) \\
y_1^* &= Y_1(x,y,y_1;\varepsilon) = y_1 + \varepsilon\eta^{(1)}(x,y,y_1) + o(\varepsilon^2) \\
y_2^* &= Y_2(x,y,y_1,y_2;\varepsilon) = y_2 + \varepsilon\eta^{(2)}(x,y,y_1,y_2) + o(\varepsilon^2) \\
&. \\
y_k^* &= Y_k(x,y,y_1,y_2 \ldots\ldots y_k;\varepsilon) = y_k + \varepsilon\eta^{(k)}(x,y,y_1,y_2,\ldots\ldots y_k) + o(\varepsilon^2)
\end{aligned}\right\} \qquad (2.3.10a)
$$

Then $k-th$ extended infinitesimal of (2.3.9) will be

$$
\left[\xi(x,y), \eta(x,y), \eta^{(1)}(x,y,y_1), \eta^{(2)}(x,y,y_1,y_2), \ldots\ldots \eta^{(k)}(x,y,y_1,y_2,\ldots y_k)\right] \qquad (2.3.10b)
$$

with corresponding $k-th$ extended infinitesimal generator

$$
\left.\begin{aligned}
V^{(k)} &= \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1} + \eta^{(2)}(x,y,y_1,y_2)\frac{\partial}{\partial y_2} + \ldots\ldots \\
&\eta^{(k)}(x,y,y_1,y_2,\ldots\ldots y_k)\frac{\partial}{\partial y_k}
\end{aligned}\right\} \qquad (2.3.10c)
$$

Theorem 2.3.2

$$
\eta^{(k)}(x,y,y_1,y_2,\ldots\ldots y_k) = \frac{D\eta^{(k-1)}}{Dx} - y_k\frac{D\xi(x,y)}{Dx}, \quad \eta^{(0)} = \eta(x,y) \qquad (2.3.11)
$$

where

$$
\frac{D}{Dx} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} + y_{n+1}\frac{\partial}{\partial y_n} + \ldots\ldots k = 1,2,3,\ldots\ldots n \qquad (2.3.12)
$$

Proof

From (2.3.7), (2.3.8), (2.3.9), (2.3.10), (2.3.10a,b,c), we have

$$Y_k(x,y,y_1,y_2,y_3,....,y_k;\varepsilon)=\frac{\frac{DY_{k-1}}{Dx}}{\frac{DX(x,y;\varepsilon)}{Dx}}=\frac{\frac{D(y_{k-1}+\varepsilon\eta^{(k-1)}+o(\varepsilon^2))}{Dx}}{\frac{D[x+\varepsilon\xi(x,y)+o(\varepsilon^2)]}{Dx}}$$

$$=\frac{y_k+\varepsilon\frac{D\eta^{(k-1)}+o(\varepsilon^2))}{Dx}}{1+\varepsilon\frac{D\varepsilon\xi(x,y)}{Dx}}+o(\varepsilon^2)$$

$$=y_k+\varepsilon\left[\frac{D\eta^{(k-1)}}{Dx}-y_k\frac{D\xi(x,y)}{Dx}\right]+o(\varepsilon^2)=y_k+\varepsilon\eta^{(k)}+o(\varepsilon^2)$$

which leads (2.3.11).

Example 2.3.2

Let us consider a rotation group having first extension y_1* as

$$y_1*=Y_1(x,y,y_1;\varepsilon)=y_1+\varepsilon\eta^{(1)}(x,y,y_1)+o(\varepsilon^2).$$

Then using (2.3.11) and (2.3.12) we obtain ,

$$\eta^{(1)}(x,y,y_1)=\frac{D\eta^{(0)}}{Dx}-y_1\frac{D\xi(x,y)}{Dx}=\frac{\partial\eta}{\partial x}+y_1\frac{\partial\eta}{\partial y}-y_1\left[\frac{\partial\xi}{\partial x}+y_1\frac{\partial\xi}{\partial y}\right]=-1-y^2{}_1$$

Therefore

$$y_1*=Y_1(x,y,y_1;\varepsilon)=y_1+\varepsilon\eta^{(1)}(x,y,y_1)=y_1+\varepsilon(-1-y^2{}_1)$$

2.4 Determination of Extended Infinitesimal Coefficient Functions $\phi^{(****......)}$.

The terms $\phi^{(*)}$,$\phi^{(**)}$, $\phi^{(***)}$,$\phi^{(****)}$, where $*$, $**$, $***$,...... represent

x , t, y, xx, yy, xt, tt, xxx, $xxxx$, ... in the prolongation are expressed as functions of

ϕ,ξ,τ,u as below by using equations (2.3.11) and (2.3.12).

$$D_x(\phi)=\phi_x+u_x\phi_u \quad :\phi=\phi(x,t,u)$$

$$D_x(\xi)=\xi_x+u_x\xi_u \quad :\xi=\xi(x,t,u)$$

$$D_x(\tau)=\tau_x+u_x\tau_u \quad :\tau=\tau(x,t,u)$$

$$D_x(\phi)=\phi_x+u_x\phi_u+u_y\phi_u \ : \ \phi=\phi(x,y,t,u)$$

$$D^2{}_x(\phi)=\phi_{xx}+2u_x\phi_{ux}+u_{xx}\phi_u+u^2{}_x\phi_{uu}$$

$$D^2{}_x(\xi)=\xi_{xx}+2u_x\xi_{ux}+u_{xx}\xi_u+u^2{}_x\xi_{uu}$$

$$D^2{}_x(\tau)=\tau_{xx}+2u_x\tau_{ux}+u_{xx}\tau_u+u^2{}_x\tau_{uu}$$

$$\phi^x=D_x(\phi-\xi u_x-\tau u_t)+\xi u_{xx}+\tau u_{xt}$$

$$\phi^t=D_t(\phi-\xi u_x-\tau u_t)+\xi u_{xt}+\tau u_{tt}$$

$$\phi^{xx}=D^2{}_x(\phi-\xi u_x-\tau u_t)+\xi u_{xxx}+\tau u_{xxt}$$

$$\phi^{tt}=D^2{}_t(\phi-\xi u_x-\tau u_t)+\xi u_{xtt}+\tau u_{ttt}$$

$$\phi^{xxx}=D^3{}_x(\phi-\xi u_x-\tau u_t)+\xi u_{xxxx}+\tau u_{xxxt}$$

$$\eta^t=\phi_t-\xi_t u_x+(\phi_u-\tau_t)u_t-\xi_u u_x u_t-\tau_u u_t{}^2 \tag{2.4.1a}$$

$$*\eta^t=D_t(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{xt}+\eta u_{yt}+\tau u_{tt} \tag{2.4.1b}$$

$$\eta^x=\phi_x-\tau_x u_t+(\phi_u-\xi_x)u_x-\xi_u u_x{}^2-\tau_u u_t u_x \tag{2.4.2a}$$

$$*\eta^x=D_x(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{xx}+\eta u_{yx}+\tau u_{xt} \tag{2.4.2b}$$

$$*\eta^y=D_y(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{xy}+\eta u_{yy}+\tau u_{ty} \tag{2.4.3b}$$

$$\begin{aligned}\phi^{xx}&=\phi_{xx}+(2\phi_{xu}-\xi_{xx})u_x-\tau_{xx}u_t+(\phi_{uu}-2\xi_{xu})u_x{}^2-2\tau_{xu}u_x u_t-\xi_{uuu}u_x{}^3\\&-\tau_{uu}u_x{}^2u_t+(\phi_u-2\xi_x)u_{xx}-2\tau_x u_{xt}-3\xi_u u_x u_{xx}-\tau_u u_t u_{xx}-2\tau_u u_x u_{xt}\end{aligned} \tag{2.4.4a}$$

$$*\phi^{xx} = \phi_{xx} + 2u_x\phi_{ux} + u_{xx}\phi_u + u^2{}_x\phi_{uu} - u_x\left(\xi_{xx} + 2u_x\xi_{ux} + u_{xx}\xi_u + u^2{}_x\xi_{uu}\right)$$
$$-u_y\left(\eta_{xx} + 2u_x\eta_{ux} + u_{xx}\eta_u + u^2{}_x\eta_{uu}\right) - u_t\left(\tau_{xx} + 2u_x\tau_{ux} + u_{xx}\tau_u + u^2{}_x\tau_{uu}\right) \quad (2.4.4b)$$
$$-2u_{xx}\left(\xi_x + u_x\xi_u\right) - 2u_{yx}\left(\eta_x + u_x\eta_u\right) - 2u_{tx}\left(\tau_x + u_x\tau_u\right)$$

$$\phi^{tt} = \phi_{tt} + \left(2\phi_{ut} - \tau_{uu}\right)u_t - \xi_{tt}u_x + \left(\phi_{uu} - 2\tau_{ut}\right)u_t^{\,2} - 2\xi_{ut}u_xu_t - \tau_{uu}u_t^{\,3} \quad (2.4.5a)$$
$$-\xi_{uu}u_xu^2{}_t + \left(\phi_u - 2\tau_t\right)u_{tt} - 2\xi_tu_{xt} - 3\tau_uu_tu_{tt} - \xi_uu_xu_{tt} - 2\xi_uu_tu_{xt}$$

$$*\phi^{tt} = \phi_{tt} + 2u_t\phi_{ut} + u_{tt}\phi_u + u^2{}_t\phi_{uu} - u_x\left(\xi_{tt} + 2u_t\xi_{ut} + u_{tt}\xi_u + u^2{}_t\xi_{uu}\right)$$
$$-u_y\left(\eta_{tt} + 2u_t\eta_{ut} + u_{tt}\eta_u + u^2{}_t\eta_{uu}\right) - u_t\left(\tau_{tt} + 2u_t\tau_{ut} + u_{tt}\tau_u + u^2{}_t\tau_{uu}\right) \quad (2.4.5b)$$
$$-2u_{xt}\left(\xi_t + u_t\xi_u\right) - 2u_{ty}\left(\eta_t + u_t\eta_u\right) - 2u_{tt}\left(\tau_t + u_t\tau_u\right)$$

$$*\eta^{yy} = \phi_{yy} + 2u_y\phi_{uy} + u_{yy}\phi_u + u^2{}_y\phi_{uu} - u_x\left(\xi_{yy} + 2u_y\xi_{uy} + u_{yy}\xi_u + u^2{}_y\xi_{uu}\right)$$
$$-u_y\left(\eta_{yy} + 2u_y\eta_{uy} + u_{yy}\eta_u + u^2{}_y\eta_{uu}\right) - u_t\left(\tau_{yy} + 2u_y\tau_{uy} + u_{yy}\tau_u + u^2{}_y\tau_{uu}\right) \quad (2.4.6b)$$
$$-2u_{yx}\left(\xi_y + u_y\xi_u\right) - 2u_{yy}\left(\eta_y + u_y\eta_u\right) - 2u_{ty}\left(\tau_y + u_y\tau_u\right)$$

$$\phi^{xxx} = \phi_{xxx} + 3u_x\phi_{uxx} + 3u_x^{\,2}\phi_{uux} + 3u_{xx}\phi_{ux} + u_x^{\,3}\phi_{uuu} + 3u_xu_{xx}\phi_{uu} + u_{xxx}\phi_u$$

$$-3u_t\left(\tau_{xxx} + 3u_x\tau_{uxx} + 3u_x^{\,2}\tau_{uux} + 3u_{xx}\tau_{ux} + 3u_xu_{xx}\tau_{uu} + u_x^{\,3}\tau_{uuu} + u_{xxx}\tau_u\right)$$

$$-3u_x\left(\xi_{xxx} + 3u_x\xi_{uxx} + 3u_x^{\,2}\xi_{uux} + 3u_{xx}\xi_{ux} + 3u_xu_{xx}\xi_{uu} + u_x^{\,3}\xi_{uuu} + u_{xxx}\xi_u\right) \quad (2.4.7)$$

$$\phi^{xxxx} = D^4{}_x\left(\phi - \xi u_x - \tau u_t\right) + \xi u_{xxxxx} + \tau u_{xxxxt}$$

$$\phi^{xxxx} = D^4{}_x(\phi) - u_xD^4{}_x(\xi) - u_tD^4{}_x(\tau) - 4u_{xx}D^3{}_x(\xi) - 4u_{xt}D^3{}_x(\tau)$$

$$-4u_{xxx}D^2{}_x(\xi) - 4u_{xxt}D^2{}_x(\tau) - 4u_{xxxx}D_x(\xi) - 4u_{xxxt}D_x(\tau) \quad (2.4.8)$$

$$D^4{}_x(\phi) = \phi_{xxxx} + u_x\phi_{uxxx} + 3\left(\ u_{xx}\phi_{uxx} + u_x\phi_{xuux} + u_x^{\,2}\phi_{uuxx}\ \right) +$$

$$3\left(\ 2u_xu_{xx}\phi_{uux} + u^2{}_x\phi_{xuux} + u^3{}_x\phi_{uuux}\ \right) +$$

$$\left(\ 3u^2{}_xu_{xx}\phi_{uuu} + u^3{}_x\phi_{xuuu} + u^4{}_x\phi_{uuuu}\ \right) + \left(\ u_{xxxx}\phi_u + u_{xxx}\phi_{xu} + u_xu_{xxx}\phi_{uu}\ \right) +$$

$$3\left(\ u_{xxx}\phi_{ux} + u_{xx}\phi_{xux} + u_xu_{xx}\phi_{uux}\ \right) + 3\left(\ \left(\ u^3{}_{xx} + u_xu_{xx}\ \right)\phi_{uu} + u_xu_{xx}\phi_{xuu} + u^2{}_xu_{xx}\phi_{uuu}\ \right)$$

$$D^4{}_x(\tau) = \tau_{xxxx} + u_x\tau_{uxxx} + 3\left(\ u_{xx}\tau_{uxx} + u_x\tau_{xuux} + u_x^{\,2}\tau_{uuxx}\ \right) +$$

$$3\left(\ 2u_xu_{xx}\tau_{uux} + u^2{}_x\tau_{xuux} + u^3{}_x\tau_{uuux}\ \right) +$$

$$(3u^2{}_x u_{xx}\tau_{uuu} + u^3{}_x\tau_{xuuu} + u^4{}_x\tau_{uuuu}) + (u_{xxxx}\tau_u + u_{xxx}\tau_{xu} + u_x u_{xxx}\tau_{uu}) +$$

$$3(u_{xxx}\tau_{ux} + u_{xx}\tau_{xux} + u_x u_{xx}\tau_{uux}) + 3((u^3{}_{xx} + u_x u_{xx})\tau_{uu} + u_x u_{xx}\tau_{xuu} + u^2{}_x u_{xx}\tau_{uuu})$$

$$D^4{}_x(\xi) = \xi_{xxxx} + u_x\xi_{uxxx} + 3(u_{xx}\xi_{uxx} + u_x\xi_{xuux} + u_x{}^2\xi_{uuxx}) +$$

$$3(2u_x u_{xx}\xi_{uux} + u^2{}_x\xi_{xuux} + u^3{}_x\xi_{uuux}) +$$

$$(3u^2{}_x u_{xx}\xi_{uuuu} + u^3{}_x\xi_{xuuu} + u^4{}_x\xi_{uuuu}) + (u_{xxxx}\xi_u + u_{xxx}\xi_{xu} + u_x u_{xxx}\xi_{uu}) +$$

$$3(u_{xxx}\xi_{ux} + u_{xx}\xi_{xux} + u_x u_{xx}\xi_{uux}) + 3((u^3{}_{xx} + u_x u_{xx})\xi_{uu} + u_x u_{xx}\xi_{xuu} + u^2{}_x u_{xx}\xi_{uuu})$$

2.5 Lie Algebras

Definition 2.5.1

Consider a k -parameter Lie group of transformations of partial differential equation with infinitesimal generators $\{v_i\}$ $i = 1,2,3,........k$

The commutator [Lie bracket] of v_i and v_j is another first order operator defined by

$$[v_i \ , v_j] = v_i v_j - v_i v_j$$

$$= \sum_{K,L=1}^{M} \left\{ \left(\xi_{ik}(x)\frac{\partial}{\partial x_k}\right)\left(\xi_{jl}(x)\frac{\partial}{\partial x_l}\right) - \left(\xi_{jk}(x)\frac{\partial}{\partial x_k}\right)\left(\xi_{il}(x)\frac{\partial}{\partial x_l}\right) \right\} \quad (2.5.1).$$

From (2.5.1). it follows that

$$[v_i \ , v_j] = -[v_j \ , v_i] \qquad \text{skew-symmetry} \qquad (2.5.2).$$

Theorem 2.5.1 [Second Fundamental Theorem of Lie]

The commutator of any two infinitesimal generators of a k -parameter Lie group of transformations is also an infinitesimal generator, in particular

$$[v_i \ , v_j] = c^n{}_{ij} v_n \qquad (2.5.3).$$

where the coefficients $c^n{}_{ij}$,……are called structure constants, i , j , n = 1,2,3,….. k

For any three . infinitesimal generators v_i , v_j , v_n it is always true that

$$[v_i , [v_j , v_n]] + [v_j , [v_n , v_i]] + [v_n , [v_i , v_j]] = 0. \qquad (2.5.4).$$

For proof, see Olver [18]

Equation.(2.5.4) is known as Jacobi's identity

Results (2.5.1) , (2.5.2) and (2.5.3) yield the third Fundamental Theorem of Lie given below.

Theorem 2.5.2 [Third Fundamental Theorem of Lie]

The structure constants ,defined by the commutation (2.5.3) satisfy the relation s

$$c^n{}_{ij} = -c_{ji}{}^n \qquad (2.5.5a).$$

$$c^k{}_{ij}\, c^l{}_{kn} + c^k{}_{jn}\, c^l{}_{kl} + c^k{}_{ni}\, c^l{}_{kj} = 0 \qquad (2.5.5b).$$

and

$$[\alpha v_i + \beta v_j , v_m] = \alpha[v_i , v_j] + \beta[v_j , v_m] , [v_i , \alpha v_j + \beta v_m] = \alpha[v_i , v_j] + \beta[v_i , v_m]$$

For proof, see Bluman and Kumei [4].

For infinitesimal generators $\{v_i\}$ i =1,2,3,……..n defined above, *bilinear* property satisfy the commutator equations

$$\left.\begin{array}{l} [\alpha v_i + \beta v_j , v_m] = \alpha[v_i , v_j] + \beta[v_j , v_m] , \\ [v_i , \alpha v_j + \beta v_m] = \alpha[v_i , v_j] + \beta[v_i , v_m] \end{array}\right\} \qquad (2.5.5c).$$

Definition 2.5.2

A Lie algebra , L ,is a vector space over some field F with an additional law of combination of elements in L (the commutator) satisfying, skew-symmetry , Jacobi' identity, and the bilinear properties.

Example 2.5.1

Consider the 8-parameter Lie group of projective transformations in R^2:

$$x^* = X(x,y;\varepsilon_i) = \frac{[(1+\varepsilon_3)x + \varepsilon_4 y + \varepsilon_5]}{[\varepsilon_1 x + \varepsilon_2 y + 1]}$$

$$y^* = Y(x,y;\varepsilon_i) = \frac{[\varepsilon_6 x + \quad (1+\varepsilon_7)y + \varepsilon_8]}{[\varepsilon_1 x + \varepsilon_2 y + 1]}$$

$i = 1,2,3,........8, \varepsilon_i \in R$ with infinitesimal generator V of the form

$$V = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

From theorem (2.2.2) we obtain the generator

$$V = (\varepsilon_1 x^2 + \varepsilon_2 xy + \varepsilon_3 x + \varepsilon_4 y + \varepsilon_5)\frac{\partial}{\partial x} + (\varepsilon_1 xy + \varepsilon_2 y^2 + \varepsilon_6 x + \varepsilon_7 y + \varepsilon_8)\frac{\partial}{\partial y}.$$

Then the infinitesimal generators for the corresponding Lie algebra , L^8 are

$$v_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad v_2 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, v_3 = x\frac{\partial}{\partial x}, v_4 = y\frac{\partial}{\partial x},$$

$$v_5 = \frac{\partial}{\partial x}, v_6 = x\frac{\partial}{\partial y}, \qquad v_7 = y\frac{\partial}{\partial y}, v_8 = \frac{\partial}{\partial y}.$$

Below is the table for the commutators of the Lie algebra L^8 whose $(i,j)^{th}$ entry is $[\, v_i , v_j \,]$

$[v_i , v_j] = v_i v_j - v_i v_j = 0$; i =j, $i = 1,2,3,........,8$.

$$[v_1 , v_5] = v_1 v_5 - v_5 v_1 = \left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \right)\left(\frac{\partial}{\partial x} \right) - \left(\frac{\partial}{\partial x} \right)\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \right)$$

$$= -2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

$[v_8 , v_2] = v_8 v_2 - v_2 v_8$

$$=\left(\frac{\partial}{\partial y}\right)\left(xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}\right)-\left(\frac{\partial}{\partial y}\right)\left(xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}\right)$$

$$=x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}=v_3+2v_7$$

Other Lie brackets are computed similarly.

The corresponding Lie brackets table constructed is shown below.

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	0	0	$-v_3$	$-v_2$	$-2v_3-v_7$	0	0	$-v_6$
v_2	0	0	0	0	$-v_4$	v_1	$-v_2$	$-v_3-2v_7$
v_3	v_1	0	0	$-v_4$	$-v_5$	v_6	0	0
v_4	v_2	0	v_4	0	0	v_7-v_3	$-v_4$	$-v_5$
v_5	v_3+v_7	v_4	v_5	0	0	v_8	0	0
v_6	0	$-v_1$	$-v_6$	$-v_7+v_3$	v_8	0	v_6	0
v_7	0	v_2	0	v_4	0	v_6	0	$-v_8$
v_8	v_6	v_3+2v_7	0	v_5	0	0	v_8	0

Table 2 [Lie bracket for (L^8)]

It should noted that the Lie bracket table can be used for finding additional infinitesimal generators.

CHAPTER 3

LIE GROUPS AND DIFFERENTIAL EQUATIONS

3.1 One Parameter Groups on The Plane.

Let us consider a change of the variables x, y involving a parameter ε:

$$T_\varepsilon : \bar{x} = \varphi(x, y, \varepsilon), \quad \bar{y} = \psi(x, y, \varepsilon) \tag{3.1.1}$$

with functions φ and ψ such that

$$T_0 :: \bar{x} = \varphi(x, y, 0), \ \bar{y} = \psi(x, y, 0) \tag{3.1.2}$$

It is assumed that $\varphi(x, y, \varepsilon)$, and $\psi(x, y, \varepsilon)$ are functionally independent, i.e. their Jacobian does

not vanish $\begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix} \neq 0$.

One can treat the equation (3.1.1) also as a transformation that carries any point $P = (x, y)$ of the (x, y) - plane into a new position $\overline{P} = (\bar{x}, \bar{y})$ and write $\overline{P} = T_\varepsilon(P)$. Accordingly, the inverse transformation:

$$T^{-1}{}_\varepsilon \text{ given by } T^{-1}{}_\varepsilon : x = \varphi^{-1}(\bar{x}, \bar{y}, \varepsilon), \ y = \psi^{-1}(\bar{x}, \bar{y}, \varepsilon) \tag{3.1.3}$$

returns $\overline{P}$ into original position P, i.e. $T^{-1}{}_\varepsilon(P) = P$.

Furthermore, the equations (3.1.2) means that T_0 is the identical transformation I, i.e. $T_0(P) = P$.

Let T_ε and T_δ be two transformations of the form (3.1.1) with different values ε and δ of the parameter. Their composition (or product) $T_\delta \, T_\varepsilon$ is defined as the consecutive application of these transformations and is given by

$$\left.\begin{aligned} \bar{\bar{x}} &= \varphi(\bar{x}, \bar{y}, \delta) = \varphi(\varphi(x, y, \varepsilon), \psi(x, y, \varepsilon), \ \delta) \\ \bar{\bar{y}} &= \psi(\bar{x}, \bar{y}, \delta) = \psi(\varphi(x, y, \varepsilon), \psi(x, y, \varepsilon), \ \delta) \end{aligned}\right\} \tag{3.1.4}$$

The geometric interpretation of the product is as follows. Since T_ε carries the point P to the point $\overline{P} = T_\varepsilon(P)$, which T_δ carries to the new position $\overline{\overline{P}} = T_\delta(\overline{P})$, the product $T_\delta\, T_\varepsilon$ is destined to carry P directly to it's final location $\overline{\overline{P}}$, without a stopover at $\overline{P}$. Thus, (3.1.4) means that

$$\overline{\overline{P}} \overset{def}{=} T_\delta(\overline{P}) = T_\delta\, T_\varepsilon(P)$$

Definition 3. 1 .1

The one parameter family G of transformations (3.1.1)

obeying the initial condition (3.1.2) is called a one parameter group if G contains

the inverse (3.1.3) and the composition $T_\delta\, T_\varepsilon$ of all its elements; $T_\delta\, T_\varepsilon = T_{\varepsilon+\delta}$.

The latter condition, invoking (3.1.4),may be written as:

$$\left.\begin{aligned} \varphi(\varphi(x,y,\varepsilon),\psi(x,y,\varepsilon),\delta) &= \varphi(x,y,\varepsilon+\delta) \\ \psi(\varphi(x,y,\varepsilon),\psi(x,y,\varepsilon),\delta) &= \psi(x,y,\varepsilon+\delta) \end{aligned}\right\} \qquad (3.\ 1\ .5).$$

3.2 Lie Groups and First Order Ordinary Differential Equations

Definition 3..2..1

The group of transformations (3.1.1) is termed a symmetry group of an ordinary differential

equation, $\dfrac{dy}{dx} = f(x,y)$

if the form of the differential remains the same after the change of variables (3.1.1). It means

that $\dfrac{d\overline{y}}{d\overline{x}} = f(\overline{x},\overline{y})$ with the same function f as in the original equation. A symmetry group of a differential equation is also termed a group of admitted operator or an infinitesimal symmetry of the equation in question.

Example 3.2.1

It is evident that the equation $y' = f(y)$ does not alter after the transformation $x^* = x + \varepsilon$ since the equation does not explicitly contain the independent variable x. Therefore the symmetry of this differential equations is given by the group translations along the x axis, $x^* = x + \varepsilon$

with the generator $X = \frac{\partial}{\partial x}$

Likewise the equation $y' = f(x)$ admits the group of translations along the y axis,

$y^* = y + \varepsilon$

with the generator $X = \frac{\partial}{\partial y}$

3.3 Lie's Integrating Factor

Consider a first order equation written in the symmetric form

$$M(x,y)dx + N(x,y)dy = 0 \qquad (3.3.1)$$

Lie showed that if

$$X = \varsigma(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

is a symmetry for equation (3.3.1) then

$$\mu = (\varsigma M - \eta N)^{-1} \qquad (3.3.1a)$$

is called Lie's integrating factor.

Example 3.3.1

We consider $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ a symmetry generator for the Riccati's equation

$y' + y^2 - \frac{2}{x^{2,}} = 0$.

Substituting into (3.3.1) $\xi = x, \eta = -y, M = y^2 - \frac{2}{x^{2,}}, N = 1$ we obtain the integrating factor

(3.3.1a) as ; $\mu = \frac{x}{x^2y^2 - xy - 2}$. After multiplication of the Riccati's equation by this factor it

becomes : $\frac{xdy + \left(xy^2 - \frac{2}{x}\right)dx}{x^2y^2 - xy - 2} = 0$

Let us rewrite it in the following form for integration:

$$\frac{xdy + ydx}{x^2y^2 - xy - 2} + \frac{dx}{x} = d(Inx + \frac{1}{3}In\frac{xy-2}{xy+1}) = 0.$$

The integration yields: $\frac{xy-2}{xy+1} = \frac{C}{x^3}$, hence solving for y, we obtain the solution of the Ricatti's

equation as

$y = \frac{2x^3 + C}{x(x^3 - C)}$, C - constant.

3.4 Lie Groups and First Order non-linear Ordinary Differential Equations

3.4 .1 Introduction

Solving non-linear ordinary differential equations(ODEs) is dominated to a large extent by various methods as may be seen from the collections by Kamke [11] , Murphy [16] and Ibragimov [10] .

The main deficiencies of these approaches are well known Roy [32]. If an equation is not used exactly as given, it is almost useless in most cases. Worse still if an equation cannot be solved by applying these collections or the various methods described there, it is by no means guaranteed that a closed form solution does not exist. For first –order equation the situation is even more intricate because in addition to the computational complexity there is the principal problem that there is no decision procedure for the existence of nontrivial symmetries of a general equation of this kind.

Contrary to(ODEs) of first order, the existence of nontrivial Lie symmetries may always be decided for higher- order equations. Some procedures concerning symmetry analysis of ordinary differential equations may be found in standard text books publications like Olver [18], Bluman and Kumei [4], Ibragimov [10] or Sophus Lie [13].

Let us now discuss the symmetry analysis . of first order (ODEs) of the form :

$$\frac{dy}{dx}+r(x,y)=0 \qquad (3.4\ .1\ )$$

The symmetry analysis procedure will require that, r (x, y) be restricted to polynomials in y and rational in x. The symmetries to be considered those point transformations that preserve the structure of the given equation; i. e. the transformed equation must again be polynomial in the pendent variable of the degree in the same degree in the first derivative.

This requirement entails the general form $x = f(u), y = g(u)v + h(u)$ for the admitted point transformations; the corresponding symmetries called structure corresponding symmetries. Abel's equation is the simplest equation in this category beyond the well known cases $r(x, y)$ linear or quadratic in y.

In this section we deal with the first order non linear (ODEs) known as Abel's equation. This equation was introduced by the Norwegian mathematician Abel and is usually written as

$$y' + \alpha_3 y^3 + \alpha_2 y^2 + \alpha_1 y + \alpha_0 = 0 , \; y' = \frac{dy}{dx}, \alpha_k \equiv \alpha_k(x) \quad for \quad k = 0,1,2,3 \qquad (3.4.2\ )$$

This equation (3.4.2) is referred to as Abel's equation of the first kind. A second equation

$$\frac{y' + \beta_3\, y^3 + \beta_2 y^2 + \beta_1 y + \beta_0}{y + g} = 0, \quad \beta_i \equiv \beta_i(x), \qquad g \equiv g(x) \qquad (3.4.3\ )$$

which is usually known as Abel's equation of the second kind, may be reduced to (3.4.2) by variable substitution change $y = \frac{1}{v(x)} - g$

such that

$$\left.\begin{aligned} \alpha_3 &= \beta_3\, g^3 - \alpha_2 g + \beta_1 g - \beta_0 \\ \alpha_2 &= g' - 3\beta_3 g^2 + 2\beta_2 g - \beta_1 \\ \alpha_1 &= 3\beta_3 g - \beta_2 \\ \alpha_0 &= -\beta_3 \end{aligned}\right\} \qquad (3.4.4\ )$$

Definition 3.4 .1

The rational normal form (RNF) of equation (3.4.1) with $r(x, y)$ rational in its arguments is the equation with the minimum number of variable coefficients that may be obtained from it by rational transformations in x and y.

Lemma 3.4.1.

There are two different possibilities for the (RNF) of Abel's equation (3.4.2).

$$\left.\begin{array}{l}\textbf{Case (a)}\ \dfrac{dy}{dx}+Ay^3+By=0\ .\\[2ex] \textbf{Case (b)}\ \dfrac{dy}{dx}+Ay^3+By+1=0\ \text{with}\ A\equiv A(x),\ B\equiv B(x).\end{array}\right\}\qquad (3.4.5)$$

The coefficients A and B are determined as follows;

Introduce a new variable function v into (3.4.2) by

$$y=v-\frac{\alpha_2}{3\alpha_3}\qquad hence\ \ y'=v'-\left(\frac{\alpha_2}{3\alpha_3}\right)'$$

and equation (3.4.2) becomes

$$v'+b_3v^3+b_1v+b_0=0 \qquad (3.4.6)$$

with $b_3=\alpha_3, b_1=\alpha_1-\dfrac{\alpha_2{}^2}{3\alpha_3}, b_0=\alpha_0-\dfrac{\alpha_1\alpha_2}{3\alpha_3}+\dfrac{2\alpha_2{}^3}{27\alpha_3{}^2}-\left(\dfrac{\alpha_2}{3\alpha_3}\right)'$

If $b_0=0$ then first alternative is obtained by, with $A=\alpha_3,\ B=b_1$,.

i.e. $v'+b_3v^3+b_1v=0$ (3.4.7)

For the other alternative we introduce again a new function w by

$v=b_0w$, $v'=b_0w'+b'w_0$,which leads to

$$\frac{dw}{dx}+Aw^3+Bw+1=0 \qquad (3.4.8)$$

$$A=b^2{}_0b_3,\quad B=b_1+\frac{b_0'}{b_0}.$$

Equation (3.4.7) is a Bernoulli equation which reduces to

$v' - 2Bv - 2A = 0$ first order linear solvable:

$$v = \overline{B}\left(2\int\frac{A}{B}dx + C\right) : \overline{B} = e^{2\int Bdx}$$

Theorem 3.4.1 (a)

The equation $(RNF)\ \frac{dy}{dx} + Ay^3 + By = 0$

always admits the two nontrivial generators:

$$V_1 = \frac{\overline{B}}{A}\left(\partial_x - By\partial_y\right)\ , V_2 = \frac{\overline{B}}{A}\int\frac{A}{B}dx\left(\partial_x - By\partial_y\right) - \frac{y}{2}\partial_y \qquad (3.4.9)$$

For proof see Schwarz [34]

Theorem 3.4.1 (b)

The equation $(RNF)\ \frac{dy}{dx} + Ay^3 + By + 1 = 0$

: $A' - 3AB \neq 0$

admits symmetry group with infinitesimal generator

$$V = \frac{1}{A' - 3AB}\left(3A\partial_x - A'y\partial_y\right) \qquad (3.4.10)$$

If and only A and B B satisfy. $A = K\left(B - \frac{A'}{3A}\right)^3 \quad K \neq 0.$ (3.4.11)

The case when

$A' - 3AB = 0$

then the equation admits the generator,

$$V = \frac{1}{A^{\frac{1}{3}}}\partial_x - \frac{A'y}{3A^{\frac{4}{3}}}\partial_y \qquad (3.4.12).$$

For proof see Schwarz [34]

3.4 Determination of Infinitesimal Transformations

for First Order Ordinary Differential Equations

In this subsection we seek to determine the two variable functions $\xi(x,y)$, $\eta(x,y)$ known as infinitesimal transformations whenever

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. ,$$

is an infinitesimal generator for the first order ordinary differential equation

$y' = f(x,y)$.

It therefore follows that ,

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y(y')^2 \qquad (3.5.1)$$

Theorem 3.5.1

Given the first order ordinary differential equation

$y' = f(x,y)$

admits one parameter Lie group of transformation with infinitesimal generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \qquad (3.5.2)$$

if and only if

$$\eta^{(1)} = \xi\ f_x + \eta f_y , \quad when \quad y' = f(x,y) \qquad (3.5.3)$$

Thus comparing (3.5.1) and (3.5.3) the first order equation admits (3.5.2) if and only if (ς, η) satisfy

$$\eta^{(1)} - \eta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y(y')^2 - (\xi\ f_x + \eta f_y) = 0$$

hence,

$$\eta_x + (\eta_y - \xi_x)y' - \xi_y(y')^2 - (\xi\ f_x + \eta f_y) = 0 \qquad (3.5.4)$$

is the desired determining equation for the infinitesimal transformation (3.5.2).

$\frac{dy}{dx}+r(x,y)=0 \Leftrightarrow \frac{dy}{dx}=-r(x,y)\equiv f(x,y)$,replacing f *by* $-r$ and

y' *by* $-r$ by we arrive at

$$\eta_x+\varsigma_x r_x-\xi_y r^2 x+\xi r_x=0 \tag{3.5.5}$$

which is the determining equation for the infinitesimal transformation for the Abel's equation

$\frac{dy}{dx}+r(x,y)=0$.

According to Bluman and Kumei [4] any substitution of the form $\eta=f\xi+\chi$ or

$\eta=-r\xi+\chi$ is known to have infinite solutions ,where as $\eta=f\xi$ yields trivial solutions.

But $\eta=\phi_1(x)+\phi_2(x)y$, $\xi=\xi(x)$ gives non-trivial solutions Schwarz [34].

Applying η , ξ for the non trivial solutions on case (a); (RNF)

$\frac{dy}{dx}=-\{\ Ay^3+By\ \}\equiv-\ r(x,y)$ and further by equating to zero coefficients of, y^m

,setting $\phi_1=0$, $\phi_2=\phi$ leads to equations

$$\phi+\frac{1}{2}\xi'+\frac{1}{2}(\log A)'\xi=0. \qquad \text{(i)}$$

$$(\phi+B\xi)'=0 \qquad \text{(ii)} \qquad (3.5.6)$$

or

$$\phi+\frac{1}{2}\xi'+\frac{1}{2}(\log A)'\xi=0. \qquad \text{(i)}$$

$$[\ \xi'+((\log A)'-2B)\xi]'=0 \qquad \text{(iii)} \qquad (3.5.7)$$

Integrating (3.5.7) we obtain,

$$\xi=\frac{1}{A}e^{2\int Bdx}[\ C_1\int Ae^{-2\int Bdx}dx+C_2\ ]$$

Case1: $C_1 = 0, C_2 = 1$ then

$$V_1 = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \; : \; \xi = \frac{\overline{B}}{A} \; , \eta = -\frac{\overline{B}}{A}By ; \overline{B} = e^{2\int Bdx}$$

Case2: $C_1 = 1, C_2 = 0$ then

$$V_2 = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \; : \; \xi = \frac{\overline{B}}{A}\int\frac{A}{B}dx \; , \eta = -\frac{B}{A}\int\frac{A}{B}dx - \frac{1}{2}y ; \overline{B} = e^{2\int Bdx}$$

i.e. $V_1 = \left(\frac{\overline{B}}{A} \right)\frac{\partial}{\partial x} - \left(\frac{\overline{B}}{A}By \right)\frac{\partial}{\partial y}$

$$V_2 = \left(\frac{\overline{B}}{A}\int\frac{A}{B}dx \right)\frac{\partial}{\partial x} - \left(\frac{B}{A}\int\frac{A}{B}dx + \frac{1}{2}y \right)\frac{\partial}{\partial y}$$

Example 3.5.1

Consider the Abel's first order ordinary differential equation

$yy' + 2y + x = 0$.

$yy' + 2y + x = 0$ is Abel's equation of second kind which is first transformed to

$y' - xy^3 - 2y^2 = 0$ - Abel's equation, first kind

then further transformed into (RNF)

$$y' - \frac{4}{724x^3}y^3 - \frac{2}{3x}y^2 + 1 = 0 \quad \text{(ii)} \qquad \text{of type (b).}$$

$$1dy + \left(-\frac{4}{724x^3}y^3 - \frac{2}{3x}y^2 + 1 \right)dx = 0$$

$$Ndy + Mdx = 0 \qquad \text{(iii)}$$

i.e. $N = 1, \qquad M = \left(-\frac{4}{724x^3}y^3 - \frac{2}{3x}y^2 + 1 \right)$

Then the infinitesimal generator

$$V = \left(\frac{3A}{A' - 3AB} \right)\frac{\partial}{\partial x} - \left(\frac{A'y}{A' - 3AB} \right)\frac{\partial}{\partial y}$$

$$V=(-3x\ )\frac{\partial}{\partial x}-(3y)\frac{\partial}{\partial y}:\xi=-3x\ ,\eta=-3y$$

The Lie's integrating factor μ for equation (ii) becomes

$$\mu=\frac{1}{\xi M+\eta N}\ =\ \left[-3x-3y\left(\ -\frac{4}{724x^3}y^3-\frac{2}{3x}y^2\ +1\ \right)\right]^{-1}$$

Multiplying equation (ii) by μ and integrating, yields the solution curve

$$\frac{y-9x}{x(2y+9x)}e^{\frac{27x}{2y+9x}}=C$$

Example 3.5.2

The first order ordinary differential equation

$$x^3y'-y^3-x^2y=0$$

is an Abel type of equation.

This is Abel's equation of first kind which is first transformed to

then further transformed into (RNF)

$$y'\ -\frac{1}{x^3}y^3-\frac{1}{x}y=0 \quad \text{(ii)} \qquad \text{of type (a).}$$

$$1dy\ +\left(\ -\frac{1}{x^3}y^3-\frac{1}{x}y\ \right)dx=0$$

$$Ndy\ +Mdx=0 \qquad \text{(iii)}$$

i.e. $N=1,\qquad M=\left(\ -\frac{1}{x^3}y^3-\frac{1}{x}y\ \right)$

Then the infinitesimal generator

$$V_1=\left(\ \frac{\overline{B}}{A}\ \right)\frac{\partial}{\partial x}-\left(\frac{\overline{B}By}{A}\right)\frac{\partial}{\partial y}$$

$$V_2=\left(\ \frac{\overline{B}}{A}\int\frac{A}{B}dx\ \right)\frac{\partial}{\partial x}-\left(\frac{\overline{B}}{A}\left(\int\frac{A}{B}dx\right)By+\frac{1}{2}y\right)\frac{\partial}{\partial y}\quad V_1=(-x\ )\frac{\partial}{\partial x}-(y)\frac{\partial}{\partial y}:\xi_1=-x\ ,\eta_1=-y$$

$$V_2=(-x\ )\frac{\partial}{\partial x}-(y)(x^{-4}+2^{-1})\frac{\partial}{\partial y}:\xi_2=-x\ ,\eta_2=-y(x^{-4}+2^{-1})$$

The Lie's integrating factor μ for equation (ii) becomes

$$\mu=\frac{1}{\xi_1 M+\eta_1 N}=\quad [-y-x(\ -\frac{1}{x^3}y^3-\frac{1}{x}y\ )\ ]^{-1}$$

Multiplying equation (ii) by μ and integrating, yields the solution curve

$$y^2=\frac{x^2}{K-2\ln x}$$

3.6 Lie Groups and Partial Differential Equations

3.6.1 Transformation Groups of Partial Differential Equations

Definition of a symmetry group for partial differential equations is the same as that for ordinary differential equations.

Let us consider partial differential equations of the $m-th$ order

$$:u_t=F\,(t,x,u,u_x,u_{xx}\ \ ....u_{x^m}),\quad \partial F\big|\partial u_{x^m}\neq 0. \qquad (3.6.1)$$

Definition 3.6.1

A set G of invertible transformations of the variables, $t,\ \ x,\ \ u,$:

$$\bar{t}=f(t,x,u,\varepsilon),\bar{x}=g(t,x,u,\varepsilon),\bar{u}=h(t,x,u,\varepsilon), \qquad (3.6.2)$$

is called a one parameter group admitted by the equation (3.6.1), if G contains the inverse to it's transformations, the identity $\bar{t}=t,\bar{x}=x,\bar{u}=u,$ as well as the composition:

$$\bar{\bar{t}}\equiv f(\bar{t},\bar{x},\bar{u},\delta)=f(t,x,u,\varepsilon+\delta),$$

$$\bar{\bar{x}}\equiv g(\bar{t},\bar{x},\bar{u},\delta)=g(t,x,u,\varepsilon+\delta),$$

$$\bar{\bar{u}}\equiv h(\bar{t},\bar{x},\bar{u},\delta)=h(t,x,u,\varepsilon+\delta),$$

and if the equation $(3.6.1)$ has the form in the new variables $\bar{t},\bar{x},\bar{u}$. :

$$\bar{u}_t = F(\bar{t},\bar{x},\bar{u},\bar{u}_{\bar{x}},\bar{u}_{\bar{x}\bar{x}}.........\bar{u}_{\bar{x}^m}) \tag{3.6.3}$$

The function F has the same form in both

equations $(3.6.1)$ and $(3.6.3)$.

Again the construction of the symmetry group G is equivalent to determination of it's infinitesimal transformations

$$\bar{t} \approx t + \varepsilon\tau(t,x,u), \quad \bar{x} \approx x + \varepsilon\xi(t,x,u), \quad \bar{u} \approx u + \varepsilon\eta(t,x,u) \tag{3.6.4}$$

obtained from $(3.6.2)$ by expanding into Taylor series with respect to the group parameter ε and keeping only the terms linear in ε. The infinitesimal transformation $(3.6.4)$ provides the generator of the group G, i.e. the differential operator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u.} \tag{3.6.5}$$

acting on any differentiable function $J(t,x,u)$ as follows:

$X(J) = \tau(t,x,u)\frac{\partial J}{\partial t} + \xi(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u}$ .The generator $(3.6.5)$ is called an operator admitted by equation $(3.6.1)$ or an infinitesimal symmetry for equation $(3.6.1)$. The group transformations $(3.6.2)$ corresponding to the generator $(3.6.5)$ are found by solving the Lie

$$\left.\begin{array}{l}\text{equations } \dfrac{d\bar{t}}{d\varepsilon} = \tau(\bar{t},\bar{x},\bar{u}), \quad \dfrac{d\bar{x}}{d\varepsilon} = \varsigma(\bar{t},\bar{x},\bar{u}), \quad \dfrac{d\bar{u}}{d\varepsilon} = \eta(\bar{t},\bar{x},\bar{u}), \\ \text{with the initial conditions: } \bar{t}_{\varepsilon=0} = t, \quad \bar{x}_{\varepsilon=0} = x, \quad \bar{u}_{\varepsilon=0} = u\end{array}\right\} \tag{3.6.6}$$

Any symmetry transformation of a differential equation carries over any solution of differential equation into it's solution. It means that, just like in the case of ordinary differential equations, the solutions of a partial differential equations, are permuted among themselves under the action of a symmetry group. The solutions may also be individually unaltered, then they are called *invariant solutions*. Accordingly, group analysis provides two basic ways for constructions of exact solutions: ***group transformations of known solutions*** and ***constructions of invariant solutions.***

3.7 Invariant Functions

Definition 3.7.1

A curve $F(x,y)=0$ is an invariant curve for transformations $(3.1.1)$ if and only if

$F(\bar{x},\bar{y})=0$ whenever $F(x,y)=0$

Theorem 3.7.1

A surface written in a solved form $F(x)=x_n - f(x,y,y_1,y_2,y_3,......y_{n-1})=0$ is an invariant surface for $(3.1.1)$ if and only if

$$VF(x)=0 \text{ whenever } F(x)=0 \ : \qquad V=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y} \tag{3.7.1}$$

For proof, see Bluman and Kumei [4]

Theorem 3.7.2

A curve written in a solved form $F(x,y)=y-f(x)=0$, is invariant curve for generator

$$V=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}$$

$$\left.\begin{array}{ll} \text{if and if} & VF(x,y)=\eta(x,y)-\xi(x,y)f'(x)=0 \\ \text{whenever} & F(x,y)=y-f(x)=0 \end{array}\right\} \tag{3.7.2}$$

For proof ,see Bluman and Kumei [4]

Definition 3.7.2

(i) A function $F(x)$ is said to be invariant function of the group of transformations $(3.1.1)$ iff for any group , $F(x^*) = F(x)$

(ii) A curve $F(x, y) = 0$, is said to be invariant curve for a one -parameter Lie group of transformations $(3.1.1)$ iff $F(x^*, y^*) = 0$ when $F(x, y) = 0$

Using results of (i) and (ii) we can solve $(3.7.1)$ for the Lie group of transformations.

Example 3.7.1

We consider the Lie group of transformations,

$$x^* = X(x, y; \varepsilon) = e^{\varepsilon} x \qquad y^* = Y(x, y; \varepsilon) = e^{2\varepsilon} y \qquad (3.7.3\ )$$

with infinitesimal generator, $V = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$.

A curve , $y - \lambda x = 0, x > 0, \lambda = cons\tan t$, is said to be invariant curve for $(3.7.3\ )$ since;

$V(y - \lambda x) = x\dfrac{\partial(y - \lambda x)}{\partial x} + y\dfrac{\partial(y - \lambda x)}{\partial y} = -\lambda x + y$ which is only equal to zero if $y - \lambda x = 0$.

3.8 Invariance of Partial Differential Equations

In this subsection we apply infinitesimal transformations to the construction of solutions of partial differential equations (PDEs). We will consider systems of (PDEs) and show that the infinitesimal criterion for their invariance leads directly to an algorithm to determine infinitesimal generators V admitted by a given partial differential equation. Invariant surfaces of the corresponding Lie group of point transformations lead to invariant solutions (similarity solutions) Ibragimov [10]. These solutions are obtained by solving partial differential equations with fewer independent variables than the given (PDEs).

First we consider a k^{th} order partial differential equation in the form

$$F(x,u,u_1,u_2,u_3.......,u_k)=0 \qquad (3.8.1)$$

where $x=(x_1,x_2,x_3,............,x_n)$ denotes n independent variables, u_j denotes the set of coordinates corresponding to all the $j-th$ order partial derivatives with respect to x.

In fact equation (3.8.1) becomes an algebraic equation which defines a hyper -surface in $(x,u,u_1,u_2,u_3.,......,u_m)$ - space. We assume that the partial differential equation (3.8.1) can be written in solvable form in terms of some k^{th} order partial derivative of u :

$$F(x,u,u_1,u_2,u_3.......,u_k)=u_{i_1i_2i_{3_4}i_5............i_l}-.\,f(x,u,u_1,u_2,u_3.......,u_k)=0, \qquad (3.8.2)$$

where $f(x,u,u_1,u_2,u_3.......,u_k)$ does not depend on $u_{i_1i_2i_{3_4}i_5............i_j}$.

Definition 3.8.1

Let $F_r(x,u,u_1,u_2,u_3.......,u_k)=0 \;, r=1,2,3............l$ (3.8.3)

be system of differential equations. The system is said to of maximal rank if Jacobian matrix

$J\,F_r(x,u,u_1,u_2,u_3.......,u_k)=\left(\dfrac{\partial F_r}{\partial x_i},\dfrac{\partial F_r}{\partial u_{kj}}\right)$ of F with respect to all the variables (x,u_i) is of rank l whenever $F(x,u,u_1,u_2,u_3.......,u_k)=0$

Definition 3.8.2

The one-parameter Lie group of transformations

$$x^*=X(x,u;\varepsilon) \qquad (3.8.4)$$

$$u^*=U(x,u;\varepsilon) \qquad (3.8.5)$$

leaves the partial differential equation (3.8.1) invariant if and only if its $k-th$ extension,

$x^*,\ u^*,\ u_1{}^*,\ u_2{}^*, \ldots\ldots\ldots\ldots u_k{}^*$ leaves the surface $F(x,u,u_1,u_2,u_3\ldots\ldots,u_k)=0$ invariant.

Theorem 3.8.1

Let G be a Lie group of transformations acting on $m-$dimensional manifold M. Let $F: M \to R^l$, $l \leq m$, define a system of algebraic equations

$$F_r(x)=0,\ r=1,2,3\ldots\ldots\ldots\ldots l$$

and assume the system is of maximal rank.

Then G is a symmetry group of the system if and only if

$V[\,F_r(x)\,]=0,\ r=1,2,3\ldots\ldots\ldots\ldots l$ whenever $F_r(x)=0$ for every infinitesimal generator V of G.

For proof ,see Olver [18].

Theorem 3.8.2

Let

$$F_r(x,u,u_1,u_2,u_3\ldots\ldots,u_k)=0,\ r=1,2,3\ldots\ldots\ldots\ldots l$$

be a system of partial differential equations of maximal rank defined on M. If G is a group of transformations acting on M and

$$V^{(k)}[\,F_r(x,u^{(k)})\,]=0,\ r=1,2,3\ldots\ldots\ldots\ldots l,$$

where

$$F_r(x,u^{(k)}) \equiv F_r(x,u,u_1,u_2,u_3\ldots\ldots,u_k) \text{ whenever } F(x,u^{(k)})=0$$

for every infinitesimal generator V of G,

then G is a symmetry group of the system of partial differential equations

$$F_r(x,u^{(k)})=0,\ r=1,2,3,\ldots\ldots,l\ .$$

For proof ,see Bluman and Kumei [4]

Now we give a criterion for the invariance of a

partial differential equation.

Theorem. 3.8.3

Let

$F_r(x,u^{(k)})=0$ be a non degenerate system of partial differential equations.

Let

$$V=\xi_i(x,u)\frac{\partial}{\partial x_i}+\eta(x,u)\frac{\partial}{\partial u} \qquad (3.8.6)$$

be the infinitesimal generator of the one-parameter Lie group of transformations

(3.8.4) , (3.8.5)

and let

$$V^{(k)}=\xi_i(x,u)\frac{\partial}{\partial x_i}+\eta(x,u)\frac{\partial}{\partial u}+\eta_i^{(1)}(x,u,u_1)\frac{\partial}{\partial u_1}+\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots+\eta^{(k)}{}_{i_1i_2\ldots\ldots i_k}(x,u,u_1,u_2,\ldots\ldots u_k)\frac{\partial}{\partial u_{i_1i_2i_3\ldots\ldots\ldots i_k}} \qquad (3.8.7)$$

be the corresponding k^{th} extended infinitesimal generator of (3.8.6) where

$$\eta_i^{(l)}=D_i\eta-(D_i\xi_j)u_j, i=1,2,3,\ldots\ldots,n; \qquad (3.8.8)$$

and $\eta^{(j)}{}_{i_1i_2i_{3_4}i_5\ldots\ldots\ldots i_j}$ .is given by

$$\eta^{(k)}{}_{i_1i_2i_{3_4}i_5\ldots\ldots\ldots i_k}.=D_{i_k}\eta^{(k-1)}{}_{i_1i_2i_3\ldots\ldots i_{k-1}}-(D_{i_k}\xi_j)u_{i_1i_2\ldots i_{k-1}j} \qquad (3.8.9)$$

$i_j=1,2,3,\ldots\ldots,n\ for, j=1,2,3,,k\ with\ k=1,2,3,\ldots\ldots$ in terms of $(\xi(x,u),\eta(x,u))$.

Then a connected local group of transformations G of the form; (3.8.4), (3.8.5) is a symmetry group of the system of partial differential equations

if and only if

$V^{(k)}[\ F_r(x,u^{(k)}\ )\ ]=0\ ,r=1,2,3............l$, whenever $F(x,u^{(k)}\ )=0$ (3.8.10)

Proof

Sufficiency condition.

We assume that

$V^{(k)}[\ F_r(x,u^{(k)}\ )\ ]=0\ ,r=1,2,3............l$, whenever $F(x,u^{(k)}\ )=0$, such that

$F_r(x,u^{(k)}\ )=0$ is a non degenerate system of partial differential equations, for every infinitesimal generator V of G.

Then we need to prove that group G is a symmetry of system of partial differential equations $F_r(x,u^{(k)}\ )=0$.

Since the system of equations $F_r(x,u^{(k)}\ )=0$ is non degenerate i.e. is of maximal rank then by theorem 3.8.2 , $V^{(k)}$ the $k-th$ extension of V leaves $F_r(x,u^{(k)}\ )=0$ invariant and hence G is a symmetry of system of partial differential equations

$F_r(x,u^{(k)}\ )=0$

Necessity of this condition.

We assume that

$F_r(x,u^{(k)}\ )=0$ is a non degenerate system of partial differential equations and that a connected local group of transformations G of the form; (3.8.4), (3.8.5) acting on open subset $M\subset X\times U$ is a symmetry group of the system of partial differential equations $F_r(x,u^{(k)}\ )=0$.

We need to prove that

$V^{(k)}[\ F_r(x,u^{(k)}\ )\ ]=0\ ,r=1,2,3............l$, whenever $F(x,u^{(k)}\ )=0$.

From theorem 3.8.1 it suffices to prove that the subset $S_F = \{ F(x,u^{(k)}) = 0 \}$ is an invariant subset the prolonged group action $G^{(k)}$ whenever G transforms solutions of the system to other solutions.

Let $(x_0, u_0^{(k)}) \in S_F$.

Using local solvability, let $u = f(x)$ be a solution of the system defined in a neighborhood of x_0 such that $u^{(k)}{}_0 = (f(x_0))^{(k)}$.

If g is a group element such that $g^{(k)}(x_0, u_0^{(k)})$ is defined, then by approximately shrinking the domain of the definition of f, we can ensure that the transformed function $\widehat{f} = g.f$ is a well defined function in a neighborhood of $\widehat{x}_0$, where

$(\widehat{x}_0, \widehat{u}_0^{(k)}) = g.\ (x_0, u_0^{(k)})$.

Since G is a symmetry group, $u = \widehat{f}(x)$ is also a solution to the system.

More so by the condition of prolonged group action,

$$g^{(k)}(x_0, u_0^{(k)}) = (x_0, (g.f)^{(k)}(\widehat{x}_0)) = (\widehat{x}_0, \widehat{u}_0^{(k)}) .$$

Hence transformed point $(\widehat{x}_0, \widehat{u}_0^{(k)})$ must again lie in S_F, thus, $F(x_0, u_0^{(k)}) = 0$.

In general

$$g^{(k)}(x_i, u_i^{(k)}) = (x_i, (g.f)^{(k)}(\widehat{x}_i)) = (\widehat{x}_i, \widehat{u}_i^{(k)}) .$$

Hence transformed point $(\widehat{x}_i, \widehat{u}_i^{(k)})$ must again lie in S_F, thus $F(x_i, u_i^{(k)}) = 0$.

Hence $g^{(k)}(F(x, u^{(k)})) = 0$, since $F(x_i, u_i^{(k)}) = 0$.

So without loss generality we obtain,

$$V^{(k)}(F_r(x, u^{(k)})) = 0 \ , r = 1,2,3,.....,l \ , \text{ given that } F(x, u^{(k)}) = 0 .$$

This completes the proof.

3.9 Invariant Solutions

If a group of transformations maps a solutions into itself, we arrive at what is called a ***self-similar*** or ***group invariant*** solution [Ibragimov [10], Stephani [36]].

Given the infinitesimal symmetry (3.6.5) of equation (3.6.1) the invariant solution under the one –parameter group generated by a generator V are obtained as follows.

We calculate two independent invariants $J_1 = k(x,t)$ and $J_2 = \mu(x,t,u)$ by solving the equation

$$V(J) \equiv \tau(t,x,u)\frac{\partial J}{\partial t} + \varsigma(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u} = 0 \qquad (3.9.1)$$

or its system of characteristics

$$\frac{dt}{\tau(x,t,u)} = \frac{dx}{\varsigma(x,t,u)} = \frac{du}{\eta(x,t,u)} \qquad (3.9.2)$$

Then we designate one of the invariants as a function of the other e.g.

$$\mu = \phi(k) \qquad (3.9.\ 3)$$

and solve (3.9.3) with respect to u .Finally we substitutes expression for u ,in equation (3.6.1) and obtain ordinary differential equation for the unknown function $\phi(k)$ of one variable. This procedure reduces the number of independent variables by one .

Example 3.9.1

We discuss the invariant solutions of the heat equation $u_t = u_{xx}$ under the group generated by the infinitesimal generator $X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$.

It can be easily shown that the heat equation $u_t = u_{xx}$ admits the infinitesimal generators

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial t}, \quad v_3 = u\frac{\partial}{\partial u}, \quad v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t},$$

$$v_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \quad v_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - [2ut + x^2u]\frac{\partial}{\partial u}$$

There are two independent invariants for X . One of them is t , while the other is obtained from the characteristic equation

$$\frac{xdx}{2t}+\frac{du}{u}=0$$

Integrating the equation yields the invariant

$J=ue^{\frac{x^2}{4t}}$ .Consequently one seeks the invariant solution in the form $J=\phi(t)$, or

$u=\phi(t)e^{-\frac{x^2}{4t}}$.

Now substitute this expression into the heat equation $u_t = u_{xx}$ .We have the first order ordinary differential equation

$$\frac{d\phi}{dt}+\frac{\phi}{2t}=0$$

It follows that $\phi(t)=\frac{C}{\sqrt{t}}, \ C-consant$.

Hence the invariant solution is

$$u=\frac{C}{\sqrt{t}}e^{-\frac{x^2}{4t}}, \ C-consant$$

Example 3.9.2

We examine the invariant solutions of the partial differential equation $u_t = uu_x + u_{xx}$ under the group generated by the infinitesimal generator $V_4 = x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}$.

Note that the partial differential equation $u_t = uu_x + u_{xx}$, admits the infinitesimal generators

$$v_1=\frac{\partial}{\partial t}, \ v_2=\frac{\partial}{\partial x}, v_3=t\frac{\partial}{\partial x}-\frac{\partial}{\partial u}, \ v_4=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u},$$

$$v_5 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - [\ ut + x\ ]\frac{\partial}{\partial u}.$$

For the infinitesimal generator $V_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$,

the corresponding characteristic system

$$\frac{dx}{x} = \frac{dt}{2t} = -\frac{du}{u}$$

provides the following invariants; $\alpha = \frac{x}{\sqrt{t}}$, $\mu = u\sqrt{t}$.

Consequently one seeks the invariant solution in the form

$u = \frac{1}{\sqrt{t}}\psi(\alpha)$, $\alpha = \frac{x}{\sqrt{t}}$. Substituting u , u_t , u_x , u_{xx} into the partial differential equation

$u_t = u u_x + u_{xx}$, we arrive at the second order variable coefficients ordinary differential equation

$$\psi'' + \psi\psi' + \frac{1}{2}(\alpha\psi' - \psi) = 0.$$

Integrating once , one has

$$\psi' + \frac{1}{2}(\psi^2 - \alpha\psi) = C.$$

Let C=0

then we obtain the invariant solutions of the partial differential equation

$u_t = u u_x + u_{xx}$ as ,

$$u = \frac{2}{\sqrt{\pi t}}\frac{e^{-\frac{x^2}{4t}}}{\beta + erf\left(\frac{x}{2\sqrt{t}}\right)} \quad , \ \beta \ arbitrary\ constant. \ .$$

3.10 Group Transformations of Solutions

The method is based on the fact that a symmetry group transforms any solutions of the equation in question into solutions of the same equation. Namely, let (3.6.2) be a symmetry transformation group of the equation (3.6.1), and let a function $u = \Phi(x,t)$ solve the equation (3.6.1). Since (3.6.2) is a symmetry transformation, the above solution can also be written in the new variables: $\bar{u} = \Phi(\bar{x}, \bar{t})$.

If $\bar{u}$, $\bar{x}$, $\bar{t}$ are group transformations of the partial differential equation (3.6.1) with $\bar{u}$, of the form $\bar{u} = \Psi(u,x,t,\varepsilon)$, for some explicit function Ψ , then applying the inverse mapping, the new symmetry solution $\hat{u}$ is defined by $\hat{u} = \Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t})\right), g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t}), \varepsilon^{-1}\right)$ where $u = \Phi(x,t)$ is any known solution of (3.6.1).

Having solved equation (3.6.6) with respect to u we obtain a one parameter family (with a parameter ε) of new solutions to the equation (3.6.1) as

$$\hat{u} = \Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t})\right), g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t}), \varepsilon^{-1}\right) \qquad (3.10.1)$$

Example 3.10.1

Consider the infinitesimal generator $V_5 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (tu + x)\frac{\partial}{\partial u}$ admitted by the partial differential equation $u_t = uu_x + u_{xx}$.

The Lie equations have the form $\frac{d\bar{t}}{d\varepsilon} = \bar{t}^2$, $\frac{d\bar{x}}{d\varepsilon} = \bar{x}\bar{t}$, $\frac{d\bar{u}}{d\varepsilon} = -(\bar{t}\bar{u} + \bar{x})$

Integrating these equations yield the groups,

$$\bar{x} = \frac{x}{1-\varepsilon t}, \bar{t} = \frac{t}{1-\varepsilon t}, \bar{u} = u(1-\varepsilon t) - \varepsilon x$$

Hence we obtain the inverse mappings,

$$\bar{x}^{-1}(x) = \frac{x}{1+\varepsilon t}, \bar{t}^{-1}(t) = \frac{t}{1+\varepsilon t}, u = \frac{1}{1-\varepsilon t}\bar{u} + \frac{\varepsilon x}{1-\varepsilon t}$$

; $\bar{u} = \overline{\Phi}(\bar{x},\bar{t})$, $\bar{u} = \Psi(u,x,t,\varepsilon)$ and finally our new solution based on the inverse groups,

$\bar{x}^{-1} = \frac{x}{1+\varepsilon t}$, $\bar{t}^{-1} = \frac{t}{1+\varepsilon t}$, $\bar{u} = u(1-\varepsilon t) - \varepsilon x$ takes the form

$$\hat{u} = \Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t})\right), g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t}), \varepsilon\right)$$

and we obtain the new symmetry solution

$$\hat{u}(x,t) = \frac{-\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t}\Phi\left[\frac{x}{1+\varepsilon t}, \frac{t}{1+\varepsilon t}\right],$$

where $\Phi[x, t]$ is any known solution of the equation.

Example 3.10.2

We examine the groups admitted by the heat equation $u_t = u_{xx}$ and its corresponding new symmetry solutions under the infinitesimal generator $X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$.

The Lie equations have the form, $\frac{d\bar{x}}{d\varepsilon} = 2\bar{t}$, $\frac{d\bar{u}}{d\varepsilon} = -(\bar{x}\bar{u})$

Integrating these equations yield the groups,

$$\bar{x} = x + 2\varepsilon t, \bar{t} = t \quad , \bar{u} = ue^{-\left(\varepsilon x + \varepsilon^2 t\right)} : \bar{u}(u,x,t,\varepsilon) \equiv \Psi(u,x,t,\varepsilon)$$

Hence we obtain the inverse mappings,

$$\bar{x}^{-1}(x) = x - 2\varepsilon t, \bar{t}^{-1}(t) = t, u = \bar{u}e^{\left(\varepsilon x + \varepsilon^2 t\right)}$$

If $u = \Phi[x, t]$ is any known solution of the heat equation then

$$\hat{u} = \Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t})\right), g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t}), \varepsilon\right)$$

$\bar{u} = e^{-\varepsilon\bar{x} + \varepsilon^2\bar{t}}\Phi(\bar{x} - 2\varepsilon\bar{t}, \bar{t})$ is the transformed function in this case and without any loss of generality, we obtain

$$\hat{u} = e^{-\varepsilon x + \varepsilon^2 t}\Phi(x - 2\varepsilon t, t)$$

as the new transformed solution.

Example 3.10.3

Consider the infinitesimal generator $v_8 = \left(x^2 - y^2 + t^2\right)\frac{\partial}{\partial x} + 2yx\frac{\partial}{\partial y} + 2xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u}$

admitted by the partial differential equation $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$.

It can be shown that the partial differential equation $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ admits the

infinitesimal generators

$$v_1 = \frac{\partial}{\partial x} \quad , \quad v_2 = \frac{\partial}{\partial y} \quad , v_3 = \frac{\partial}{\partial t} \quad , \quad v_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} \quad ,$$

$$v_5 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad t\frac{\partial}{\partial t} \quad , v_6 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \quad v_7 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} \quad ,$$

$$v_8 = \left(x^2 - y^2 + t^2\right)\frac{\partial}{\partial x} + 2yx\frac{\partial}{\partial y} + 2xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u},$$

$$v_9 = 2xy\frac{\partial}{\partial x} + \left(-x^2 + y^2 + t^2\right)\frac{\partial}{\partial y} + 2yt\frac{\partial}{\partial t} - yu\frac{\partial}{\partial u},$$

$$v_{10} = 2xt\frac{\partial}{\partial x} + 2yt\frac{\partial}{\partial y} + \left(x^2 + y^2 + t^2\right)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u},$$

$$v_{11} = u\frac{\partial}{\partial u} \quad , \quad v_\alpha = \alpha(x,y,t)\frac{\partial}{\partial u} \; v_\alpha = \alpha(x,y,t)$$

For the infinitesimal generator

$$v_8 = \left(x^2 - y^2 + t^2\right)\frac{\partial}{\partial x} + 2yx\frac{\partial}{\partial y} + 2xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u},$$

the Lie equations have the form

$$\frac{d\bar{t}}{d\varepsilon} = 2\bar{x}\bar{t}\,, \quad \frac{d\bar{x}}{d\varepsilon} = \bar{x}^2 - \bar{y}^2 - \bar{t}^2\,, \quad \frac{d\bar{y}}{d\varepsilon} = 2\bar{x}\bar{y}\,, \quad \frac{d\bar{u}}{d\varepsilon} = -(\bar{x}\bar{u})$$

Integrating the corresponding Lie equations yield the groups:

$$\bar{t} = \frac{t}{1 - 2\varepsilon x - \varepsilon^2\left(t^2 - x^2 - y^2\right)}, \quad \bar{x} = \frac{x + \varepsilon\left(t^2 - x^2 - y^2\right)}{1 - 2\varepsilon x - \varepsilon^2\left(t^2 - x^2 - y^2\right)}, \quad \bar{y} = \frac{y}{1 - 2\varepsilon x - \varepsilon^2\left(t^2 - x^2 - y^2\right)},$$

$$\bar{u} = u\sqrt{1 - 2\varepsilon x - \varepsilon^2\left(t^2 - x^2 - y^2\right)}.$$

Then the inverse mappings are;

$$\bar{t}^{-1} = \frac{t}{1+2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)}, \quad \bar{x}^{-1} = \frac{x-\varepsilon\left(t^2-x^2-y^2\right)}{1+2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)},$$

$$\bar{y}^{-1} = \frac{y}{1+2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)} \text{ and } u = \frac{\bar{u}}{\sqrt{1-2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)}}$$

If $u = \Phi[\ x, y, t\ ]$;is any known solution of the wave equation with $\bar{u} = \Psi(u, x, t, \varepsilon)$ then

$$\hat{u} = \Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t})\right), g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t}), \varepsilon\right)$$

i.e., $$\hat{u} = \frac{\Phi[\bar{x}^{-1}, \bar{y}^{-1}, \bar{t}^{-1}]}{\sqrt{1+2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)}}$$

is a new symmetry solution.

The next chapter illustrates the basic methods for determination of infinitesimals ,infinitesimal symmetry and symmetry transformations where we discuss: Korteweg-de-Fries, two-dimensional wave and the Boussinesq equations.

CHAPTER 4

DETERMINATION OF SYMMETRY OF PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we illustrate the procedure for determining symmetry of partial differential equations by discussing examples involving ; Korteweg-de-Vries, two-dimensional wave and Boussinesq equations.

4.1 Korteweg-de-Fries Equation

Korteweg-de-Fries equation is a nonlinear third order partial differential equation of the form

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x}=0 \qquad (4.1.1\ )$$

We need to determine its infinitesimals, infinitesimal generators and all the groups it admits. This equation arises in the theory of long waves in shallow water and other physical systems. Here the required symmetry groups of transformations are of the form

$$t^*=T(t,x,u;\varepsilon),\ x^*=X(t,x,u;\varepsilon),\ u^*=U(t,x,u;\varepsilon) \qquad (4.1.1\text{a})$$

with corresponding infinitesimals

$$\xi(t,x,u)=\frac{\partial X(t,x,u;\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0},\ \tau(t,x,u)=\frac{\partial T(t,x,u;\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}\ \phi(t,x,u)=\frac{\partial U(t,x,u;\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}$$

We let the generator V ,of (4.1.1) be of the form

$$V=\xi(t,x,u)\frac{\partial}{\partial x}+\tau(t,x,u)\frac{\partial}{\partial t}+\phi(t,x,u)\frac{\partial}{\partial u} \qquad (4.1.1\text{b})$$

We determine all the coefficient functions ξ , τ , ϕ so that the corresponding one-parameter Lie group of transformations $t^*=T(t,x,u;\varepsilon)$, $x^*=X(t,x,u;\varepsilon)$, $u^*=U(t,x,u;\varepsilon)$ form a symmetry group of (4.1.1).

For the symmetry condition to be satisfied by (4.1.1) then

$$V^{(3)}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]=0 \qquad (4.1.2)$$

;such that $V^{(3)}$ is the third prolongation with

$$V^{(3)} = \xi(t,x,u)\frac{\partial}{\partial x}+\tau(t,x,u)\frac{\partial}{\partial t}+\phi(t,x,u)\frac{\partial}{\partial u}+\eta^{x}\frac{\partial}{\partial u_x}+\eta^{t}\frac{\partial}{\partial u_t}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+$$

$$\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{tt}\frac{\partial}{\partial u_{tt}}+\eta^{xxx}\frac{\partial}{\partial u_{xxx}}+\eta^{xtx}\frac{\partial}{\partial u_{xtx}}+\eta^{xxt}\frac{\partial}{\partial u_{xxt}}+\eta^{ttx}\frac{\partial}{\partial u_{ttx}}+.\eta^{ttt}\frac{\partial}{\partial u_{ttt}}\ .$$

Equation (4.1.2) becomes

$$\Big[\ \xi(t,x,u)\frac{\partial}{\partial x}+\tau(t,x,u)\frac{\partial}{\partial t}+\phi(t,x,u)\frac{\partial}{\partial u}+\eta^{x}\frac{\partial}{\partial u_x}+\eta^{t}\frac{\partial}{\partial u_t}+$$

$$\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{tt}\frac{\partial}{\partial u_{tt}}+\eta^{xxx}\frac{\partial}{\partial u_{xxx}}+\eta^{xtx}\frac{\partial}{\partial u_{xtx}}+$$

$$\eta^{xxt}\frac{\partial}{\partial u_{xxt}}+\eta^{ttx}\frac{\partial}{\partial u_{ttx}}+.\eta^{ttt}\frac{\partial}{\partial u_{ttt}}\Big]\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]=0. \qquad (4.1.2i)$$

This can further be simplified to give

$$\xi(t,x,u)\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+\tau(t,x,u)\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+$$

$$\phi(t,x,u)\frac{\partial}{\partial u}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+\eta^{x}\frac{\partial}{\partial u_x}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+$$

$$\eta^{t}\frac{\partial}{\partial u_t}+\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+\eta^{xx}\frac{\partial}{\partial u_{xx}}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+$$

$$\eta^{xt}\frac{\partial}{\partial u_{xt}}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+\eta^{tt}\frac{\partial}{\partial u_{tt}}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+$$

$$\eta^{xxx}\frac{\partial}{\partial u_{xxx}}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+\eta^{xtx}\frac{\partial}{\partial u_{xtx}}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+$$

$$\eta^{xxt}\frac{\partial}{\partial u_{xxt}}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+\eta^{ttx}\frac{\partial}{\partial u_{ttx}}\left[\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x} \right]+.$$

$$\eta^{ttt}\frac{\partial}{\partial u_{ttt}}[\ \frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x}\ ]=0. \qquad (4.1.2ii)$$

Here we differentiate partially with respect to the partial variables u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxt}, u_{xxx}, u_{xtx}, u_{xtt}, u_{ttt}, and u, t, x as algebraic variables.

We obtain the infinitesimals condition to be

$$\phi u_x+\eta^x u+\eta^t+\eta^{xxx}=0 \qquad (4.1.3)$$

which must be satisfied whenever $u_t=-u_{xxx}-uu_x$.

When (2.4.1a), (2.4.2a),and (2.4.7) are substituted into (4.1.3) we obtain:

$$\begin{aligned}
&\phi_t-\xi_t u_x+(\phi_u-\tau_t)u_t-\xi_u u_x u_t-\tau_u u_t^2+u_x\phi+\\
&u[\ \phi_x-\tau_x u_t+(\phi_u-\xi_x)u_x-\xi_u u_x^2-\tau_u u_t u_x\ ]+\\
&\phi_{xxx}+3u_x\phi_{uxx}+3u_x^2\phi_{uux}+3u_{xx}\phi_{ux}+u_x^3\phi_{uuu}+3u_x u_{xx}\phi_{uu}+u_{xxx}\phi_u\\
&-3u_t\left(\tau_{xxx}+3u_x\tau_{uxx}+3u_x^2\tau_{uux}+3u_{xx}\tau_{ux}+3u_x u_{xx}\tau_{uu}+u_x^3\tau_{uuu}+u_{xxx}\tau_u\right)\\
&-3u_x\left(\xi_{xxx}+3u_x\xi_{uxx}+3u_x^2\xi_{uux}+3u_{xx}\xi_{ux}+3u_x u_{xx}\xi_{uu}+u_x^3\xi_{uuu}+u_{xxx}\xi_u\right)\\
&-3u_{xx}\left(\xi_{xx}+2u_x\xi_{ux}+u_{xx}\xi_u+u_x^2\xi_{uu}\right)\\
&-3u_{xt}\left(\tau_{xx}+2u_x\tau_{ux}+u_{xx}\tau_u+u_x^2\tau_{uu}\right)-3u_{xxx}\left(\xi_x+u_x\xi_u\right)\\
&-3u_{xxt}\left(\tau_x+u_x\tau_u\right)+\phi_t-\xi_t u_x+(\phi_u-\tau_t)u_t-\xi_u u_x u_t-\tau_u u_t^2\\
&-[\ \phi_{xx}+(2\phi_{xu}-\xi_{xx})u_x-\tau_{xx}u_t+(\phi_{uu}-2\xi_{xu})u_x^2-2\tau_{xu}u_x u_t-\xi_{uu}u_x^3\\
&-\tau_{uu}u_x^2 u_t+(\phi_u-2\xi_x)u_{xx}-2\tau_x u_{xt}-3\xi_u u_x u_{xx}-\tau_u u_t u_{xx}-2\tau_u u_x u_{xt}\ ]=0
\end{aligned} \qquad (4.1.3i)$$

On replacing u_t by $-u_{xxx}-uu_x$ wherever it occurs ,and equating the coefficients of the various monomials in the first , second and third order partial derivatives of u ,we obtain the resulting determining equations for the infinitesimals for the Korteweg –de Vries equation (4.1.1): to be

Monomial terms	Equation	
u_{xxt} :	$\tau_x + u_x\tau_u \quad =0$	(i)
$u^2{}_{xxx}$:	$\tau_u \quad =0$	(ii)
$u_{xx}{}^2$:	$-3\xi_u \;=0$	(iii)
$u_x u_{xxt}$:	$\tau_u \;=0$	(iv)
$u_x u_{xxx}$:	$\xi_u = 0$	(v)
u_{xx} :	$3\xi_{xx} = 3\phi_{ux}$	(vi)
$u_x u_{xx}$:	$\xi_u + 3\phi_{uu} - 6\xi_{ux} - 9\xi_{ux} = 0$	(vii)
$u_x{}^2$:	$3\phi_{uux} = 0$	(viii)
u_x :	$\phi - \xi_t + [\;\phi_u - \xi_x\;]u \;+ 3\phi_{xuu} = 0$	(ix)
1 :	$\phi_{xxx} + \; u\phi_x + \;\; \phi_t = 0$	(x)

Solutions of equations (i)-(x) yield the infinitesimals ξ, τ, ϕ as follows:.

$$\xi = c_1 + c_3 t + c_4 x \tag{4.1.4a}$$

$$\tau = c_2 + 3c_4 t \tag{4.1.4b}$$

$$\phi = c_3 + (-2c_4 u). \tag{4.1.4c}$$

We express ξ ,τ ,ϕ in the standard basis form as

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\left.\begin{aligned} \xi &= 1.c_1 + 0.c_2 + t.c_3 + 1.c_4 x &&= c_1 + c_3 t + c_4 x \\ \tau &= 0.c_1 + 1.c_2 + 0.c_3 + 3.c_4.t &&= c_2 + 3c_4 t \\ \phi &= 0.c_1 + 0.c_2 + 1.c_3 - 2.c_4.u &&= c_3 + (-2c_4 u). \end{aligned}\right\} \tag{4.1.5}$$

We form the corresponding Lie algebra of the basis generators v_1, v_2, v_3, v_4 in (4.1.5) of the form

$v_i = \hat{\xi}_i \frac{\partial}{\partial x} + \hat{\tau}_i \frac{\partial}{\partial t} + \hat{\phi}_i \frac{\partial}{\partial u}$: $\hat{\xi}_i$, $\hat{\tau}_i$, $\hat{\phi}_i$ are the coefficients c_i in the standard solutions of ξ, τ, ϕ. Hence the v_i's $i = 1,2,3,4$ are obtained from the tabulation in equation (4.1.5) as follows:

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial t}, \quad v_3 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial x}, \quad v_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, \tag{4.1.6}$$

4.2 Lie Groups Admitted by Equation (4.1.1)

The one-parameter groups G_i admitted the by the infinitesimal generators, v_i, are determined by solving the corresponding Lie equations which yield groups (4.2.1) shown below

We now use v_1, v_2, v_3, v_4 to solve for each G_i.

Thus

$$v_1 = \frac{\partial}{\partial x};\ G_1:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_1(x+\varepsilon, t, u) \tag{4.2.1a}$$

$$v_2 = \frac{\partial}{\partial t};\ G_2:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_2(x, t+\varepsilon, u) \tag{4.2.1b}$$

$$v_3 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial x};\ G_3:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_3(x+\varepsilon t, t, u+\varepsilon) \tag{4.2.1c}$$

$$v_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u};,\ G_4:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_4\left(e^{\varepsilon} x, e^{3\varepsilon} t, e^{-2\varepsilon} u\right) \tag{4.2.1d}$$

4.3 The Wave Equation

The wave equation described in two dimensions is of the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.3.1}$$

We need to determine its infinitesimals, infinitesimal generators and all the groups it admits.

.We let the infinitesimal generator V for (4.3.1) , be of the form

$$V = \xi(t,x,y,u)\frac{\partial}{\partial x} + \tau(t,x,y,\ u)\frac{\partial}{\partial \tau} + \eta(t,x,y,u)\frac{\partial}{\partial y} + \phi(t,x,y,u)\frac{\partial}{\partial u} \tag{4.3.2}$$

Then we now determine infinitesimals ξ , τ , η , ϕ so that the corresponding one-parameter Lie group of transformations,

$$x^* = X(t,x,y,u;\varepsilon),\ y^* = Y(t,x,y,u;\varepsilon)\ ,\ \tau^* = T(t,x,y,u;\varepsilon),\ u^* = U(t,x,y,u;\varepsilon)$$

form a symmetry group of (4.3.1).

We know that the equation

$$V^{(2)}\left[\ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\ \right] = 0 \tag{4.3.3}$$

is the symmetry condition for (4.3.1) and we observe that $V^{(2)}$ is the second prolongation

with $V^{(2)} = \xi(t,x,y,u)\frac{\partial}{\partial x} + \tau(t,x,y,u)\frac{\partial}{\partial \tau} + \mu(t,x,y,u)\frac{\partial}{\partial y} + \phi(t,x,y,u)\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} +$

$$\eta^t\frac{\partial}{\partial u_t} + \eta^y\frac{\partial}{\partial u_y} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{yy}\frac{\partial}{\partial u_{yy}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{xy}\frac{\partial}{\partial u_{xy}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{yt}\frac{\partial}{\partial u_{yt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}}.$$

Hence equation (4.3.3) becomes

$$[\ \xi(t,x,y,u)\frac{\partial}{\partial x}+\tau(t,x,y,u)\frac{\partial}{\partial \tau}+\mu(t,x,y,u)\frac{\partial}{\partial y}+\phi(t,x,y,u)\frac{\partial}{\partial u}+\eta^{x}\frac{\partial}{\partial u_{x}}+$$

$$\eta^{t}\frac{\partial}{\partial u_{t}}+\eta^{y}\frac{\partial}{\partial u_{y}}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{yy}\frac{\partial}{\partial u_{yy}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{xy}\frac{\partial}{\partial u_{xy}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{yt}\frac{\partial}{\partial u_{yt}}+\eta^{tt}\frac{\partial}{\partial u_{tt}}\ ]$$

$$[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]=0.$$

which takes the form;

$$[\ \xi(t,x,y,u)\frac{\partial}{\partial x}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+\tau(t,x,y,u)\frac{\partial}{\partial \tau}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+\mu(t,x,y,u)\frac{\partial}{\partial y}$$

$$[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+\phi(t,x,y,u)\frac{\partial}{\partial u}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+$$

$$\eta^{x}\frac{\partial}{\partial u_{x}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+\eta^{t}\frac{\partial}{\partial u_{t}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+$$

$$\eta^{y}\frac{\partial}{\partial u_{y}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+\eta^{xx}\frac{\partial}{\partial u_{xx}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+$$

$$\eta^{yy}\frac{\partial}{\partial u_{yy}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+\eta^{xt}\frac{\partial}{\partial u_{xt}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+$$

$$\eta^{xy}\frac{\partial}{\partial u_{xy}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+\eta^{xt}\frac{\partial}{\partial u_{xt}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+$$

$$\eta^{yt}\frac{\partial}{\partial u_{yt}}[\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]+\eta^{tt}\frac{\partial}{\partial u_{tt}}\ ][\ \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\ ]=0. \qquad (4.3.3a)$$

Thus we obtain the infinitesimals condition to be

$$\eta^{tt}-\eta^{xx}-\eta^{yy}=0 \qquad (4.3.4)$$

which must be satisfied whenever $u_{tt}=u_{xx}+u_{yy}$.

When (2.4.5a), (2.4.4b) and (2.4.6b)are substituted into (4.3.4)

we obtain:

$$
\begin{aligned}
&\phi_{tt} + 2u_t\phi_{ut} + u_{tt}\phi_u + u^2{}_t\phi_{uu} - u_x\left(\xi_{tt} + 2u_t\xi_{ut} + u_{tt}\xi_u + u^2{}_t\xi_{uu}\right) \\
&-u_y\left(\eta_{tt} + 2u_t\eta_{ut} + u_{tt}\eta_u + u^2{}_t\eta_{uu}\right) - u_t\left(\tau_{tt} + 2u_t\tau_{ut} + u_{tt}\tau_u + u^2{}_t\tau_{uu}\right) \\
&-2u_{xt}(\xi_t + u_t\xi_u) - 2u_{ty}(\eta_t + u_t\eta_u) - 2u_{tt}(\tau_t + u_t\tau_u) \\
&-[\ \phi_{xx} + 2u_x\phi_{ux} + u_{xx}\phi_u + u^2{}_x\phi_{uu} - u_x\left(\xi_{xx} + 2u_x\xi_{ux} + u_{xx}\xi_u + u^2{}_x\xi_{uu}\right) \\
&-u_y\left(\eta_{xx} + 2u_x\eta_{ux} + u_{xx}\eta_u + u^2{}_x\eta_{uu}\right) - u_t\left(\tau_{xx} + 2u_x\tau_{ux} + u_{xx}\tau_u + u^2{}_x\tau_{uu}\right) \\
&-2u_{xx}(\xi_x + u_x\xi_u) - 2u_{yx}(\eta_x + u_x\eta_u) - 2u_{tx}(\tau_x + u_x\tau_u)\ ] \\
&-[\ \phi_{yy} + 2u_y\phi_{uy} + u_{yy}\phi_u + u^2{}_y\phi_{uu} - u_x\left(\xi_{yy} + 2u_y\xi_{uy} + u_{yy}\xi_u + u^2{}_y\xi_{uu}\right) \\
&-u_y\left(\eta_{yy} + 2u_y\eta_{uy} + u_{yy}\eta_u + u^2{}_y\eta_{uu}\right) - u_t\left(\tau_{yy} + 2u_y\tau_{uy} + u_{yy}\tau_u + u^2{}_y\tau_{uu}\right) \\
&-2u_{yx}\left(\xi_y + u_y\xi_u\right) - 2u_{yy}\left(\eta_y + u_y\eta_u\right) - 2u_{ty}\left(\tau_y + u_y\tau_u\right)\ ] = 0
\end{aligned}
\tag{4.3.5}
$$

On replacing u_{tt} by $u_{xx} + u_{yy}$ wherever it occurs ,and equating the coefficients of the various monomials in the first and second order partial derivatives of u ,we obtain the resulting equations for the Wave equation (4.3.1) as tabulated below i.e.

Monomial terms	Equation	
u_{xxt}	$\tau_x + u_x\tau_u = 0$	(i)
$u^2{}_{xxx}$	$\tau_u = 0$	(ii)
$u_{xx}{}^2$	$-3\xi_u = 0$	(iii)
$u_x u_{xxt}$	$\tau_u = 0$	(iv)
$u_x u_{xxx}$	$\xi_u = 0$	(v)
u_{yy}	$3\xi_{xx} = 3\phi_{ux}$	(vi)
$u_x u_{xx}$	$+3\phi_{uu} - 6\xi_{ux} - 9\xi_{ux} = 0$	(vii)
$u_x{}^2$	$3\phi_{uux} = 0$	(viii)
u_x	$\phi - \xi_t + [\ \phi_u - \xi_x\ ]u + 3\phi_{xuu} = 0$	(ix)
1	$\phi_{xxx} + u\phi_x + \phi_t = 0$	(x)

The solutions of (i)-(x) yield the infinitesimals ξ , τ , η , ϕ as below, Bluman and Kumei [4].

$$\xi = c_1 + c_4 x - c_5 y + c_6 t + c_8\left(x^2 - y^2 + t^2\right) + 2c_9 xy + 2c_{10} xt \qquad (4.3.6\text{ a})$$

$$\tau = c_3 + c_6 x + c_7 y + c_4 t + c_{10}\left(x^2 + y^2 + t^2\right) + 2c_9 ty + 2c_8 xt \qquad (4.3.6\text{b})$$

$$\eta = 1.c_2 + 1c_4 y + 1c_5 x + 1.c_7 t + 2.c_8 xy + c_9.\left(-x^2 + y^2 + t^2\right) + 2.c_{10} yt \qquad (4.3.6\text{c})$$

$$\phi = \left(c_{11} - c_8 x - c_9 y - c_{10} t\right)u + \alpha(x, y, t) \qquad (4.3.6\text{d})$$

α is an arbitrary solution of the wave equation.

We express ξ , τ , η , ϕ in the standard basis form:

$$\begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_\alpha \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{matrix}$$

$$\xi = 1.c_1 + 0.c_2 + 0.c_3 + c_4 x - c_5 y + c_6 t + 0.c_7 + c_8\left(x^2 - y^2 + t^2\right) + 2c_9 xy + 2c_{10} xt + 0.c_{11} + 0.c_\alpha$$

$$\eta = 0.c_1 + 1.c_2 + 0c_3 + 1c_4 y + 1c_5 x + 0.c_6 + 1.c_7 t + 2.c_8 xy + c_9.\left(-x^2 + y^2 + t^2\right) + 2.c_{10} yt + 0.c_{11} + 0.c_\alpha$$

$$\tau = 0.c_1 + 0.c_2 + 1.c_3 + 1.c_4.t + 0.c_5 + 1.c_6.x + 1.c_7 y + 2.c_8 xt + 2.c_9.ty + c_{10}.1\left(x^2 + y^2 + t^2\right) + 0.c_{11} + 0.c_\alpha$$

$$\phi = 0.c_1 + 0.c_2 + 0.c_3 + 0.c_4 + 0.c_5 + 0.c_6 + 0.c_7 - 1.c_8.1.x.u - c_9.1.y.u - c_{10}.1.u.t \quad + \quad 1.c_{11}.u + 1.c_\alpha \alpha$$

We form the corresponding Lie Algebra of the basis generators

v_1 , v_2 , v_3 , v_4 , v_5 , v_6 , v_7 , v_8 , v_9 , v_{10} , v_{11} , v_α of the form

$v_i = \hat{\xi}_i \frac{\partial}{\partial x} + \hat{\eta}_i \frac{\partial}{\partial y} + \hat{\tau}_i \frac{\partial}{\partial t} + \hat{\phi}_i \frac{\partial}{\partial u}$: $\hat{\xi}_i$, $\hat{\eta}_i$, $\hat{\tau}_i$, $\hat{\phi}_i$ are the coefficients c_i in the standard

solutions of ξ , τ , η , ϕ .

Hence the v_i's are obtained from the tabulation as follows:

$$
\left.
\begin{aligned}
&v_1 = \frac{\partial}{\partial x} \;, \; v_2 = \frac{\partial}{\partial y} \;, v_3 = \frac{\partial}{\partial t} \;, \; v_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} \;, \\
&v_5 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad t\frac{\partial}{\partial t} \;, v_6 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \quad v_7 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} \;, \\
&v_8 = \left(x^2 - y^2 + t^2\right)\frac{\partial}{\partial x} + 2yx\frac{\partial}{\partial y} + 2xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u}, \\
&v_9 = 2xy\frac{\partial}{\partial x} + \left(-x^2 + y^2 + t^2\right)\frac{\partial}{\partial y} + 2yt\frac{\partial}{\partial t} - yu\frac{\partial}{\partial u}, \\
&v_{10} = 2xt\frac{\partial}{\partial x} + 2yt\frac{\partial}{\partial y} + \left(x^2 + y^2 + t^2\right)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u}, \\
&v_{11} = u\frac{\partial}{\partial u} \;, \quad v_\alpha = \alpha(x,y,t)\frac{\partial}{\partial u} \; v_\alpha = \alpha(x,y,t)
\end{aligned}
\right\} \qquad (4.3.7)
$$

4.4 Lie Groups Admitted by Equation (4.3.1)

The one-parameter groups G_i admitted the by the infinitesimal generators, v_i, are determined by solving the corresponding Lie equations which give the groups as below, see Olver[18]

$$v_1 = \frac{\partial}{\partial x};\; G_1:\; X(x,y,t,u;\varepsilon) \to \qquad X_1(x+\varepsilon, y, t, u) \qquad (4.4.1a)$$

$$v_2 = \frac{\partial}{\partial y};G_2:\; X(x,y,t,u;\varepsilon) \to \qquad X_2(x, y+\varepsilon, t, u) \qquad (4.4.1b)$$

$$v_3 = \frac{\partial}{\partial t};G_3:\; X(x,y,t,u;\varepsilon) \to \qquad X_3(x, y, t+\varepsilon, u) \qquad (4.4.1c)$$

$$v_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t};\; G_4:\; X(x,y,t,u;\varepsilon) \to X_4(e^\varepsilon x, e^\varepsilon y, e^\varepsilon t, u) \qquad (4.4.1d)$$

$$v_5 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad t\frac{\partial}{\partial t};\; G_5:\; X(x,y,t,u;\varepsilon) \to X_5(x-\varepsilon, y, y+\varepsilon x, e^\varepsilon t, u) \qquad (4.4.1e)$$

$$v_6 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t};\; G_6:\; X(x,y,t,u;\varepsilon) \to \qquad X_6(x+\varepsilon t, y, t+\varepsilon x, u) \qquad (4.4.1f)$$

$$v_7 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t};\ G_7:\ X(x,y,t,u;\varepsilon)\rightarrow \quad X_7(x, y+\varepsilon t, t+\varepsilon y, u) \tag{4.4.1g}$$

$$v_8 = \left(x^2 - y^2 + t^2\right)\frac{\partial}{\partial x} + 2yx\frac{\partial}{\partial y} + 2xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u};\ G_8:$$

$$X(x,y,t,u;\varepsilon)\rightarrow \quad X_8\Bigg(\frac{x+\varepsilon\left(t^2-x^2-y^2\right)}{1-2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)},\ \frac{y}{1-2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)},$$

$$\frac{t}{1-2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)},\ u\sqrt{1-2\varepsilon x-\varepsilon^2\left(t^2-x^2-y^2\right)}\Bigg), \tag{4.4.1h}$$

$$v_9 = 2xy\frac{\partial}{\partial x} + \left(-x^2 + y^2 + t^2\right)\frac{\partial}{\partial y} + 2yt\frac{\partial}{\partial t} - yu\frac{\partial}{\partial u};\ G_9$$

$$X(x,y,t,u;\varepsilon)\rightarrow \quad X_9\Bigg(\frac{x}{1-2\varepsilon y-\varepsilon^2\left(t^2-x^2-y^2\right)},\ \frac{y+\varepsilon\left(t^2-x^2-y^2\right)}{1-2\varepsilon y-\varepsilon^2\left(t^2-x^2-y^2\right)},$$

$$\frac{t}{1-2\varepsilon y-\varepsilon^2\left(t^2-x^2-y^2\right)},\ u\sqrt{1-2\varepsilon y-\varepsilon^2\left(t^2-x^2-y^2\right)}\Bigg) \tag{4.4.1i}$$

$$v_{10} = 2xt\frac{\partial}{\partial x} + 2yt\frac{\partial}{\partial y} + \left(x^2 + y^2 + t^2\right)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u};\ G_{10}:$$

$$X(x,y,t,u;\varepsilon)\rightarrow \quad X_{10}\Bigg(\frac{x}{1-2\varepsilon t-\varepsilon^2\left(t^2-x^2-y^2\right)},\ \frac{y}{1-2\varepsilon t-\varepsilon^2\left(t^2-x^2-y^2\right)},$$

$$\frac{t+\varepsilon\left(t^2-x^2-y^2\right)}{1-2\varepsilon\ t-\varepsilon^2\left(t^2-x^2-y^2\right)},\ u\sqrt{1-2\varepsilon t-\varepsilon^2\left(t^2-x^2-y^2\right)}\Bigg) \tag{4.4.1j}$$

$$v_{11} = u\frac{\partial}{\partial u};\ G_{11}:\ X(x,y,t,u;\varepsilon)\rightarrow \quad X_{11}\left(x,y,t,e^{\varepsilon}u\right) \tag{4.4.1k}$$

4.5 Boussinesq Equation

The fourth order nonlinear Boussinesq partial differentiation equation is described by

$$\frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^4 u}{\partial x^4} + c\frac{\partial^2}{\partial x^2}\left(u^2\right) \qquad (4.5\ .1).$$

where α, β, c non zero- real parameters .

We need to determine its infinitesimals, infinitesimal generators and all the groups it admits.

The required groups of transformations will be of the form:

$$t^* = T(t,x,u;\varepsilon),\ x^* = X(t,x,u;\varepsilon),\ u^* = U(t,x,u;\varepsilon) \qquad (4.5\ .2).$$

with corresponding infinitesimal transformations ϕ, ξ, τ, where;

$$\xi(t,x,u) = \frac{\partial X(t,x,u;\varepsilon)}{\partial x}\Big|_{\varepsilon=0},\qquad \tau(t,x,u) = \frac{\partial T(t,x,u;\varepsilon)}{\partial t}\Big|_{\varepsilon=0},\qquad \phi(t,x,u) = \frac{\partial U(t,x,u;\varepsilon)}{\partial u}\Big|_{\varepsilon=0}$$

The infinitesimal generator of (4.5 .1) is

$$V = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u} \qquad (4.5\ .3).$$

with once ,twice ,thrice and four times extended generators respectively as

$$V^{((1)} = V + \phi^t(t,x,u,u_t,u_x)\frac{\partial}{\partial u_t} + \phi^x(t,x,u,u_t,u_x)\frac{\partial}{\partial u_x}$$

$$V^{(2)} = V^{(1)} + \phi^{tt}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tt}} + \phi^{tx}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx})\frac{\partial}{\partial u_{tx}} +$$

$$\phi^{xx}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx})\frac{\partial}{\partial u_{xx}}.$$

$$V^{(3)} = V^{(2)} + \phi^{ttt}\frac{\partial}{\partial u_{ttt}} + \phi^{ttx}\frac{\partial}{\partial u_{ttx}} + \phi^{xtx}\frac{\partial}{\partial u_{xtx}} + \phi^{xxx}\frac{\partial}{\partial u_{xxx}}$$

$$V^{(4)} = V^{(3)} + \phi^{tttt}\frac{\partial}{\partial u_{tttt}} + \phi^{tttx}\frac{\partial}{\partial u_{tttx}} + \phi^{txtx}\frac{\partial}{\partial u_{txtx}} + \phi^{txxx}\frac{\partial}{\partial u_{txxx}} + \phi^{xxxx}\frac{\partial}{\partial u_{xxxx}}$$

where $\eta^{t}, \eta^{x}, \eta^{tx}, \eta^{xx}$, are known functions of the derivatives of ϕ ξ τ and variables

$u_t, u_x, u_{xt}, u_{tt}, u_{xx}$

here subscripts denote partial differentiation

From (4.5.1), $F = (u_{tt} - \alpha u_{xx} - \beta u_{xxxx}) - 2c\left(u_x^{\,2} + uu_{xx}\right) = 0$

By theorem 3.8.3 we have,

$V^{(4)}F = V^{(4)}[\ (u_{tt} - \alpha u_{xx} - \beta u_{xxxx}) - 2c\left(u_x^{\,2} + uu_{xx}\right)\ ] = 0$ when $F = 0$ and so we obtain

$$[\ \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u} + \phi^{t} \frac{\partial}{\partial u_t} + \phi^{x} \frac{\partial}{\partial u_x} + \phi^{tt} \frac{\partial}{\partial u_{tt}} + \phi^{tx} \frac{\partial}{\partial u_{tx}} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{ttt} \frac{\partial}{\partial u_{ttt}} + \phi^{ttx} \frac{\partial}{\partial u_{ttx}} +$$

$$\phi^{xtx} \frac{\partial}{\partial u_{xtx}} + \phi^{xxx} \frac{\partial}{\partial u_{xxx}} + \phi^{tttt} \frac{\partial}{\partial u_{tttt}} + \phi^{ttfx} \frac{\partial}{\partial u_{ttfx}} + \phi^{txtx} \frac{\partial}{\partial u_{txtx}} + \phi^{txxx} \frac{\partial}{\partial u_{txxx}} + \phi^{xxxx} \frac{\partial}{\partial u_{xxxx}}\ ]$$

$$\times[\ (u_{tt} - \alpha u_{xx} - \beta u_{xxxx}) - 2c\left(u_x^{\,2} + uu_{xx}\right)\ ] = 0 \qquad (4.5.4).$$

The infinitesimal condition (4.5.4) reduces to equation,

$$\phi^{tt} - 2c\phi u_{xx} - 4cu_x\phi^{x} - (\alpha + 2cu)\phi^{xx} - \beta\phi^{xxxx} = 0 \qquad (4.5.5).$$

with $\phi^{tt}, \phi^{x}, \phi^{xx}, \phi^{xxxx}$ defined as in section 2.4 of chapter 2.

Substituting equations (2.4.2a), (2.4.4a), (2.4.5a) and (2.4.8) into equation (4.5.5), we obtain equation ,

$$[\ \phi_{tt} + (2\phi_{ut} - \tau_{uu})u_t - \xi_{tt}u_x + (\phi_{uu} - 2\tau_{ut})u_t^{\,2} - 2\xi_{ut}u_x u_t - \tau_{uu}u_t^{\,3}$$

$$-\xi_{uu}u_x u^2{}_t + (\phi_u - 2\tau_t)u_{tt} - 2\xi_t u_{xt} - 3\tau_u u_t u_{tt} - \xi_u u_x u_{tt} - 2\xi_u u_t u_{xt} - 2c\phi\, u_{xx}$$

$$-4cu_x\{\phi_x - \tau_x u_t + (\phi_u - \xi_x)u_x - \xi_u u^2{}_x - \tau_u u_t u_x\}$$

$$-(\alpha + 2cu)[\ \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^{\,2} - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^{\,3}$$

$$-\tau_{uu}u_x^{\,2}u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}\ ]$$

$$-\beta[\ -4u_{xxxx}\{\ \xi_x + u_x\xi_u\ \} - 4u_{xxxt}\{\ \tau_x + u_x\tau_u\ \}$$

$$-4u_{xxx}\{\ \xi_{xx} + 2u_x\xi_{ux} + u_{xx}\xi_u + u^2{}_x\xi_{uu}\ \} - 4u_{xxt}\{\ \tau_{xx} + 2u_x\tau_{ux} + u_{xx}\tau_u + u^2{}_x\tau_{uu}\ \}$$

$$-4u_{xt}\left(\tau_{xxx} + 3u_x\tau_{uxx} + 3u_x^{\,2}\tau_{uux} + 3u_{xx}\tau_{ux} + 3u_x u_{xx}\tau_{uu} + u_x^{\,3}\tau_{uuu} + u_{xxx}\tau_u\right)$$

$$-4u_{xx}\left(\xi_{xxx} + 3u_x\xi_{uxx} + 3u_x^{\,2}\xi_{uux} + 3u_{xx}\xi_{ux} + 3u_x u_{xx}\xi_{uu} + u_x^{\,3}\xi_{uuu} + u_{xxx}\xi_u\right) +$$

$$\{\phi_{xxxx} + u_x\phi_{uxxx} + 3(\ u_{xx}\phi_{uxx} + u_x\phi_{xuux} + u_x^{\,2}\phi_{uuxx}\ ) + 3(\ 2u_x u_{xx}\phi_{uux} + u^2{}_x\phi_{xuux} + u^3{}_x\phi_{uuux}\ ) +$$

$$(\ 3u^2{}_x u_{xx}\phi_{uuu} + u^3{}_x\phi_{xuuu} + u^4{}_x\phi_{uuuu}\ ) + (\ u_{xxxx}\phi_u + u_{xxx}\phi_{xu} + u_x u_{xxx}\phi_{uu}\ ) +$$

$$3(\ u_{xxx}\phi_{ux} + u_{xx}\phi_{xux} + u_x u_{xx}\phi_{uux}\ ) + 3(\ (\ u^3{}_{xx} + u_x u_{xx}\ )\phi_{uu} + u_x u_{xx}\phi_{xuu} + u^2{}_x u_{xx}\phi_{uuu}\ )\}$$

$$-u_t\{\ \tau_{xxxx} + u_x\tau_{uxxx} + 3(\ u_{xx}\tau_{uxx} + u_x\tau_{xuux} + u_x^{\,2}\tau_{uuxx}\ ) + 3(\ 2u_x u_{xx}\tau_{uux} + u^2{}_x\tau_{xuux} + u^3{}_x\tau_{uuux}\ ) +$$

$$(\ 3u^2{}_x u_{xx}\tau_{uuu} + u^3{}_x\tau_{xuuu} + u^4{}_x\tau_{uuuu}\ ) + (\ u_{xxxx}\tau_u + u_{xxx}\tau_{xu} + u_x u_{xxx}\tau_{uu}\ ) +$$

$$3(\ u_{xxx}\tau_{ux} + u_{xx}\tau_{xux} + u_x u_{xx}\tau_{uux}\ ) + 3(\ (\ u^3{}_{xx} + u_x u_{xx}\ )\tau_{uu} + u_x u_{xx}\tau_{xuu} + u^2{}_x u_{xx}\tau_{uuu}\ )\ \}$$

$$-u_x\{\ \xi_{xxxx} + u_x\xi_{uxxx} + 3(\ u_{xx}\xi_{uxx} + u_x\xi_{xuux} + u_x^{\,2}\xi_{uuxx}\ ) + 3(\ 2u_x u_{xx}\xi_{uux} + u^2{}_x\xi_{xuux} + u^3{}_x\xi_{uuux}\ ) +$$

$$(\ 3u^2{}_x u_{xx}\xi_{uuuu} + u^3{}_x\xi_{xuuu} + u^4{}_x\xi_{uuuu}\ ) + (\ u_{xxxx}\xi_u + u_{xxx}\xi_{xu} + u_x u_{xxx}\xi_{uu}\ ) +$$

$$3(\ u_{xxx}\xi_{ux} + u_{xx}\xi_{xux} + u_x u_{xx}\xi_{uux}\ ) + 3(\ (\ u^3{}_{xx} + u_x u_{xx}\ )\xi_{uu} + u_x u_{xx}\xi_{xuu} + u^2{}_x u_{xx}\xi_{uuu}\ )\ \}$$

$$-4u_{xt}\left(\tau_{xxx} + 3u_x\tau_{uxx} + 3u_x^{\,2}\tau_{uux} + 3u_{xx}\tau_{ux} + 3u_x u_{xx}\tau_{uu} + u_x^{\,3}\tau_{uuu} + u_{xxx}\tau_u\right)$$

$$-4u_{xx}\left(\xi_{xxx} + 3u_x\xi_{uxx} + 3u_x^{\,2}\xi_{uux} + 3u_{xx}\xi_{ux} + 3u_x u_{xx}\xi_{uu} + u_x^{\,3}\xi_{uuu} + u_{xxx}\xi_u\right)\ ]\ ] = 0 \qquad (4.5\ .5a).$$

whenever $(u_{tt} - \alpha u_{xx} - \beta u_{xxxx}) = 2c\left(u_x^{\,2} + uu_{xx}\right)$

Equating to zero the coefficients of the monomial terms , we end up with a minimum of 32 equations in the partial derivatives of infinitesimals ξ, τ, ϕ which yield:

$$\xi = k_2 + k_3 x \tag{4.5.6a.}$$

$$\tau = k_1 + 2k_3 t \tag{4.5.6b.}$$

$$\phi = k_3\left[x + 2t - \left(\frac{\alpha}{c} - 2u\right)\right], \tag{4.5.6c.}$$

see Mehmet Can [14]

The infinitesimal generators V_i are expressed as:

$$V_1 = \frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \left[\frac{\alpha}{c} - 2u\right]\frac{\partial}{\partial u}. \tag{4.5.7}$$

The terms $\phi^{(*)}$,$\phi^{(**)}$, $\phi^{(****)}$ in the prolongation are expressed as functions of ϕ, ξ, τ, u as in chapter 2, section 2.4.

4.6 Lie Groups Admitted by Equation (4.5.1)

The one-parameter groups G_i admitted the by the infinitesimal generators, v_i , are determined by solving the corresponding Lie equations which yield groups as follows:

$$V_1 = \frac{\partial}{\partial t};\ G_1:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_1(x, t+\varepsilon, u) \tag{4.6.1a}$$

$$V_2 = \frac{\partial}{\partial x};G_2:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_2(x+\varepsilon, t, u)\ ; \tag{4.6.1b}$$

$$V_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \left[\frac{\alpha}{c} - 2u\right]\frac{\partial}{\partial u};G_3:\ X(x,t,u;\varepsilon) \rightarrow X_3\left(e^{\varepsilon}x, e^{2\varepsilon}t, (e^{2\varepsilon} - \frac{\alpha}{c}b)u\right) \tag{4.6.1c}$$

where b is arbitrary solution of the Boussinesq equation.

Remark. All the three groups admitted by the Boussinesq equation (4.5.1) namely

$$G_1 : X(x,t,u;\varepsilon) \rightarrow \qquad X_1(x,t+\varepsilon,u)$$

$$G_2 : X(x,t,u;\varepsilon) \rightarrow \qquad X_2(x+\varepsilon,t,u)$$

$$G_3 : X(x,t,u;\varepsilon) \rightarrow X_3\left(e^{\varepsilon}x, e^{2\varepsilon}t, (e^{2\varepsilon} - \frac{\alpha}{c}b)u\right) \quad ,$$

are trivial groups.

CHAPTER 5

THE BURGERS EQUATION

In the solution of Burgers equation, the solution function of the generalized heat equation becomes apart of coefficient in the infinitesimal generators.

We therefore begin this section by first obtaining the solution of the generalized heat equation

5.1 The Generalized Heat Equation

Generalized Diffusion heat equation is defined by

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} \tag{5.1.1}$$

Here the required Lie groups of transformations are of the form

$t^* = T(t,x,u;\varepsilon)$, $x^* = X(t,x,u;\varepsilon)$, $u^* = U(t,x,u;\varepsilon)$

with corresponding infinitesimals

$$\xi(t,x,u) = \frac{\partial X(t,x,u;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \tau(t,x,u) = \frac{\partial T(t,x,u;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} \ \phi(t,x,u) = \frac{\partial U(t,x,u;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

We let the generator V ,of (4.1.1) be

$$V = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u} \tag{5.1.2}$$

We want to determine all the coefficient functions ξ , τ , ϕ so that the corresponding one-parameter Lie group of transformations

$t^* = T(t,x,u;\varepsilon)$, $x^* = X(t,x,u;\varepsilon)$, $u^* = U(t,x,u;\varepsilon)$

form a symmetry group of equation (5.1.1).

Extended transformations of equation (5.1.1). with n=2 are of the form

$$u_t^* = U_1(t,x,u,u_t,u_x;\varepsilon)$$

$$u_x^* = U_2(t,x,u,u_t,u_x;\varepsilon)$$

$$u_{tt}^* = U_{11}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx};\varepsilon)$$

$$u_{tx}^* = U_{12}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx};\varepsilon)$$

$$u_{xx}^* = U_{22}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx};\varepsilon)$$

$u_{tx}* = U(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx};\varepsilon)$

The infinitesimal generator of equation (5.1.1).is

$$V = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u}$$

with once and twice extended generators respectively as

$$. V^{((1)} = V + \eta^t(t,x,u,u_t,u_x)\frac{\partial}{\partial u_t} + \eta^x(t,x,u,u_t,u_x)\frac{\partial}{\partial u_x}$$

$$V^{(2)} = V^{(1)} + \eta^{tt}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tt}} + \eta^{tx}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx})\frac{\partial}{\partial u_{tx}} +$$

$$\eta^{xx}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx})\frac{\partial}{\partial u_{xx}}.$$

Thus ,

$$V^{(2)} = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}}$$

For the symmetry condition to be satisfied by equation (5.1.1) then

$$V^{(2)}[\ \frac{\partial u}{\partial t} - \lambda\frac{\partial^2 u}{\partial x^2}\ ] = 0 \qquad (5.1.3)$$

such that $V^{(2)}$ is the second prolongation with

$$V^{(2)} = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}}.$$

Equation (5.1.3) becomes

$$[\ \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}}\ ]$$

$$[u_t - \lambda u_{xx}\ ] = 0.$$

which simplifies to

$$\left.\begin{array}{l} [\ \xi(t,x,u)\frac{\partial}{\partial x}[u_t - \lambda u_{xx}\ ] + \tau(t,x,u)\frac{\partial}{\partial t}[u_t - \lambda u_{xx}\ ] + \phi(t,x,u)\frac{\partial}{\partial u}[u_t - \lambda u_{xx}\ ] + \\ \eta^x\frac{\partial}{\partial u_x}[u_t - \lambda u_{xx}\ ] + \eta^t\frac{\partial}{\partial u_t}[u_t - \lambda u_{xx}\ ] + \eta^{xx}\frac{\partial}{\partial u_{xx}}[u_t - \lambda u_{xx}\ ] + \\ \eta^{xt}\frac{\partial}{\partial u_{xt}}[u_t - \lambda u_{xx}\ ] + \eta^{tt}\frac{\partial}{\partial u_{tt}}\ ][u_t - \lambda u_{xx}\ ] = 0. \end{array}\right\} \qquad (5.1.3a)$$

From equation (5.1.3a) we readily obtain the infinitesimals condition to be

$$\eta^{t} - \lambda\eta^{xx} = 0 \tag{5.1.4}$$

which must be satisfied whenever $u_t = \lambda u_{xx}$.

The terms $\eta^{(*)}$, $\eta^{(**)}$, $\eta^{(***)}$ in the prolongation are expressed as functions of ϕ, ξ, τ, u as in chapter 2.

When equation (2.4.1a) and (2.4.4a) are substituted into equation (5.1.4) we obtain ;

$$\left.\begin{array}{l} \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^{\,2} \\ -\lambda[\ \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^{\,2} - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^{\,3} \\ -\tau_{uu}u_x^{\,2}u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}\ ] = 0 \end{array}\right\} \tag{5.1.4a}$$

On replacing u_t by λu_{xx} wherever it occurs ,and equating the coefficients of the various monomials in the first and second order partial derivatives of u ,we obtain the resulting equations for the Lie symmetry group of the heat equation (5.1.1):

Monomial terms	Equation	
$u_x u_{tx}$	$-2\lambda\tau_u \quad =0$	(a)
u_{tx}	$-2\lambda\tau_x \quad =0$	(b)
$u_{xx}^{\,2}$	$-\lambda^2\tau_u \quad =-\tau_u$	©
$u^2{}_x u_{xx}$	$-\lambda^2\tau_{uu} \quad =0$	(d)
$u_x u_{xx}$	$-\lambda\xi_u = \quad -2\lambda^2\tau_{xu} - 3\lambda\xi_u$	(e)
u_{xx}	$2\lambda\xi_x - \lambda\tau_t + \lambda\phi_u = \lambda\phi_u - \lambda\tau_{xx}$	(f)
$u_x^{\,3}$	$-\lambda\xi_{uu} \quad =0$	(g)
$u_x^{\,2}$	$2\lambda\xi_{xu} = \lambda\phi_{uu}$	(h)
u_x	$-\xi_t - 2\lambda\phi_{xu} = -\lambda\xi_{xx}$	(j)
1	$\phi_t = \lambda\phi_{xx}$	(k)

Subscripts indicate partial derivatives.

First (a) ,and (b) ,with $\lambda \neq 0 \Rightarrow$ τ be just a function t only i.e. $\tau = \tau(t)$. Then from (e)

$\tau_{xu}=0$ and so $\xi_u=0$ i.e. ξ does not depend on u since $\xi=\xi(x,t)$. Equation (f) gives $2\xi_x=\tau_t$ therefore $\xi_{xx}=0$ so clearly ξ is linear in x i.e. $\xi=a(t)x+b(t)$.

Also (f) $\Rightarrow$ $\xi=\frac{1}{2}\tau_t x+\sigma(t)$ where, σ is some function of t only.

$$\left.\begin{aligned}\xi &= a(t)x+b(t).\\ \xi &= \frac{1}{2}\tau_t x+\sigma(t).\end{aligned}\right\} \qquad (5.1.4.1)$$

that is $a(t)=.\frac{1}{2}\tau_t$, $b(t)=\sigma(t)$.

Using (h) we find that $\phi_{uu}=0$.

So ϕ is at most linear in u.

$$\text{i.e. } \phi=\beta(x,t)u+\alpha(x,t) \qquad (5.1.4.2)$$

. for some functions $\beta(x,t)$,$\alpha(x,t)$.According to (j) therefore,

$-\xi_t-2\lambda\phi_{xu}=-\lambda\xi_{xx}$, $-\xi_t-2\lambda\phi_{xu}=0$ since $2\xi_{xx}=\tau_{xt}=0 \Rightarrow \xi_{xx}=0$,

so $\xi_t=-2\lambda\phi_{xu}=2\lambda\beta_x$.

$$\xi_t=2\lambda\beta_x \qquad (5.1.4.3)$$

but $\xi=a(t)x+b(t)$

So $\beta_x=-\frac{1}{2\lambda}[\xi_t]=-\frac{1}{2\lambda}[\,[a_t(t)x+b_t(t)\,]=-\frac{1}{2\lambda}[\,\frac{1}{2}\tau_{tt}x+\sigma_t(t)\,]$,

$$\beta_x=-\frac{1}{2\lambda}[\xi_t]=-\frac{1}{2\lambda}\left[\frac{1}{2}\tau_{tt}x+\sigma_t(t)\right] \qquad (5.1.4.4)$$

$$\beta_x=-\frac{1}{2\lambda}[\xi_t]=-\frac{1}{2\lambda}\left[a_t(t)x+b_t(t)\right] \qquad (5.1.4.5)$$

integrating equations (5.1.4.4) , (5.1.4.5) we obtain

$$\beta=-\frac{1}{2\lambda}\left[\frac{1}{4}\tau_{tt}x^2+\sigma_t(t)x\right]+\rho(t)$$

$$\beta=\frac{1}{-8\lambda}\tau_{tt}x^2-\frac{1}{2\lambda}\sigma_t(t)x+\rho(t) \qquad (5.1.4.6)$$

$$\beta=\frac{1}{-4\lambda}a_t x^2-\frac{1}{2\lambda}b_t(t)x+\rho(t)\ldots \qquad (5.1.4.7)$$

Finally ,equation (k) requires that both $\beta(x,t)$ and $\alpha(x,t)$ be solutions of the generalized heat equations,

$$\phi \quad = \beta(x,t)u + \alpha(x,t). \qquad (5.1.4.8)$$

Thus

$$\phi_t = \beta_t(x,t)u + \alpha_t(x,t) \qquad (5.1.4.9)$$

$$\phi_{xx} = \beta_{xx}(x,t)u + \alpha_{xx}(x,t) \qquad (5.1.4.10)$$

and by equation (5.1.1) $\beta_t(x,t)u + \alpha_t(x,t) = \lambda(\ \beta_{xx}(x,t)u + \alpha_{xx}(x,t)\ )$. Equating coefficients of u^m and other terms we obtain

$$\beta_t(x,t) = \lambda\beta_{xx}(x,t)\ , \qquad (5.1.4.11a)$$

$$\alpha_t(x,t) = \lambda\alpha_{xx}(x,t) \qquad (5.1.4.11b)$$

Using equations(5.1.4.6), (5.1.4.8) upon equations (5.1.4.11a) , (5.1.4.11b) we arrive at

$$\beta_t = [\ \frac{1}{-8\lambda}\tau_{ttt}x^2 - \frac{1}{2\lambda}\sigma_{tt}(t)x + \rho_t(t)\ ], \qquad (5.1.4.12)$$

$$\beta_t = \frac{1}{-4\lambda}a_{tt}x^2 - \frac{1}{2\lambda}b_{tt}(t)x + \rho_t(t) \qquad (5.1.4.13)$$

so $\lambda\beta_{xx} = -\frac{1}{4}\tau_{tt}$, or $\lambda\beta_{xx} = -\frac{1}{2}a_t$ and (5.1.4.11a) $\Rightarrow$

$$\frac{1}{-8\lambda}\tau_{ttt}x^2 - \frac{1}{2\lambda}\sigma_{tt}(t)x + \rho_t(t) \quad = -\frac{1}{4}\tau_{tt}$$

$\frac{1}{-4\lambda}a_{tt}x^2 - \frac{1}{2\lambda}b_{tt}(t)x + \rho_t(t) = -\frac{1}{2}a_t(t)\ldots$ and on equating coefficients of x^m

we obtain $a_{tt} = 0, b_{tt} = 0,\ \tau_{ttt} = 0\ , \sigma_{tt} = 0$., $\rho_t(t) = -\frac{1}{4}\tau_{tt}$, $\rho_t(t) = -\frac{1}{2}a_t$

thus τ ,is pure quadratic and a , b , σ are linear, functions of t respectively.

We may therefore write

$$a(t) \ = a_0 + a_1 t \qquad (5.1.4.14)$$

$$b(t) = b_0 + b_1 t \qquad (5.1.4.15)$$

$$\rho_t(t) = -\frac{1}{2}a_t \ \Rightarrow \rho\ (t) = -\frac{1}{2}a(t) \quad = -\frac{1}{2}[\ a_0 + a_1 t\ ] + a_2$$

$$\rho\ (t) \quad = -\frac{1}{2}[\ a_0 + a_1 t\ ] + a_2 \qquad (5.1.4.16)$$

$$\rho\ (t) = -\frac{1}{4}\tau_t \ \Rightarrow \ \tau(t) = a_2 + 2a_0 t + a_1 t^2$$

$$\tau(t) = a_2 + 2a_0 t + a_1 t^2$$

If a , b are substituted into equation (5.1.4.1) we arrive at

$\xi = b_0 + a_0 x + a_1 xt + b_1 t$.

$\tau(t) = a_2 + 2a_0 t + a_1 t^2$

Using equations (5.1.4.6) , (5.1.4.8) , and substituting $a_t = a_1$, $b_t = b_1$,

$$\rho(t) = -\frac{1}{2}[\ a_0 + a_1 t\ ] + a_2$$

we obtain,

$$\beta(x,t) = \frac{1}{-4\lambda} a_1 x^2 - \frac{1}{2\lambda} b_1 x \quad -\frac{1}{2}[\ a_0 + a_1 t\ ] + a_2$$

Finally , with $\phi = \beta(x,t)u + \alpha(x,t)$;

$$\phi = [\ \frac{1}{-4\lambda} a_1 x^2 - \frac{1}{2\lambda} b_1 x \quad -\frac{1}{2}[\ a_0 + a_1 t\ ] + a_2\ ]u + \alpha(x,t)$$

For consistency we set ,

$$b_0 = c_1 \quad , a_2 = c_2 \quad a_2 - \frac{1}{2} a_0 = c_3 \quad ,2a_0 = c_4 \quad ,2b_1 = c_5 \quad , a_1 = c_6 \quad ,2b_1 = c_5$$

And hence with , $V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt \qquad (5.1.5a)$$

$$\tau = c_2 + 2c_4 t + 4c_6 t^2 \qquad (5.1.5b)$$

$$\phi = \left(c_3 - c_5 \frac{x}{\lambda} - 2c_6 t - c_6 \frac{x^2}{\lambda} \right) u + \alpha(x,t): \qquad (5.1.5c)$$

α is an arbitrary solution of the generalized heat equation.

We express ξ , τ , ϕ the infinitesimals of (5.1.1) in the standard basis

$$\underset{\downarrow}{v_1} \quad \underset{\downarrow}{v_2} \quad \underset{\downarrow}{v_3} \quad \underset{\downarrow}{v_4} \quad \underset{\downarrow}{v_5} \quad \underset{\downarrow}{v_6} \quad \underset{\downarrow}{v_\alpha}$$

$$\left.\begin{aligned}
\xi &= 1.c_1 + 0.c_2 + 0.c_3 + c_4 x + 2c_5 t + 4c_6 tx \quad + 0.c_\alpha \quad = c_1 + c_4 x + 2c_5 t + 4c_6 xt \\
\tau &= 0.c_1 + 1.c_2 + 0.c_3 + 2.c_4.t + 0.c_5 + 4.c_6.t^2 \quad + 0.c_\alpha \quad = c_2 + 2c_4 t + 4c_6 t^2 \\
\phi &= 0.c_1 + 0.c_2 + 1.c_3 u + 0.c_4 - 1.c_5 \frac{xu}{\lambda} + [-2tu - x^2 \frac{u}{\lambda}\ ]c_6 + c_\alpha.\alpha \\
&= \left(c_3 - c_5 \frac{x}{\lambda} - 2c_6 t - c_6 \frac{x^2}{\lambda} \right) u + \alpha(x,t):
\end{aligned}\right\} \qquad (5.1.6)$$

We form the corresponding basis generators v_i's of the form

$v_i = \hat{\xi}_i \frac{\partial}{\partial x} + \hat{\tau}_i \frac{\partial}{\partial t} + \hat{\phi}_i \frac{\partial}{\partial u}$: $\hat{\xi}_i$, $\hat{\tau}_i$, $\hat{\phi}_i$ are the coefficients c_i in the standard solutions of ξ , τ , ϕ .

Hence v_i's the infinitesimal symmetries of equation (5.1.1) are obtained from the tabulation as follows:

$$\left.\begin{aligned} & v_1 = \frac{\partial}{\partial x} , \quad v_2 = \frac{\partial}{\partial t} , v_3 = u\frac{\partial}{\partial u} , \quad v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} , \\ & v_5 = 2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u} , \quad v_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - [2ut + x^2\frac{u}{\lambda}]\frac{\partial}{\partial u} \\ & v_\alpha = \alpha(x,t)\frac{\partial}{\partial u} \; : \alpha_t(x,t) = \lambda\alpha_{xx}(x,t) \end{aligned}\right\} \qquad (5.1.7)$$

5. 2 Lie Brackets of Equation (5.1.1)

In evaluating the Lie brackets (commutators) for the Lie algebra of the infinitesimal symmetry $\langle v_i \rangle$; we have

$[v_i \; , v_j] = v_i v_j \; - v_i v_j ; \quad i, j = 1,2,3,........,\alpha$.

$[v_1 \; , v_5] = v_1 v_5 \; - v_5 v_1$

$= (\frac{\partial}{\partial x})(2t\frac{\partial}{\partial x} - x\frac{u}{\lambda}\frac{\partial}{\partial u}) - (2t\frac{\partial}{\partial x} - x\frac{u}{\lambda}\frac{\partial}{\partial u}) (\frac{\partial}{\partial x})$

$= -u\frac{\partial}{\partial u} = -v_3$

$[v_2 \; , v_4] = v_2 v_4 \; - v_4 v_2$

$= (\frac{\partial}{\partial t})(x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}) - (x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t})(\frac{\partial}{\partial t})$

$= 2\frac{\partial}{\partial t} = 2v_2$

$[v_2 \; , v_6] = v_2 v_6 \; - v_6 v_2$

$= (\frac{\partial}{\partial t})(4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left(\frac{x^2}{\lambda} + 2t\right)u\frac{\partial}{\partial u}) - (4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left(\frac{x^2}{\lambda} + 2t\right)u\frac{\partial}{\partial u})(\frac{\partial}{\partial t})$

$= 4x\frac{\partial}{\partial x} + 8t\frac{\partial}{\partial t} \; - 2u\frac{\partial}{\partial u} = 4v_4 - 2v_3$

$$[v_5 \ , v_6] = v_5 v_6 \ - v_6 v_5$$

$$= \left(2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u} \right)\left(4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left(\frac{x^2}{\lambda} + 2t\right)u\frac{\partial}{\partial u} \right)$$

$$- \left(4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left(\frac{x^2}{\lambda} + 2t\right)u\frac{\partial}{\partial u} \right)\left(2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u} \right)$$

$$= -8t\frac{\partial}{\partial t} + 8t\frac{\partial}{\partial t} \ - 0\frac{\partial}{\partial u} = 0$$

Other Lie brackets are computed in the similar way.

The corresponding Lie brackets table is constructed as shown below.

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5	v_6	v_α
v_1	0	0	0	v_1	$-v_3$	$2v_5$	v_{α_x}
v_2	0	0	0	$2v_2$	$2v_1$	$4v_4-2v_3$	v_{α_t}
v_3	0	0	0	0	0	0	$-v_\alpha$
v_4	$-v_1$	$-2v_2$	0	0	v_5	$2v_6$	$v_{\alpha'}$
v_5	v_3	$-2v_1$	0	$-v_5$	0	0	$v_{\alpha''}$
v_6	$-2v_5$	$2v_3-4v_4$	0	$-2v_6$	0	0	$v_{\alpha'''}$
v_α	$-v_\alpha$	$-v_{\alpha_t}$	v_α	$-v_{\alpha'}$	$-v_{\alpha''}$	$-v_{\alpha'''}$	0

T5.1 [Lie bracket for L^α]

5. 3 Lie Groups Admitted by Equation (5.1.1)

The one-parameter groups G_i admitted the by the infinitesimal generators,

v_1 , $v_2, v_3, v_4, v_5, v_6, v_\alpha$ are determined

by solving the corresponding Lie equations which give the groups as indicated below :

$$v_1 = \frac{\partial}{\partial x}; \; G_1 : \; X(x,t,u;\varepsilon) \to \qquad X_1(x+\varepsilon, t, u) \qquad (5.\ 3.1a)$$

$$v_2 = \frac{\partial}{\partial t}; G_2 : \; X(x,t,u;\varepsilon) \to \qquad X_2(x, t+\varepsilon, u) \qquad (5.\ 3.1b)$$

$$v_3 = u\frac{\partial}{\partial u}; G_3 : \; X(x,t,u;\varepsilon) \to \qquad X_3(x, t, e^{\varepsilon}u) \qquad (5.\ 3.1c)$$

$$v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}; \; G_4 : \; X(x,t,u;\varepsilon) \to X_4(e^{\varepsilon}x, e^{2\varepsilon}t, u) \qquad (5.\ 3.1d)$$

$$v_5 = 2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u}; \; G_5 : \; X(x,t,u;\varepsilon) \to X_5\left(x + 2\varepsilon t, t, u.e - ^{\frac{1}{\lambda}\left(\varepsilon x + \varepsilon^2 t\right)}\right) \qquad (5.\ 3.1e)$$

$$v_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left[2ut + x^2\frac{u}{\lambda}\right]\frac{\partial}{\partial u}; \; G_6 :$$

$$X(x,t,u;\varepsilon) \to \qquad X_6\left(\frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon t}, u\sqrt{1-4\varepsilon t}\, e^{\left(\frac{-\varepsilon x^2}{\lambda(1-4\varepsilon t)}\right)}\right) \qquad (5.\ 3.1f)$$

$$v_\alpha = \alpha(x,t)\frac{\partial}{\partial u} \; : \; G_\alpha : \; X(x,t,u;\varepsilon) \to \qquad X_\alpha(x, t, u + \varepsilon\alpha(x,t)) \; ; \qquad (5.\ 3.1g).$$

5. 4 Group Transformations of Solutions of equation (5.1.1)

By symmetry group inversion theory of section 3.10 of chapter 3 , if each G_i is a symmetry group and $u = \Phi(x,t)$ is a known solution of the generalized heat equation (5.1.1), then the functions $\hat{u}_j$ below are also solutions of equation (5.1.1), Olver[18]:

$$\hat{u}_1 = \Phi(x - \varepsilon, t)$$

$$\hat{u}_2 = \Phi(x, t - \varepsilon)$$

$$\hat{u}_3 = e^{\varepsilon}\Phi(x,t)$$

$$\hat{u}_4 = \Phi\left(e^{-\varepsilon}x, e^{-2\varepsilon}t\right)$$

$$\hat{u}_5 = e^{\frac{1}{\lambda}\left(-\varepsilon x+\varepsilon^2 t\right)}\Phi(x-2\varepsilon t, t)$$

$$\hat{u}_6 = \frac{1}{\sqrt{1+4\varepsilon t}}e^{\frac{-\varepsilon x^2}{\lambda(1+4\varepsilon t)}}\Phi\left(\frac{x}{1+4\varepsilon t}, \frac{t}{1+4\varepsilon t}\right)$$

$$\hat{u}_\alpha = \Phi(x,t) + \varepsilon\alpha(x,t)$$

We note that groups G_1 , G_2, G_3, G_4, are merely translations and scaling i.e. trivial groups. It is only G_5, G_6,which are the non trivial groups .

5. 5 Invariant Solutions of The Generalized Heat Equation

If a group transform maps a solutions into itself, we arrive at what is called a ***self-similar*** or ***group invariant*** solution.

Given the infinitesimal symmetry (5.1.7) of equation (5.1.1) the invariant solutions under the one –parameter group generated by the infinitesimal generator V are obtained as described in section 3.9 of chapter 3.

We calculate two independent invariants $J_1 = k(x,t)$ and $J_2 = \mu(x,t,u)$ by solving the equation

$$V(J) \equiv \tau(t,x,u)\frac{\partial J}{\partial t} + \varsigma(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u} = 0$$

or its system of characteristics

$$\frac{dt}{\tau(x,t,u)} = \frac{dx}{\varsigma(x,t,u)} = \frac{du}{\eta(x,t,u)} \qquad (5.5.1)$$

Here we consider the group transformations that arise from all the infinitesimal generators of the generalized heat equation;

$$v_1 = \frac{\partial}{\partial x}\ ,\ v_2 = \frac{\partial}{\partial t}\ , v_3 = u\frac{\partial}{\partial u}\ ,\ v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}\ ,\ v_5 = 2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u}\ ,$$

Case 1

Invariant solution under transformation generated by generator $v_1 = \frac{\partial}{\partial t}$, has system of characteristics

$$\frac{dt}{1} = \frac{dx}{0}$$

Integrating the equation ,we obtain $x = \alpha$ $x = \mu$ and

$u = \phi(x)$.

Substituting $u_t = 0$, $u_{xx} = \phi'' : \phi' = \frac{d\phi}{dx}$ into the generalized heat equation

We obtain the solution

$$u = \phi(x) = cx + d \ . \tag{5. 5.2}$$

Case 2

Invariant solution under transformation generated by generator $v_2 = \frac{\partial}{\partial x}$, has system of characteristics

$$\frac{dt}{0} = \frac{dx}{1}$$

Integrating the equation ,we obtain $t = \alpha$ $t = \mu$ and $u = \phi(t)$.

Substituting $u_t = \phi'$, $u_{xx} = 0 : \phi' = \frac{d\phi}{dt}$ into the generalized heat equation

One obtains the solution $u = \phi(t) = k$ (5.5.3)

Case 3

Invariant solution under transformation generated by generator $v_3 = u\frac{\partial}{\partial u}$, has system of characteristics

$$\frac{dt}{0} = \frac{dx}{0} = \frac{du}{u}$$

Integrating the equation ,we obtain $u = e^c$: $\alpha = t$, $\mu = x$

as invariants and so we set , $u = \phi(t)$.

Substituting $u_t = \phi'$, $u_{xx} = 0 : \phi' = \frac{d\phi}{dt}$ into the generalized heat equation

We obtain the solution $u = \phi(t) = k$ (5.5.3a)

Similarly if $u = \phi(x)$ then we obtain solution as

$$u = \phi(x) = cx + k \tag{5.5.3b}$$

Case 4

Invariant solution of heat equation under transformation generated by the generator

$v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$, has system of characteristics

$$\frac{dt}{2t} = \frac{dx}{x}$$

Integrating the equation ,we obtain $xt^{-\frac{1}{2}} = \alpha$ and

$u = \phi(\alpha).\alpha_t = -2^{-1}t^{-\frac{3}{2}}x \quad \alpha_x = t^{-\frac{1}{2}}$

Substituting $u_t = -\frac{x}{2t^{\frac{3}{2}}}$,$u_{xx} = t^{-1}\phi'' : \phi' = \frac{d\phi}{d\alpha}$,into the generalized heat equation (5.1.1)

we obtain the equation

$$\phi'' + \frac{\alpha\phi'}{2\lambda} = 0 \tag{5.5.4}$$

According to Wylie [42] equation (5.15.4) reduces to

$$\left.\begin{aligned} & b_0\phi' + b_1\phi = c: \\ & b_0 = e^{\frac{\alpha^2}{4\lambda}}, \qquad and \qquad b_1 = c_1 \end{aligned}\right\} \tag{5.5.5}$$

If we set $c = 0$ into equation (5.15.5) and finally integrating it ,we obtain

$$\phi(\alpha) = \frac{K}{2\sqrt{\lambda}}\int e^{-\frac{\alpha^2}{4\lambda}}d\alpha .$$

$$\phi(\alpha) = \frac{K\sqrt{\pi}}{4\sqrt{\lambda}}erf\left(\frac{x}{2\sqrt{\lambda t}}\right) \qquad + C_2 ,$$

where the error function $erf(x)$ is defined as ,

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

and the complementary error function $erfc(x)$ defined as

$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du$, see Abramowitz and Stegun [1].

Hence we obtain,

$$u = \phi(\alpha) = \frac{C\sqrt{\pi}}{4\sqrt{\lambda}}erf\left(\frac{x}{2\sqrt{\lambda t}}\right) \qquad + C_2 \tag{5.5.6}$$

Case 5

Invariant solution under transformation generated by the infinitesimal generator

$$v_5 = 2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u}$$

has , system of characteristics

$$\frac{dx}{2t} = \frac{dt}{0} = \frac{\lambda du}{-xu}$$

Integrating the equation ,we obtain $ue^{\frac{x^2}{4\lambda t}} = \alpha$, t invariant, and

$$u = \phi(t).e^{-\frac{x^2}{4\lambda t}}$$

Substituting u_t , u_{xx} $:\phi' = \frac{d\phi}{dt}$ into the generalized heat equation (5.1.1)

we obtain the equation

$$\phi' + \frac{\phi}{2t} = 0$$

which on integration yields

$$\phi(t) = Ct^{-\frac{1}{2}}$$

Hence $u = \frac{c}{\sqrt{t}}.e^{-\frac{x^2}{4\lambda t}}$ (5.5.7)

Case 6

Invariant solution under transformation generated by the infinitesimal generator

$$v_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - [2ut + x^2\frac{u}{\lambda}]\frac{\partial}{\partial u},$$

has system of characteristics

$$\frac{dt}{4t^2} = \frac{dx}{4x} = \frac{-\lambda du}{2\lambda tu + ux^2}.$$

Integrating ,we obtain $xe^{-\frac{1}{t}} = \alpha$ and $\mu = \frac{2tu\lambda + x^2u}{\lambda}$: $\mu = \phi(\alpha)$

$$u = \phi(\alpha)\left(\frac{\lambda}{2\lambda t + x^2}\right). \_ \ \alpha_x = e^{-\frac{1}{t}} ,\alpha_t = \frac{x}{t^2}e^{-\frac{1}{t}};\ \phi_t = \phi_\alpha\alpha_t ,\phi_x = \phi_\alpha\alpha_x$$

Substituting u_t , u_{xx} into the generalized heat equation (5.1.1), we get

$$\phi'' + A(x,t)\phi' + B(x,t)\phi = 0$$

which reduces to the first order linear equation

$$b_0\phi' + b_1\phi = c$$

where $b_0 = e^{\frac{\alpha^2}{2}a^*} + K$

.

Hence we get

$$\phi(\alpha) = C\int e^{-\frac{\alpha^2}{2}a^*}\, d\alpha \quad \text{where} \quad a^* = \frac{(2\lambda t + x^2)^2 - 2t^2(1+\lambda)}{t^2 e^{\frac{2}{t}}(2\lambda t + x^2)}$$

Finally we arrive at the solution

$$u(x,t) = \frac{C\lambda\sqrt{\pi}}{\left(2\lambda t + x^2\right)\sqrt{2a^*\lambda}}\, erf\left(xe^{\frac{1}{t}}\frac{\sqrt{a^*}}{\sqrt{2}}\right) + C_1\, \frac{\lambda}{(2\lambda t + x^2)} \tag{5.5.8}$$

5.6 Symmetry Solutions of Equation (5.1.1)

According to section 3.10 of chapter 3, symmetry transformations convert known solutions into new solutions, Bluman and Kumei[4],Olver[18].

We consider the group transformations that arise from the infinitesimal generators admitted by equation (5.1.1);

$$v_1 = \frac{\partial}{\partial x}\ ,\ v_2 = \frac{\partial}{\partial t}\ , v_3 = u\frac{\partial}{\partial u}\ ,\ v_4 = 2x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t}\ ,\ v_5 = 2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u}\ ,$$

$$v_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left[2ut + x^2\frac{u}{\lambda}\ \right]\frac{\partial}{\partial u},$$

thus

$$v_1 = \frac{\partial}{\partial x};\ G_1:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_1(x+\varepsilon, t, u)$$

$$v_2 = \frac{\partial}{\partial t};G_2:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_2(x, t+\varepsilon, u)$$

$$v_3 = u\frac{\partial}{\partial u};G_3:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_3(x, t, e^{\varepsilon}u)$$

$$v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t};\ G_4:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_4(e^{\varepsilon}x, e^{2\varepsilon}t, u)$$

$$v_5 = 2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u};\ G_5:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_5\left(x + 2\varepsilon t, t, u.e^{-\frac{1}{\lambda}(\varepsilon x + \varepsilon^2 t)}\right)$$

$$v_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left[2ut + x^2\frac{u}{\lambda}\right]\frac{\partial}{\partial u};\ G_6:$$

$$X(x,t,u;\varepsilon) \rightarrow \qquad X_6\left(\frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon t}, u\sqrt{1-4\varepsilon t}\, e^{\left(\frac{-\varepsilon x^2}{\lambda(1-4\varepsilon t)}\right)}\right)$$

We note that groups G_1 , G_2, G_3, G_4, are merely translations and scaling i.e. trivial groups. It is only G_5, G_6,which are the non trivial groups .

Thus the genuine and therefore significant transformation groups we consider are only G_5 and, G_6.

Case1

First we consider the group G_5:

$$v_5 = 2t\frac{\partial}{\partial x} - \frac{xu}{\lambda}\frac{\partial}{\partial u};\ G_5:\ X(x,t,u;\varepsilon) \rightarrow \qquad X_5\left(x + 2\varepsilon t, t, u.e^{-\frac{1}{\lambda}(\varepsilon x + \varepsilon^2 t)}\right)$$

Then the new symmetry solution of (5.1.1) under G_5 becomes

$$u = \phi(x - 2\varepsilon t, t)e^{\frac{(-\varepsilon x + \varepsilon^2 t)}{\lambda}} \qquad (5..6.1)$$

whenever $u = \phi(x,t)$ is a known solution of the generalized heat equation, see Olver [18] .

Solution (i)

Consider the simple invariant solution of generalized heat equation. $u = c$

Substituting $u = c$ into (5.6.1) we get

$$u = ce^{\frac{(-\varepsilon x + \varepsilon^2 t)}{\lambda}} \qquad (5.6.1i)$$

Solution (ii)

Inserting invariant solution $u = cx$ into (5.6.1) we obtain,

$$u = (cx - 2c\varepsilon t)e^{\frac{(-\varepsilon x + \varepsilon^2 t)}{\lambda}} \tag{5.6.1ii}$$

Solution (iii)

Inserting invariant solution $u = \frac{c}{\sqrt{t}}.e^{-\frac{x^2}{4\lambda t}}$ into (5.6.1) we obtain,

$$u = \frac{c}{\sqrt{t}}.e^{-\frac{(x-2\varepsilon t)^2}{4\lambda t}} \quad .e^{\frac{(-\varepsilon x + \varepsilon^2 t)}{\lambda}} \tag{5.6.1iii}$$

Solution (iv)

Inserting invariant solution $u = \frac{C\sqrt{\pi}}{4\sqrt{\lambda}} erf\left(\frac{x}{2\sqrt{\lambda t}}\right) + C_2$

into (5.6.1) we obtain,

$$u = \frac{C\sqrt{\pi}}{4\sqrt{\lambda}} erf\left(\frac{x - 2\varepsilon t}{2\sqrt{\lambda t}}\right) e^{\frac{(-\varepsilon x + \varepsilon^2 t)}{\lambda}} \tag{5.6.1iv}$$

Solution (v)

Inserting invariant solution

$$u(x,t) = \frac{C\lambda\sqrt{\pi}}{(2\lambda t + x^2)\sqrt{2a * \lambda}} erf\left(xe^{\frac{1}{t}}\frac{\sqrt{a*}}{\sqrt{2}}\right) + C_1 \frac{\lambda}{(2\lambda t + x^2)}$$

into equation (5.6.1) we obtain,

$$u(x,t) = [\ \frac{C\lambda\sqrt{\pi} erf\left((x - 2\varepsilon t)e^{\frac{1}{t}}\frac{\sqrt{\overline{a}*}}{\sqrt{2}}\right)}{(2\lambda t + (x - 2\varepsilon t)^2)\sqrt{2\overline{a} * \lambda}} + \frac{C_1\lambda}{(2\lambda t + (x - 2\varepsilon t)^2)}\ ]\times e^{\frac{(-\varepsilon x + \varepsilon^2 t)}{\lambda}} \tag{5.6.1v}$$

Case 2

Secondly we consider the group G_6:

$X(x,t,u;\varepsilon)\rightarrow \qquad X_6\left(\frac{x}{1-4\varepsilon t},\frac{t}{1-4\varepsilon t},u\sqrt{1-4\varepsilon t}\,e^{\left(\frac{-\varepsilon x^2}{\lambda(1-4\varepsilon t)}\right)}\right)$, from which we develop new symmetry solutions of equation (5.1.1):

$$u(x,t)=\Phi\left(\frac{x}{1+4\varepsilon t},\frac{t}{1+4\varepsilon t}\right)\frac{1}{\sqrt{1+\varepsilon t}}e^{\frac{-\varepsilon x^2}{\lambda(1+4\varepsilon t)}} \qquad (5.6.2)$$

where $\Phi(x,t)$ is a known solution, of the generalized heat equation (5.1.1).

Solution (i)

Consider the simple invariant solution of generalized heat equation, $u=c$

Substituting $u=c$ into equation (5.6.2) we obtain,

$$u=\frac{C}{\sqrt{1+\varepsilon t}}e^{\frac{-\varepsilon x^2}{\lambda(1+4\varepsilon t)}} \qquad (5.6.2i)$$

Solution (ii)

inserting invariant solution $u=cx$ into equation (5.6.2) we obtain,

$$u(x,t)=\frac{Cx}{1+4\varepsilon t}\frac{1}{\sqrt{1+\varepsilon t}}e^{\frac{-\varepsilon x^2}{\lambda(1+4\varepsilon t)}} \qquad (5.6.2ii)$$

Solution (iii)

Inserting invariant solution $u=\frac{c}{\sqrt{t}}.e^{-\frac{x^2}{4\lambda t}}$ into equation (5.6.2) we obtain,

$$u(x,t)=.\frac{C\sqrt{1+4\varepsilon t}}{\sqrt{t}}e^{-\frac{(1+4\varepsilon)x^2}{\lambda(1+4\varepsilon t)}} \qquad (5.6.2iii)$$

Solution (iv)

Inserting invariant solution $u=\frac{C\sqrt{\pi}}{4\sqrt{\lambda}}erf\left(\frac{x}{2\sqrt{\lambda t}}\right)+C_2$

into equation (5.6.2) we obtain,

$$u(x,t)=\left[\frac{C\sqrt{\pi}}{4\sqrt{\lambda}}erf\left(\frac{x}{\sqrt{4\lambda\, t(1+4\varepsilon t)}}\right)+C_2\right]\frac{1}{\sqrt{1+\varepsilon t}}e^{\frac{-\varepsilon x^2}{\lambda(1+4\varepsilon t)}} \qquad (5.6.2iv)$$

Solution (v)

Inserting invariant solution $u(x,t) = \frac{C\lambda\sqrt{\pi}}{\left(2\lambda t + x^2\right)\sqrt{2a^* \lambda}} erf\left(xe^{\frac{1}{t}} \frac{\sqrt{a^*}}{\sqrt{2}} \right) + C_1 \frac{\lambda}{(2\lambda t + x^2)}$

into equation (5.6.2) we obtain,

$$u(x,t) = \left[\frac{\left(1+4\varepsilon t\right)^2 C\lambda\sqrt{\pi}\, erf\left(\frac{x}{1+4\varepsilon t} e^{\frac{1+4\varepsilon t}{t}} \frac{\sqrt{\bar{a}^*}}{\sqrt{2}} \right)}{\left(2\lambda t(1+4\varepsilon t)+x^2\right)\sqrt{2\bar{a}^* \lambda}} + C_1 \right] \times \frac{1}{\sqrt{1+\varepsilon t}} e^{\frac{-\varepsilon x^2}{\lambda(1+4\varepsilon t)}} \qquad (5.6.2v)$$

5.7 Lie Group Solution of The Burgers Equation

The Burgers equation we are solving which is referred to as (1.1.0) in chapter 1 is the partial differential equation

$u_t + uu_x = \lambda u_{xx}$ $\qquad\lambda$ - real parameter

Here the required symmetry groups of transformations are of the form

$t^* = T(t,x,u;\varepsilon)$, $x^* = X(t,x,u;\varepsilon)$, $u^* = U(t,x,u;\varepsilon)$

with corresponding infinitesimals

$$\xi(t,x,u) = \frac{\partial X(t,x,u;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \tau(t,x,u) = \frac{\partial T(t,x,u;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \phi(t,x,u) = \frac{\partial U(t,x,u;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

We let the generator V ,of (1.1.0) be of the form

$$V = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u}$$

We determine all the coefficient functions ξ , τ , ϕ so that the corresponding one-parameter Lie group of transformations

$t^* = T(t,x,u;\varepsilon)$, $x^* = X(t,x,u;\varepsilon)$, $u^* = U(t,x,u;\varepsilon)$

form a symmetry group of equation (1.1.0).

Extended transformations of equation (1.1.0) with n=2 are of the form

$u_t^* = U_1(t,x,u,u_t,u_x;\varepsilon)$

$u_x{}^* = U_2(t,x,u,u_t,u_x;\varepsilon)$

$u_{tt}{}^* = U_{11}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx};\varepsilon)$

$u_{tx}{}^* = U_{12}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx};\varepsilon)$

$u_{xx}{}^* = U_{22}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx};\varepsilon)$

$u_{tx}{}^* = U(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx};\varepsilon)$

The infinitesimal generator of equation (1.1.0) is

$$V = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u}$$

with once and twice extended generators respectively as

$$V^{(1)} = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \phi(t,x,u)\frac{\partial}{\partial u} + \eta^{x}\frac{\partial}{\partial u_x} +$$

$$\eta^{t}\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}}. \tag{5.7.1}$$

$$V^{((1)} = V + \eta^{t}(t,x,u,u_t,u_x)\frac{\partial}{\partial u_t} + \eta^{x}(t,x,u,u_t,u_x)\frac{\partial}{\partial u_x}$$

$$V^{(2)} = V^{(1)} + \eta^{tt}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tt}} + \eta^{tx}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx})\frac{\partial}{\partial u_{tx}} +$$

$$\eta^{xx}(t,x,u,u_t,u_x,u_{xt},u_{tt},u_{xx})\frac{\partial}{\partial u_{xx}}. \tag{5.7.2}$$

where $\eta^{t},\eta^{x},\eta^{tx},\eta^{xx}$, are known functions of the derivatives of ϕ $,\xi,\tau$ and variables

$u_t,u_x,u_{xt},u_{tt},u_{xx}$.

Here subscripts denote partial differentiation.

From equation (1.1.0), $F = u_t + uu_x - \lambda u_{xx} = 0$

By theorem (3.8.3.) it follows that

$$V^{(2)}F = V^{(2)}(u_t + uu_x - \lambda u_{xx}) = 0 \tag{5. 7.3}$$

when $F = 0$, and so we obtain

$$\left[\ \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \eta^{t}\frac{\partial}{\partial u_t} + \eta^{x}\frac{\partial}{\partial u_x} + \eta^{tt}\frac{\partial}{\partial u_{tt}} + \eta^{tx}\frac{\partial}{\partial u_{tx}} + \eta^{xx}\frac{\partial}{\partial u_{xx}}\ \right]\left[\ u_t + uu_x - \lambda u_{xx}\ \right]=0$$

$$0 + 0 + \phi u_x + \eta^{t} + \eta^{x}u + 0 + 0 + 0 - \lambda\eta^{xx} = 0$$

The infinitesimal condition above reduces to

$$\phi u_x + \eta^t + \eta^x u - \lambda\eta^{xx} = 0 . \qquad (5.\ 7.4.)$$

with $\eta^t, \eta^x, \eta^{xx}$ defined in section 2.4 of chapter 2.

Substituting $\eta^t, \eta^x, \eta^{xx}$ into equation (5.2.4), we obtain equation

$$[\ \phi u_x + \{\phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u^2{}_t\} + u\{\phi_x - \tau_x u_t + (\phi_u - \xi_x)u_x - \xi_u u^2{}_x - \tau_u u_t u_x\}$$

$$-\lambda[\ \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3$$
$$-\tau_{uu}u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}\ ] = 0 \qquad (5.7.4a)$$

Equate to zero the coefficients of monomials in the first and second partial derivatives of u and on substituting $u_t + uu_x = \lambda u_{xx}$; $u_t = \lambda u_{xx} - u_{xx}$; $uu_x = \lambda u_{xx} - u_t$; wherever it occurs in (5.7.4.a) we arrive at the determining equations:

$u_x u_{xx}$: $\quad +3\lambda\xi_u - 2\lambda\xi_u + 2\lambda\tau_{xu} = 0$ (a)

$u_t u_{xx}$: $\quad \tau_u\lambda - \tau_u = 0$ (b)

$uu_x u_{xx}$ $\quad -2\lambda\tau_u - \lambda\tau_u = 0$ ©

$u^2{}_{xx}$: $\quad -\lambda^2\tau_u = 0$ (d)

u_{xx} $:\lambda(\phi_u - \tau_t) + \lambda^2\tau_{xx} - \lambda(\phi_u - 2\xi_x) + \lambda(\phi_u - \xi_x) = 0$ (e)

$u^3{}_x$: $\quad \lambda\xi_{uu} = 0$ (f)

$u^2{}_x$: $\quad -\lambda(\phi_{uu} - 2\xi_{xu}) = 0$ (g)

u_t $:(\phi_u - \tau_t) - (\phi_u - \xi_x) + \lambda\tau_{xx} - (\phi_u - 2\xi_x) = 0$ (h)

u_x $: \phi - \xi_t - \lambda(2\phi_{xu} - \xi_{xx}) = 0$ (i)

u_{tx}: $\quad 2\lambda\tau_x = 0$ (j)

$u_x u_{tx}$: $\quad 2\lambda\tau_u = 0$ (k)

$u_x u_t$: $\quad 2\lambda\tau_{xu} + 3\xi_u + \xi_u = 0$ (l)

1: $\phi_t = \lambda\phi_{xx}$ (m)

We see that (a) , (b) ,and (c) $\Rightarrow$ τ , ξ , are independent of u ,using (j) then τ , is further independent of x and so $\tau = \tau(t)$ only but, ξ , = $\xi(x,t)$. Equation (g) $\Rightarrow$ ϕ is linear in u

i.e. $\phi = \beta(x,t)u + \alpha(x,t)$ (5.7.4.1)

.From equation (e) we obtain

$\phi_u = \tau_t - \xi_x$ (5.7.4. 2)

and (h) $\Rightarrow$

$\phi_u = -\tau_t + 3\xi_x$ (5.7.4.3)

Then equation (5. 7.4.2) add equation (5. 7.4.3) , we arrive at

$\phi_u = \xi_x$ (5.7.4.4)

Similarly equation (5. 7.4.2) subtract equation (5. 7.4.3) gives ,

$\tau_t = 2\xi_x$ (5.7.4.5)

Partial differentiating equation (5. 7.4.5) with respect to , x we get ,

$\tau_{tx} = 0 = 2\xi_{xx}$ thus, $\xi_{xx} = 0 \Rightarrow$

$\xi = a(t)x + b(t)$.

Applying equation (5. 7.4.4) on ϕ we obtain

$$\beta(x,t) = \tau_t - \xi_x \quad (5.7.4.6)$$

Also, $\beta(x,t) = -\tau_t + 3\xi_x$ (5.7.4.7)

Differentiating partially equations (5. 7.4. 6) , (5. 7.4. 7) , we obtain

$\beta_x = \tau_{tx} - \xi_{xx} = -\xi_{xx}$ (5.7.4.8)

$$\beta_x = -\tau_{tx} + 3\xi_{xx} = 3\xi_{xx} \quad (5.7.4.\ 9)$$

$$\beta_t = \tau_{tt} - \xi_{tx} \quad (5.7.4.10\ )$$

Equations (5.7.4.8) , (5.7.4.9) imply ;

$$\beta_x = \xi_{xx} = 0 \quad (5.7.4.11)$$

It follows from equation (5.7.4.11) that ξ is linear x and, β is independent of x and therefore depends on , t, only i.e.

$$\beta = rt + k \quad (5.7.4.12\ )$$

$$\xi = a(t)x + b(t) \quad (5.7.4.13\ )$$

Equations (5.7.4.1) , (m) give

$$\beta_t = \lambda\beta_{xx} \quad (5.7.4.14\ )$$

$$\alpha_t = \lambda\alpha_{xx} \quad (5.\ 7.4.15\ )$$

Equations (5.7.4. 14), (5.7.4.15) implies α , β are solutions of heat equation.

Equations (5. 7.4.14), (5. 7.4.7) implies that ,

$$\tau_{tt} = 3\xi_{tx} \quad (5.7.4.16\ )$$

Differentiating partially twice equations (5.7.4.7), (5.7.4.14) we get

$$\beta_{tt} = \lambda\beta_{xxt} = 3\lambda\ \xi_{xxxt} = 0 \quad (5.7.4.17\ )$$

$$\beta_{tt} = -\tau_{ttt} + 3a_{tt} \quad (5.7.4.18)$$

Comparing equations (5.7.4.17), (5.7.4.18) we get

$$\tau_{ttt} = 3a_{tt} \quad (5.7.4.19\ )$$

Comparing equations (5.7.4.6), (5.7.4.7) we get

$$\beta = \xi_x \quad (5.7.4.20)$$

Differentiating partially twice equation (5.7.4.6) ,we get

$$\tau_{ttt} - \xi_{ttx} = \beta_{tt} = 0 \qquad (5.7.4.21)$$

Differentiating partially twice equation (5.7.4. 19) ,we get

$$\tau_{ttt} - 3\xi_{ttx} = -\beta_{tt} = 0 \qquad (5.7.4.22)$$

Comparing equations (5.7.4. 21), (5.7.4.22) we get

$$\tau_{ttt} = 0, \xi_{ttx} = 0 \qquad (5.7.4.23)$$

From equation (5.7.4.23) we deduce that

τ is purely quadratic in t , and ξ is linear in both , x and t.

We note that β is linear in t hence we deduce:

$$\xi = m + nx + lt + jxt \qquad (5.7.4.24)$$

$$\tau = m_0 + n_0 t + l_0 t^2 \qquad (5.7.4.25)$$

$$\phi = (m_1 + n_1 t)u + l_1 x + j_0 \qquad (5.7.4.26)$$

Differentiate partially equations (5.7.4.24), (5.7.4.25), (5.7.4. 26) and substitute into equations (a),(h) and (e).

On comparing the coefficients of , x and t. tx , u, tu we get:

$l = 0, n_0 = 2n + j_0 \ , n_1 = -j \ , l_1 = j, m_1 = -n; \ m \ , m_0 \ arbitrary.$

For uniformity we set $m_0 = c_1, m = c_2, j_0 = c_3, n = c_4, j = c_5$

Hence the relations (5.7.4.24), (5.7.4.25) ,(5.7.4.26) yield the infinitesimals τ , ξ , ϕ as:

$$\xi = c_2 + c_4 x + c_5 xt \qquad (5.7.5a)$$

$$\tau = c_1 + (c_3 + 2c_4)t + c_5 t^2 \qquad (5.7.5b)$$

$$\phi = c_3 - c_4 u + c_5 x - c_5 tu + \alpha(x,t): \ \alpha_t = \lambda \alpha_{xx} \qquad (5.7.5c)$$

α , is an arbitrary solution the generalized heat equation.

We express ξ , τ , ϕ in the standard basis

$$\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_\alpha \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\left.\begin{aligned} \xi &= 0.c_1 + 1.c_2 + 0.c_3 + c_4 x + c_5 tx + 0.c_\alpha \\ \tau &= 1.c_1 + 0.c_2 + 1.c_3 t + 2.c_4 t + 1.c_5 t^2 + 0.c_\alpha \\ \phi &= 0.c_1 + 0.c_2 + 1.c_3 - 1.c_4 u + 1.c_5 (x - tu) + 1.c_\alpha .\alpha \end{aligned}\right\} \quad (5.7.6)$$

We form the corresponding basis infinitesimal generators v_i's of the form

$v_i = \widehat{\xi}_i \frac{\partial}{\partial x} + \widehat{\tau}_i \frac{\partial}{\partial t} + \widehat{\phi}_i \frac{\partial}{\partial u}$: $\widehat{\xi}_i$, $\widehat{\tau}_i$, $\widehat{\phi}_i$ are the coefficients c_i in the standard solutions of ξ , τ , ϕ .Hence v_1 , v_2 , v_3 , v_4 , v_5 , v_α are listed below as the infinitesimal generators for the Burgers equation.

$$v_1 = \frac{\partial}{\partial t}, \quad v_2 = \frac{\partial}{\partial x}, v_3 = t\frac{\partial}{\partial t} + \frac{\partial}{\partial u}, \quad v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

$$v_5 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - [\, ut - x \,]\frac{\partial}{\partial u}, v_\alpha = \omega\frac{\partial}{\partial u} : \ \omega_t = \lambda\omega_{xx} .$$

5.8 Lie Brackets of The Infinitesimal Generators of The Burgers Equation

The set $\langle v_i \rangle$ of the generators of the Burgers equation forms a Lie algebra, we therefore construct a Lie bracket table for $\langle v_i \rangle$.

A Lie bracket $\lfloor v_i , v_j \rfloor$ for any two operators , v_i , v_j is give by;

$\lfloor v_i , v_j \rfloor = v_i v_j - v_i v_j = 0$; i =j, i =1,2,3,........,α.

Construction of the Lie brackets table for the infinitesimal generators $\langle v_i \rangle$;

employs the definition

$\lfloor v_i , v_j \rfloor = v_i v_j - v_i v_j = 0$; $i, j = 1,2,3,.......5,\alpha$.

It follows that $\lfloor v_j \ , v_j \rfloor = v_j v_j - v_j v_j = 0$, $\lfloor v_i \ , v_j \rfloor = -\lfloor v_j \ , v_i \rfloor$; $i, j = 1,2,3,\ldots\ldots\ldots,\alpha$.

$$v_1 = \frac{\partial}{\partial t} \ , \ v_2 = \frac{\partial}{\partial x} \ , v_3 = t\frac{\partial}{\partial t} + \frac{\partial}{\partial u} \ , \ v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} \ ,$$

$$v_5 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - [\ ut - x\ ]\frac{\partial}{\partial u}, v_\alpha = \omega\frac{\partial}{\partial u}: \ \omega_{\ t} = \lambda\,\omega_{xx}$$

$$[v_1 \ , v_4] = v_1 v_4 - v_4 v_1$$

$$=(\frac{\partial}{\partial t})[\ x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\ ]\text{-}[\ x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\ ](\frac{\partial}{\partial t}) = 2(\frac{\partial}{\partial t}) = 2v_1$$

$$[v_1 \ , v_3] = v_1 v_3 - v_3 v_1 = (\frac{\partial}{\partial t})[\ t\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\ ]\text{-}[\ t\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\ ](\frac{\partial}{\partial t}) = \frac{\partial}{\partial x} = v_2$$

$$[v_1 \ , v_5] = v_1 v_5 - v_5 v_1$$

$$= (\frac{\partial}{\partial t})[\ tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - [\ ut - x\ ]\frac{\partial}{\partial u}\ ],\text{-}[\ tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - [\ ut - x\ ]\frac{\partial}{\partial u}\ ](\frac{\partial}{\partial t})$$

$$= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} = v_4$$

$$[v_2 \ , v_4] = v_2 v_4 - v_4 v_2$$

$$= (\frac{\partial}{\partial x})[\ x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\ ]\text{-}[\ x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\ ](\frac{\partial}{\partial x}) = (\frac{\partial}{\partial x}) = v_2$$

$$[v_2 \ , v_3] = v_2 v_3 - v_3 v_2 = (\frac{\partial}{\partial x})[\ t\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\ ]\text{-}[\ t\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\ ](\frac{\partial}{\partial x}) = 0$$

Other Lie brackets are computed in the similar way. On application of skew symmetry property of the Lie brackets , Lie brackets table is fully constructed as shown below .

$[v_i , v_j]$	v_1	v_2	v_3	v_4	v_5	v_α
v_1	0	0	v_2	$2v_2$	v_4	v_{α_x}
v_2	0	0	0	v_2	v_3	v_{α_t}
v_3	v_2	0	0	$2v_5-2v_3$	v_5	$-v_\alpha$
v_4	$-2v_2$	$-v_2$	$-2v_5+2v_3$	0	0	$v_{\alpha'}$
v_5	$-v_4$	$-v_3$	$-v_5$	0	0	$v_{\alpha''}$
v_α	$-v_\alpha$	$-v_{\alpha_t}$	v_α	$-v_{\alpha'}$	$-v_{\alpha''}$	0

Table5.8 [Lie brackets for L^α]

5.9 Lie Groups Admitted by Equation (1.1.0).

The one-parameter groups G_i admitted the by the Burgers equation , are determined by solving the corresponding Lie equations

$$v_1:\frac{d\bar{t}}{d\varepsilon}=1;\quad v_2:\frac{d\bar{x}}{d\varepsilon}=1;\quad v_3:\frac{d\bar{x}}{d\varepsilon}=\bar{t},\quad \frac{d\bar{u}}{d\varepsilon}=1;\ v_4:\frac{d\bar{t}}{d\varepsilon}=2\bar{t},\quad \frac{d\bar{x}}{d\varepsilon}=\bar{x},\quad \frac{d\bar{u}}{d\varepsilon}=-\bar{u};$$

$$v_5:\frac{d\bar{x}}{d\varepsilon}=\bar{x}\bar{t},\quad \frac{d\bar{t}}{d\varepsilon}=\bar{t}^2,\quad \frac{du}{d\varepsilon}=\bar{x}-\bar{t}\bar{u}$$

with initial conditions: $\bar{t}_{\varepsilon=0}=t,\ \bar{x}_{\varepsilon=0}=x,\ \bar{u}_{\varepsilon=0}=u$

which lead to;

$$v_1;G_1: X(x,t,u;\varepsilon)\rightarrow \qquad X_1(x,t+\varepsilon,u)$$

$$v_2;G_2: X(x,t,u;\varepsilon)\rightarrow \qquad X_2(x+\varepsilon,t,u)$$

$$v_3;G_3: X(x,t,u;\varepsilon)\rightarrow \qquad X_3(x+\varepsilon t,t,\ u+\varepsilon)$$

$$v_4;G_4: X(x,t,u;\varepsilon)\rightarrow \qquad X_4\left(e^{\varepsilon}x,\ e^{2\varepsilon}t,e^{-\varepsilon}u\right)$$

$$v_5;G_5: X(x,t,u;\varepsilon)\rightarrow \qquad X_5\left(\frac{x}{1-\varepsilon t},\frac{t}{1-\varepsilon t},u(1-\varepsilon t)+\varepsilon x\right)$$

$$v_\alpha;G_\alpha: X(x,t,u;\varepsilon)\rightarrow \qquad X_\alpha(x,t,u+\varepsilon\alpha(x,t))\ ;$$

5.10 Group Transformations of Solutions

By symmetry group inversion theory of section (3.10) of chapter 3 , if each G_i is a symmetry group and $u=\Phi(x,t)$ is a solution of the Burgers equation (1.1.0), then transformation groups of the Burgers equation (1.1.0), solve the equation (1.1.0).

The above solution can also be written in the new variables: $\bar{u}=\Phi(\bar{t},\bar{x})$.

If $\bar{u}$, $\bar{x}$, $\bar{t}$ are group transformations of the Burgers equation (1.1.0) with $\bar{u}$, of the form $\bar{u} = \Psi(u,x,t,\varepsilon)$, for some explicit function Ψ , then applying the_ inverse mapping, the new symmetry solution $\hat{u}$ satisfies relation $\hat{u} = \Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t})\right), g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t}), \bar{\varepsilon}^{-1}\right)$ where $u = \Phi(x,t)$ is any known solution of (1.1.0).

$$v_1; G_1 : X(x,t,u;\varepsilon) \to \qquad X_1(x,t+\varepsilon,u)$$

$$\bar{x} = x; \bar{t} = t + \varepsilon; \bar{u} = u \quad : \bar{x}^{-1} = x, \bar{t}^{-1} = t - \varepsilon, \bar{u}^{-1} = u$$

$$\bar{u}(x,t,u,\varepsilon) \equiv \Psi(u,x,t,\varepsilon) \Rightarrow \Psi(x,t,u,\varepsilon) = u$$

Then the new symmetry solution $\hat{u}_1$, is defined by

$$\hat{u}_1 = \Psi\left(\Phi\left(\bar{x}^{-1}, \bar{t}^{-1}\right), \bar{x}^{-1}, \bar{t}^{-1}, \bar{\varepsilon}^{-1}\right) = \Phi\left(\bar{x}^{-1}, \bar{t}^{-1}\right) = \Phi(x, t-\varepsilon)$$

Similarly

$$\hat{u}_2 = \Phi(x-\varepsilon,t)$$

$$\hat{u}_3 = \Phi(x-\varepsilon,t) - \varepsilon$$

$$\hat{u}_4 = e^{-\varepsilon}\Phi\left(e^{-\varepsilon}x, e^{-2\varepsilon}t\right)$$

$$\hat{u}_5 = \frac{\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t}\Phi\left(\frac{x}{1+\varepsilon t}, \frac{t}{1+\varepsilon t}\right)$$

$$\hat{u}_\alpha = \Phi\left(\bar{x}^{-1}, \bar{t}^{-1}\right) + \varepsilon\alpha(x,t)$$

5.11 Invariant Solutions of The Burgers Equation (1.1.0)

(a) The invariant solution of the Burgers equation $u_t + uu_x = \lambda u_{xx}$ under the transformation group generated by generator

$$V = \frac{\partial}{\partial x} \tag{5.11.1}$$

has the system of characteristics

$\frac{du}{u} = \frac{dt}{0} = \frac{dx}{1}$, with invariant as $\phi = t \quad \mu = t$ hence

$$u = \phi(t) \tag{5.11.2}$$

Substituting u , u_t , u_x , u_{xx} into equation (1.1.0), we obtain the ordinary differential equation

$$\phi'(t) = 0$$

which leads to solution

$$u = C \tag{5.11.3}$$

(b) The invariant solution of the Burgers equation $u_t + uu_x = \lambda u_{xx}$ under the group generated by generator $V = \frac{\partial}{\partial t}$

has the system of characteristics

$\frac{du}{u} = \frac{dt}{1} = \frac{dx}{0}$, with invariant as $\phi = x \quad \mu = x$,hence

$$u = \phi(x)$$

Substituting u , u_t , u_x , u_{xx} into equation (1.1.0), we obtain the ordinary differential equation

$$\lambda\phi''(t) - \phi\phi' = 0 \qquad .(5.11.4)$$

Integrate it twice, we get

$\int \frac{2\lambda d\phi}{\phi^2 + C} = \int dx$ where C is a constant .

Hence we obtain the solution:

$$u = \frac{2\lambda}{x + C_1} \quad : \qquad C = 0$$

(5.11.5)

$$u = k \tan \frac{kx + C_1}{2\lambda} : \qquad C > 0 : C = k^2 : \qquad (5.11.6)$$

$$u = -s \coth \frac{sx + C_1}{2\lambda} : \qquad C < 0 : C = -s^2 \qquad (5.11.7)$$

(c) The invariant solution of the Burgers equation $u_t + uu_x = \lambda u_{xx}$ under the Galilean transformation group generated by generator

$$V = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$$

has the system of characteristics

$\frac{du}{1} = \frac{dt}{0} = \frac{dx}{t}$, with invariant as $\phi = t$ satisfying

$u = \frac{x}{t} + \phi(t)$,with

$$u_t = -\frac{x}{t^2} + \phi'(t) \;, u_x = \frac{1}{t} \;, u_{xx} = 0$$

Substituting u $,u_t$ $,u_x$ $,u_{xx}$ into equation (1.1.0), we obtain the ordinary differential equation

$$\phi'(t) + \frac{\phi}{t} = 0 \;. \qquad (5.11.8)$$

Integrate it, we get

$$u = \frac{x}{t} - \frac{c}{t} \qquad (5.11.9)$$

(d) The invariant solution of the Burgers equation $u_t + uu_x = \lambda u_{xx}$ under the group of dilation group generated by

$$V = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$$

has the system of characteristics

$$\frac{du}{-u}=\frac{dt}{2t}=\frac{dx}{x} ,$$

such that the invariant α takes the form

$$\alpha=\frac{x}{\sqrt{t}}. \tag{5.11.10}$$

and so

$$u=\frac{\phi(\alpha)}{\sqrt{t}}. \tag{5.11.11}$$

Substituting u ,u_t ,u_x ,u_{xx} into equation (1.1.0), we obtain

the second order ordinary differential equation with variable coefficients

$$\lambda\phi''+\frac{\phi+\alpha\phi'}{2}-\phi\phi'=0 . \tag{5.11.12}$$

Integrate it, we get the first order nonlinear equation

$$\phi'=\frac{\phi^2}{2\lambda}-\frac{\alpha\phi}{\lambda t}+C \tag{5.11.13}$$

which is Abel's (Riccati's) equation of the first kind.

We integrate it to , yield solution

$$u=[\ \sqrt{t}\,C_3-\left(2\sqrt{2\lambda}\right)^{-1}\sqrt{t\pi}\,e^{\frac{\alpha^2}{2}}erf(\alpha)\ ]^{-1}\ :C=0 \tag{5.11.14}$$

(e) The invariant solution of the Burgers equation $u_t+uu_x=\lambda u_{xx}$ under the transformation group generated by

$$V=t^2\frac{\partial}{\partial t}+xt\frac{\partial}{\partial x}+\left(x-tu\right)\frac{\partial}{\partial u}$$

has the system of characteristics

$$\frac{du}{x-tu}=\frac{dt}{t^2}=\frac{dx}{xt} ,$$

such that the invariant α is defined as

$$,\frac{x}{t}=C=\alpha\ ,u=\frac{x}{t}-\frac{\phi}{t} \tag{5.11.15}$$

Substituting u ,u_t ,u_x ,u_{xx} into equation (1.1.0), we obtain

the second order ordinary differential equation

$$.\lambda\phi''+\phi\phi'=0 . \tag{5.11.16}$$

Integrate it, one has the first order nonlinear equation

$$\phi' + \frac{\phi^2}{2\lambda} = C \tag{5.11.17}$$

We integrate it, to give solution:

$$\left.\begin{aligned} \phi &= \frac{2\lambda t}{x + c_1 t} \\ u &= \frac{x}{t} - \frac{2\lambda}{x + tc_1} \end{aligned}\right\} \quad : C = 0 \tag{5.11.18}$$

$$\left.\begin{aligned} \phi &= k\sqrt{2}\tanh\left[\frac{\sqrt{2}kx}{2\lambda t} + C_2\right] \qquad C = k^2 \qquad : C > 0 \\ u &= \frac{x}{t} - \frac{k}{t}\sqrt{2}\tanh\left[\frac{\sqrt{2}kx}{2\lambda t} + C_2\right] : \; C = k^2 \quad : C > 0 \end{aligned}\right\} \tag{5.11.19}$$

$$\left.\begin{aligned} \phi &= r\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}rx}{2\lambda t}\right] : \; C = -r^2 \qquad : C < 0 \\ u &= \frac{x}{t} - \frac{r}{t}\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}rx}{2\lambda t}\right] : \; C = -r^2 \qquad : C < 0 \end{aligned}\right\} \tag{5.11.20}$$

(f) The invariant solution of the Burgers equation $u_t + uu_x = \lambda u_{xx}$ under the infinite-dimensional group generated by generator $v_\alpha = \omega \frac{\partial}{\partial u}$: $\omega_t = \lambda \omega_{xx}$.

(i) $\omega = \frac{c}{\sqrt{t}} e^{-\frac{x^2}{4\lambda t}}$ has the system of characteristics

$\frac{du}{\omega} = \frac{dt}{0} = \frac{dx}{0}$, which is invariant under both t, x

and we integrate to give,

$$u\left[\frac{c}{\sqrt{t}} e^{-\frac{x^2}{4\lambda t}} \phi(t)\right]^{-1} = \alpha \;, \mu = \phi(t) : \alpha = constant \tag{5.11.21}$$

which we reduce to equation

$$u = \frac{c}{\sqrt{t}} e^{-\frac{x^2}{4\lambda t}} \phi(t). \tag{5.11.22}$$

Substituting u $,u_t$ $,u_x$ $,u_{xx}$ with $c=1$ into equation (1.1.0), we obtain the ordinary differential equation

$$\phi' - \left(\frac{xt^{-\frac{3}{2}}}{2\lambda} e^{-\frac{x^2}{4\lambda t}} \right)\phi(t) = 0 \qquad \phi' \equiv \frac{d\phi}{dt} \tag{5.11.23}$$

Integrating the above equation leads to

$$\phi(t) = e^{\int \left(\frac{t^{-\frac{3}{2}} x e^{-\frac{x^2}{4\lambda t}}}{\lambda} \right) dt} = e^{-\left[\frac{4\lambda e^{-\frac{x^2}{4\lambda t}}}{(4\lambda t + x^2)\sqrt{t}} \right]} \tag{5.11.23a}$$

and finally we get

$$u = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4\lambda t}} \left[e^{-\left[\frac{4\lambda e^{-\frac{x^2}{4\lambda t}}}{(4\lambda t + x^2)\sqrt{t}} \right]} + K \right] \tag{5.11.24}$$

(ii) $V_\omega = \omega \frac{\partial}{\partial u}$ $: \omega_t = \lambda \omega_{xx}, \omega = cx$ has the system of characteristics

$\frac{du}{cx} = \frac{dt}{0} = \frac{dx}{0}$, which is invariant under both t $,x$

and on integrating ,gives

$$ux^{-1} = \alpha \quad , \mu = \phi(t) : \alpha = t \tag{5.11.25}$$

which leads to equations.

$$u = x\phi(t),. \text{ or } u = x\phi(x). \tag{5.11.26}$$

Case1

$$u = x\phi(t). \tag{5.11.27}$$

Substituting u $,u_t$ $,u_x$ $,u_{xx}$ into equation (1.1.0), we obtain the ordinary differential equation

$$\phi' + \phi^2 = 0 \qquad \phi' \equiv \frac{d\phi}{dt}. \tag{5.11.28}$$

Integrating the above equation leads to $\varphi(t) = (t-c)^{-1}$,and finally we get

$$u(x,t) = x(t-c)^{-1} \tag{5.11.29}$$

Case2

$u \ = x\phi(x).$

Substituting $u \ ,u_t \ ,u_x \ ,u_{xx}$ into equation (1.1.0), we obtain

the ordinary differential equation

$$x\lambda\phi'' + 2\lambda\phi' \ - \frac{d}{dx}\left[\frac{x^2 \ \phi^2}{2}\right] = 0 \qquad \phi' \equiv \frac{d\phi}{dx} \ .$$

i.e. $\lambda\frac{d}{dx}(\ x\phi'\ ) + \lambda\phi' \ - \frac{d}{dx}\left[\frac{x^2 \ \phi^2}{2}\right] = 0$ (5.11.30)

We integrate the above equation to get

$$\lambda x\phi' + \lambda\phi \ - \left[\frac{x^2 \ \phi^2}{2}\right] = C \qquad (5.11.31)$$

or

$$\phi' + \frac{1}{x}\phi \ - \left[\frac{x \ \ \phi^2}{2\lambda}\right] = \frac{1}{x}C \qquad (5.11.32)$$

This is Bernoulli equation which yields

$$\phi = \frac{6\lambda}{6c\lambda x - x^4} \ \text{: with } \ C = 0, \qquad (5.11.33)$$

hence

$$u = \frac{6\lambda}{6c\lambda - x^3} \qquad (5.11.34)$$

(iii) $V_\omega = \omega\frac{\partial}{\partial u} \ : \omega_t = \lambda\omega_{xx} : \omega = k(x,t)erf\left[l(x,t)xe^{\frac{1}{t}}\right]$ has the system of characteristics

$\frac{du}{\omega} = \frac{dt}{0} = \frac{dx}{0}$, which is invariant under both $t \ ,x$

and on integrating , we get

$u = \omega c \quad ,\mu = \phi(t) : \alpha = t \ or \, c = \phi(t)$, (5.11.35)

which leads to equations

$u \ = \omega\phi(t),$.(5.11.36)

Substituting $u \ ,u_t \ ,u_x \ ,u_{xx}$ into equation (1.1.0), we obtain

the ordinary differential equation

$$\phi' + \left[\frac{\omega_t - \lambda\omega_{xx}}{\omega}\right]\phi + \omega_x \phi^2 = 0 \qquad \phi' \equiv \frac{d\phi}{dt} \qquad .(5.11.37)$$

This is a Bernoulli or Abel's equation, hence .(5.3.37) reduces to

$$\phi' + \omega_x \phi^2 = 0 \quad . \qquad (5.11.38)$$

$$\phi(t) = [\ K \int a_2 I(x,t)dt + C\ ]^{-1} \qquad (5.11.39)$$

and finally we get

$$u = Ck(x,t)erf\left[\ l(x,t)xe^{\frac{1}{t}}\ \right]\left[c_3 + \int \omega_x dt\right]^{-1} \qquad .(5.11.40)$$

$$u = \frac{Ck(x,t)erf\left(\ l(x,t)xe^{\frac{1}{t}}\ \right) + L}{C_3 \ + F(x,t)} \quad :,. \qquad .(5.11.41)$$

(iv) $V_{\omega} = \omega\frac{\partial}{\partial u} \quad : \omega_t = \lambda\omega_{xx} : \omega = \sqrt{\frac{\pi}{\lambda}} erf\left[x\sqrt{\frac{\lambda}{4t}}\right]$ has the system of characteristics

$\frac{du}{\omega} = \frac{dt}{0} = \frac{dx}{0}$, which is invariant under both t , x

and on integrating , we get

$$u = \omega c \quad , \mu = \phi(t) : \alpha = t \ or \ c = \phi(t) \quad . \qquad (5.11.42)$$

which leads to equations.

$u = \omega\phi(t)$,. or $u = \omega\phi(x)$

Substituting u , u_t , u_x , u_{xx} into equation (1.1.0), we obtain

the ordinary differential equation

$$\phi' + \left[\frac{\omega_t - \lambda\omega_{xx}}{\omega}\right]\phi + \omega_x \phi^2 = 0 \qquad \phi' \equiv \frac{d\phi}{dt} \qquad (5.11.43)$$

This is a Bernoulli or Abel's equation, hence

$$\phi' + +\omega_x \phi^2 = 0 \qquad (5.11.44)$$

$$\phi(t) = [\ K \int \omega_x dt + C\ ]^{-1}$$

and finally we obtain

$$u=\frac{K\sqrt{\frac{\pi}{\lambda}}erf\left(x\sqrt{\frac{\lambda}{4t}}\right)+L}{C_3+F_1(x,t)} \quad :, \tag{5.11.45}$$

(v) $V_\omega=\omega\frac{\partial}{\partial u}$ $:\omega_t=\lambda\omega_{xx},\omega=c$ has the system of characteristics

$\frac{du}{c}=\frac{dt}{0}=\frac{dx}{0}$, which is invariant under both t , x

and on integrating ,gives

$$u=\alpha \quad ,\mu=\phi(t):\alpha=t \tag{5.11.46}$$

which leads to equations

$$.u=\phi(t),.\text{or}\quad u=\phi(x). \tag{5.11.47}$$

Case1

$u=\phi(t),$

Substituting u $,u_t$ $,u_x$ $,u_{xx}$ into equation (1.1.0), we obtain

$\phi'(t)=0$ which leads to trivial solution

$$u=\text{C} \tag{5.11.48}$$

Case2

$u=\phi(x),$

Substituting u $,u_t$ $,u_x$ $,u_{xx}$ into equation (1.1.0), we obtain

the second order ordinary differential equation

$$\lambda\phi''-\phi\phi'=0 \quad . \tag{5.11.49}$$

Integrate it, we get the first order nonlinear equation

$$\phi'-\frac{\phi^2}{2\lambda}=C \tag{5.11.50}$$

We integrate it, and obtain solution:

$$\left.\begin{aligned}&\phi=\frac{-2\lambda}{x+c_1} \qquad :C=0\\&u=\frac{-2\lambda}{x+c_1}\end{aligned}\right\} \tag{5.11.51}$$

$$\left.\begin{aligned}&\phi = k\tan\left[\frac{kx}{2\lambda t}+C_2\right] \quad C = k^2 \quad :C>0\\&u = k\tan\left[\frac{kx}{2\lambda t}+C_2\right] \quad C = k^2 \quad :C>0\end{aligned}\right\} \tag{5.11.52}$$

$$\left.\begin{aligned}&\phi = r\coth\left[C_3+\frac{rx}{2\lambda t}\right] \quad C=-r^2 \quad :C<0\\&u=\phi = r\coth\left[C_3+\frac{rx}{2\lambda t}\right] \quad C=-r^2 \quad :C<0\end{aligned}\right\} \tag{5.11.53}$$

5.12 Symmetry Solutions of The Burgers Equation (1.1.0)

We consider the infinitesimal generator

$$v_5 = tx\frac{\partial}{\partial x}+t^2\frac{\partial}{\partial t}+[\ x-tu\ ]\frac{\partial}{\partial u},$$

of the Burgers equation as the only non trivial symmetry ;

$$G_5:\ X(x,t,u;\varepsilon)\rightarrow X_5\left(\frac{x}{1-\varepsilon t},\frac{t}{1-\varepsilon t},u(1-\varepsilon t)+\varepsilon x\right)$$

which has the groups:

$$\bar{x}=\frac{x}{1-\varepsilon t},\bar{t}=\frac{t}{1-\varepsilon t},\bar{u}=u(1-\varepsilon t)+\ \varepsilon x.$$

We apply the _ inverse mapping, in section 3.10 of chapter 3.

If $\bar{u}$, $\bar{x}$, $\bar{t}$ are group transformations of the partial differential equation (3.8.1) with $\bar{u}$, of the form $\bar{u}=\Psi(u,x,t,\varepsilon)$, for some explicit function Ψ , then the new symmetry solution $\hat{u}$ is defined by $\hat{u}=\Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}),g_\varepsilon^{-1}(\bar{t})\right),g_\varepsilon^{-1}(\bar{x}),g_\varepsilon^{-1}(\bar{t}),\varepsilon\right)$ where $u=\Phi(x,t)$ is any known solution of (3.8.1).

If $u=\Phi(x,t)$ is any known solution of (1.1.0) then,

$\bar{u}=\overline{\Phi}(\bar{x},\bar{t})$, $\hat{u}=\Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}),g_\varepsilon^{-1}(\bar{t})\right),g_\varepsilon^{-1}(\bar{x}),g_\varepsilon^{-1}(\bar{t}),\varepsilon\right)$

and finally our new solution based on the inverse groups,

$\bar{x}^{-1} = \dfrac{x}{1+\varepsilon t}$, $\bar{t}^{-1} = \dfrac{t}{1+\varepsilon t}$, $\bar{u} = u(1-\varepsilon t) + \varepsilon x$ takes the form

$$\hat{u} = \Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t})\right), g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t}), \varepsilon\right)$$

and we obtain the symmetry solution of equation (1.1.0) as

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t}\Phi\left[\frac{x}{1+\varepsilon t}, \frac{t}{1+\varepsilon t}\right], \qquad (5.12.1)$$

where $u = \Phi(x,t)$ is a known solution of the Burgers equation .

Solution (i)

Consider the simple invariant solution of the Burgers equation. $u = c$

Substituting $u = K,\ say\ K = \lambda$ into equation (5.12.1) we obtain,

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \lambda\left[\frac{1}{1+\varepsilon t}\right] \qquad (5.12.2)$$

Solution (ii)

Inserting invariant solution

$$u = \frac{-2\lambda}{x + c_1}$$

into equation (5.12.1) we obtain,

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t}\left[\frac{-2\lambda(1+\varepsilon t)}{x + c_1(1+\varepsilon t)}\right] \qquad (5.12.3)$$

Solution (iii)

Inserting invariant solution $u = k\tan\left[\dfrac{kx}{2\lambda t} + C_2\right]$ into equation (5.12.1) we obtain,

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t}\left[k\tan\left[\frac{kx}{2\lambda t} + C\right]\right] \qquad (5.12.4)$$

Solution (iv)

Inserting invariant solution $u = r\coth\left[C_3 + \dfrac{rx}{2\lambda}\right]$ into equation (5.12.1) we obtain,

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \left[\frac{1}{1+\varepsilon t}\right] r\coth\left[C_3 + \frac{rx}{2\lambda(1+\varepsilon t)}\right] \qquad (5.12.5)$$

Solution (v)

Inserting invariant solution $u = \frac{x}{t} - \frac{k}{t}\sqrt{2}\tanh\left[\frac{\sqrt{2}kx}{2\lambda t} + C_2\right]$ into equation (5.12.1) we obtain,

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}][\ \frac{x}{t} - \frac{k(1+\varepsilon t)}{t}\sqrt{2}\tanh\left[\frac{\sqrt{2}kx}{2\lambda t(1+\varepsilon t)} + C_2\right]\ ] \tag{5.12.6}$$

Solution (vi)

Inserting the invariant solution of the Burgers equation

$u = \frac{x}{t} - \frac{r}{t}\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}rx}{2\lambda\ t}\right]$ into equation (5.12.1) we obtain

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}][\ \frac{x}{t} - \frac{r(1+\varepsilon t)}{t}\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}rx}{2\lambda\ t}\right]\ ] \tag{5.12.7}$$

Solution (vii)

Inserting invariant solution $u = [\ \sqrt{t}\,C_3 - \left(2\sqrt{2\lambda}\right)^{-1}\sqrt{t\pi}\,e^{\frac{\alpha^2}{2}} erf(\alpha)\ ]^{-1}$ into equation (5.12.1) we obtain,

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}][\ \sqrt{\frac{t}{1+\varepsilon t}}\,C_3 - \left(2\sqrt{2\lambda}\right)^{-1}\sqrt{\frac{t\pi}{1+\varepsilon t}}\,e^{\frac{\bar{\alpha}^2}{2}} erf(\bar{\alpha})\ ]^{-1} \tag{5.12.8}$$

Solution (viii)

Inserting invariant solution $u = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4\lambda t}} \left[\ e^{-[\frac{4\lambda e^{-\frac{x^2}{4\lambda t}}}{(4\lambda t + x^2)\sqrt{t}}]} + K\right]$

into equation (5.12.1) we obtain

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}][\ \frac{1}{\sqrt{\frac{t}{1+\varepsilon t}}} e^{-\frac{x^2}{4\lambda(1+\varepsilon t)^2 t}}\ [\ e^{-[\frac{4\lambda(1+\varepsilon t)^2 e^{-\frac{x^2}{4\lambda t(1+t\varepsilon)}}}{(4\lambda(1+\varepsilon t)t + x^2)\sqrt{\frac{t}{1+\varepsilon t}}}]} + K\ ] \tag{5.12.9}$$

Solution (ix)

Inserting invariant solution $u(x,t) = x(t-c)^{-1}$

Into equation (5.12.1) we obtain,

$$u(x,t)=\frac{\varepsilon x}{1+\varepsilon t}+[\frac{1}{1+\varepsilon t}][x(t-c(1+\varepsilon t))^{-1}] \qquad (5.12.10)$$

Solution (x)

Inserting invariant solution $u=\dfrac{6\lambda}{6c\lambda-x^3}$ into equation (5.12.1) we obtain ,

$$u(x,t)=\frac{\varepsilon x}{1+\varepsilon t}+[\frac{1}{1+\varepsilon t}]\left[\frac{6\lambda(1+\varepsilon t)^3}{6c\lambda(1+\varepsilon t)^3-x^3}\right] \qquad (5.12.11)$$

Solution (xi)

Inserting invariant solution $u=\dfrac{x}{t}-\dfrac{c}{t}$ into equation (5.12.1) we obtain

$$u(x,t)=\frac{\varepsilon x}{1+\varepsilon t}+[\frac{1}{1+\varepsilon t}]\left[\frac{x-c(1+\varepsilon t)}{t}\right] \qquad (5.12.12)$$

Solution (xii)

Inserting invariant solution

$$u=\frac{Ck(x,t)erf\left(l(x,t)xe^{\frac{1}{t}}\right)+L}{C_3+F(x,t)}$$

into equation (5.12.1) we obtain,

$$u(x,t)=\frac{\varepsilon x}{1+\varepsilon t}+[\frac{1}{1+\varepsilon t}]\frac{C\bar{k}(x,t)erf\left(\bar{l}(x,t)\bar{x}e^{\frac{1}{\bar{t}}}\right)+L}{C_3+\bar{F}(x,t)} \qquad (5.12.13)$$

Solution (xiii)

Inserting invariant solution

$$u=\frac{K\sqrt{\frac{\pi}{\lambda}}erf\left(x\sqrt{\frac{\lambda}{4t}}\right)+L}{C_3+F_1(x,t)}$$

into equation (5.12.1) we obtain,

$$u(x,t)=\frac{\varepsilon x}{1+\varepsilon t}+[\frac{1}{1+\varepsilon t}]\frac{K\sqrt{\frac{\pi}{\lambda}}erf\left(\frac{x}{(1+\varepsilon t)}\sqrt{\frac{\lambda(1+\varepsilon t)}{4t}}\right)+L}{C_3+\bar{F}_1(x,t)} \qquad (5.12.14)$$

Solution (iv)

Inserting invariant solution $u=\dfrac{x}{t}-\dfrac{2\lambda}{x+tc_1}$

into equation (5.12.1) we obtain,

$$u = [\ \frac{x}{t} - \frac{2\lambda(1+\varepsilon t)}{x+tc_1}\ ] \tag{5.12.15}$$

5.13 General Symmetry Solutions of the Burgers Equation (1.1.0)

Consider the Lie groups G_1, G_2, G_3, G_4, G_5 admitted by equation (1.1.0)

$$v_1 = \frac{\partial}{\partial t};\ G_1 : X(x,t,u;\varepsilon) \to \qquad X_1(x,t+\varepsilon,u)$$

$$v_2 = \frac{\partial}{\partial x};\ G_2 : X(x,t,u;\varepsilon) \to \qquad X_2(x+\varepsilon,t,u)$$

$$v_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u};\ G_3 : X(x,t,u;\varepsilon) \to X_3(x+\varepsilon t, t, u+\varepsilon)$$

$$v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u};\ G_4 : X(x,t,u;\varepsilon) \to X_4(e^{\varepsilon}x, e^{2\varepsilon}t, e^{-\varepsilon}u)$$

$$v_5 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t}(x-tu)\frac{\partial}{\partial u};\ G_5 : X(x,t,u;\varepsilon) \to X_5(\frac{x}{1-\varepsilon t}, \frac{t}{1-\varepsilon t}, u(1-\varepsilon t)+\varepsilon x)$$

We transform solution (5.12.1) using G_3

Thus

$$u(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t}\Phi[\ \frac{x}{1+\varepsilon t}, \frac{t}{1+\varepsilon t}\ ],$$

is further mapped by G_3 into a new solution

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ] * \Phi[\ \frac{x - \varepsilon t - \varepsilon^2 t^2}{1+\varepsilon t}, \frac{t}{1+\varepsilon t}\ ] \tag{5.13.1}$$

where $u = \Phi[x,t]$ as a known solution of the Burgers equation (1.1.0).

Solution (i)

Consider the simple invariant solution of the Burgers equation. $u = c$

Substituting $u = K,\ say\ K = \lambda$ into (5.13.1) we obtain,

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon + \lambda}{1+\varepsilon t} \tag{5.13.2}$$

Solution (ii)

Inserting invariant solution

$$u = \frac{-2\lambda}{x + c_1}$$

into (5.13.1) we obtain,

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*[\ \frac{-2\lambda(1+\varepsilon t)}{x - \varepsilon^2 t^2 - \varepsilon t + c_1(1+\varepsilon t)}\ ] \qquad (5.13.3)$$

Solution (iii)

Inserting invariant solution $u = k\tan\left[\frac{kx}{2\lambda t} + C_2\right]$ into equation (5.13.1) we obtain,

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*\left[k\tan\left[\frac{k(x - \varepsilon^2 t^2 - \varepsilon t)}{2\lambda t} + C_2\right]\right] \qquad (5.13.4)$$

Solution (iv)

Inserting invariant solution $u = r\coth\left[C_3 + \frac{rx}{2\lambda}\right]$ into equation (5.13.1) we obtain,

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*\ r\coth\left[C_3 + \frac{r(x - \varepsilon^2 t^2 - \varepsilon t)}{2\lambda(1+\varepsilon t)}\right] \qquad (5.13.5)$$

Solution (v)

Inserting invariant solution $u = \frac{x}{t} - \frac{k}{t}\sqrt{2}\tanh\left[\frac{\sqrt{2}kx}{2\lambda t} + C_2\right]$ into (5.13.1) we obtain,

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*$$

$$[\ \frac{x - \varepsilon^2 t^2 - \varepsilon t - k(1+\varepsilon t)}{t}\sqrt{2}\tanh\left[\frac{\sqrt{2}k(x - \varepsilon^2 t^2 - \varepsilon t)}{2\lambda t} + C_2\right]\ ] \qquad (5.13.6)$$

Solution (vi)

Inserting the invariant solution of the Burgers equation

$$u = \frac{x}{t} - \frac{r}{t}\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}rx}{2\lambda\, t}\right]$$

into equation (5.13.1) we obtain

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*$$

$$[\ \frac{x-\varepsilon^2 t^2 - \varepsilon t - r(1+\varepsilon t)}{t}\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}r(x-\varepsilon^2 t^2 - \varepsilon t)}{2\lambda\, t}\right]\ ] \tag{5.13.7}$$

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*\ \Phi[\ \frac{x-\varepsilon t - \varepsilon^2 t^2}{1+\varepsilon t}, \frac{t}{1+\varepsilon t}\ ]$$

Solution (vii)

Inserting invariant solution $u = [\ \sqrt{t}\, C_3 - \left(2\sqrt{2\lambda}\right)^{-1}\sqrt{t\pi}\, e^{\frac{\alpha^2}{2}} erf(\alpha)\ ]^{-1}$ into equation (5.13.1) we obtain,

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*[\ \sqrt{\frac{t}{1+\varepsilon t}}C_3 - \left(2\sqrt{2\lambda}\right)^{-1}\sqrt{\frac{t\pi}{1+\varepsilon t}}e^{\frac{\overline{\alpha}^2}{2}} erf(\overline{\alpha})\ ]^{-1} \tag{5.13.8}$$

Solution (viii)

Inserting invariant solution $u = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4\lambda t}} [\ e^{-[\ \frac{4\lambda e^{-\frac{x^2}{4\lambda t}}}{(4\lambda t + x^2)\sqrt{t}}\ ]} + K]$

into equation (5.13.1) we obtain

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*$$

$$[\ \frac{1}{\sqrt{\frac{t}{1+\varepsilon t}}} e^{-\frac{\hat{x}^2}{4\lambda(1+\varepsilon t)^2 t}}\ [\ e^{-[\ \frac{4\lambda(1+\varepsilon t)^2 e^{-\frac{\hat{x}^2}{4\lambda t(1+t\varepsilon)}}}{(4\lambda(1+\varepsilon t)t + \hat{x}^2)\sqrt{\frac{t}{1+\varepsilon t}}}\ ]} + K\ ] \tag{5.13.9}$$

where

$\hat{x} = x - \varepsilon^2 t^2 - \varepsilon t$

Solution (ix)

Inserting invariant solution $u(x,t) = x(t-c)^{-1}$

into equation (5.13.1) we obtain,

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*[(x-\varepsilon^2 t^2 - \varepsilon t)(t - c(1+\varepsilon t))^{-1}] \tag{5.13.10}$$

Solution (x)

Inserting invariant solution $u=\dfrac{6\lambda}{6c\lambda-x^3}$ into equation (5.13.1) we obtain ,

$$u(x,t)=\frac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\ \frac{1}{1+\varepsilon t}\ ]*[\ \frac{6\lambda(1+\varepsilon t)^3}{6c\lambda(1+\varepsilon t)^3-(x-\varepsilon^2t^2-\varepsilon t)^3}\ ] \qquad (5.13.11)$$

Solution (xi)

Inserting invariant solution $u=\dfrac{x}{t}-\dfrac{c}{t}$ into equation (5.13.1) we obtain

$$u(x,t)=\frac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\ \frac{1}{1+\varepsilon t}\ ]*[\ \frac{x-\varepsilon^2t^2-\varepsilon t-c(1+\varepsilon t)}{t}\ ] \qquad (5.13.12)$$

Solution (xii)

Inserting invariant solution

$$u=\frac{Ck(x,t)erf\left(l(x,t)xe^{\frac{1}{t}}\right)+L}{C_3\ +F(x,t)}$$

into equation (5.13.1) we obtain,

$$u(x,t)=\frac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\ \frac{1}{1+\varepsilon t}\ ]*\frac{Ck(\hat{x},t)erf\left(l(\hat{x},t)\hat{x}e^{\frac{1}{t}}\right)+L}{C_3\ +F(\hat{x},t)} \qquad (5.13.13)$$

where

$\hat{x}=x-\varepsilon^2t^2-\varepsilon t$

Solution (xiii)

Inserting invariant solution

$$u=\frac{K\sqrt{\frac{\pi}{\lambda}}erf\left(x\sqrt{\frac{\lambda}{4t}}\right)+L}{C_3\ +F_1(x,t)}$$

into equation (5.13.1) we obtain,

$$u(x,t)=\frac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\ \frac{1}{1+\varepsilon t}\ ]*\frac{K\sqrt{\frac{\pi}{\lambda}}erf\left(\frac{\hat{x}}{(1+\varepsilon t)}\sqrt{\frac{\lambda(1+\varepsilon t)}{4t}}\right)+L}{C_3\ +F_1(\hat{x},t)} \qquad (5.13.14)$$

Solution (xiv)

Inserting invariant solution $u = \dfrac{x}{t} - \dfrac{2\lambda}{x + tc_1}$

into equation (5.13.1) we obtain,

$$u(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ]*[\ \frac{x - \varepsilon^2 t^2 - \varepsilon t}{t} - \frac{2\lambda(1+\varepsilon t)}{x - \varepsilon^2 t^2 - \varepsilon t + tc_1}\ ] \qquad (5.13.15)$$

5.14 Global Symmetry Solutions of the Burgers Equation (1.1.0)

Consider the Lie groups G_1, G_2, G_3, G_4, G_5 admitted by equation (1.1.0)

$$v_1 = \frac{\partial}{\partial t};\ G_1 : X(x,t,u;\varepsilon) \rightarrow \qquad X_1(x, t+\varepsilon, u)$$

$$v_2 = \frac{\partial}{\partial x};G_2 : X(x,t,u;\varepsilon) \rightarrow \qquad X_2(x+\varepsilon, t, u)$$

$$v_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u};G_3 : X(x,t,u;\varepsilon) \rightarrow X_3(x+\varepsilon t, t,\ u+\varepsilon)$$

$$v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u};\ G_4 : X(x,t,u;\varepsilon) \rightarrow X_4(e^{\varepsilon}x,\ e^{2\varepsilon}t, e^{-\varepsilon}u)$$

$$v_5 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t}\ (x - tu)\frac{\partial}{\partial u};\ G_5 : X(x,t,u;\varepsilon) \rightarrow X_5(\frac{x}{1-\varepsilon t}, \frac{t}{1-\varepsilon t}, u(1-\varepsilon t) + \varepsilon x)$$

$$v_\alpha = \alpha(x,t)\frac{\partial}{\partial u}\ :\ G_\alpha : X(x,t,u;\varepsilon) \rightarrow \qquad X_\alpha(x, t, u + \varepsilon\alpha(x,t))\ ;$$

By symmetry group inversion theory of section (3.10) of chapter 3 , if each G_i is a symmetry group and $u = \Phi(x,t)$ is a known solution of the Burgers equation (1.1.0), then the functions $\hat{u}_j$ below are also solutions of the Burgers equation (1.1.0), see Olver [18].

By applying the new symmetry solution inversion formula;

$$\hat{u} = \Psi\left(\Phi\left(g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t})\right), g_\varepsilon^{-1}(\bar{x}), g_\varepsilon^{-1}(\bar{t}), \varepsilon\right)$$

on each group G_i we obtain the new symmetry solutions

$$\hat{u}_1 = \Phi(x, t-\varepsilon)$$

$$\hat{u}_2 = \Phi(x-\varepsilon, t)$$

$$\hat{u}_3 = \Phi(x-\varepsilon t, t) - \varepsilon$$

$$\hat{u}_4 = e^{-\varepsilon} \; \Phi(e^{-\varepsilon} x, e^{-2\varepsilon} t)$$

$$\hat{u}_5 = \frac{\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t} \Phi \left[\frac{x}{1+\varepsilon t}, \frac{t}{1+\varepsilon t} \right]$$

$$\hat{u}_\alpha = \Phi(x,t) + \varepsilon\alpha(x,t)$$

The most general one-parameter group of symmetries is obtained by considering a general linear combination

$V_g = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + c_5 v_5 + c_\alpha v_\alpha$ of the given vector fields. We may represent an arbitrary group transformation g as the composition of the transformations in the various one-parameter subgroups G_1, G_2, G_3, G_4, G_5, G_α. In particular if g is near the identity, it can be represented uniquely in the exponential form

$$g = \exp(\varepsilon_\alpha V_\alpha) * \exp(\varepsilon_5 V_5) * \exp(\varepsilon_4 V_4) * \exp(\varepsilon_3 V_3) * \exp(\varepsilon_2 V_2) * \exp(\varepsilon_1 V_1).$$

Thus the most general solution u_g, .i.e. **global** solution of equation (1.1.0) is obtained by

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + \left[\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t} \right] \times \Phi \left[\frac{x e^{-\varepsilon_4} - (\varepsilon_2\varepsilon_5 + \varepsilon_3)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2}{1+\varepsilon_5 t}, \frac{t e^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}{1+\varepsilon_5 t} \right] \quad (5.14.1)$$

where $u = \Phi[x,t]$ as a known solution of the Burgers equation (1.1.0).

Solution (i)

Consider the simple invariant solution of the Burgers equation. $u = c$

Substituting $u = K,\ say\ K = \lambda$ into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3 + \lambda e^{-\varepsilon_4}}{1+\varepsilon_5 t} \quad (5.14.2)$$

Solution (ii)

Inserting invariant solution

$$u = \frac{-2\lambda}{x + c_1}$$

into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]\times$$

$$[\frac{-2\lambda(1+\varepsilon_5 t)}{xe^{-\varepsilon_4} - (\varepsilon_2\varepsilon_5 + \varepsilon_3)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2 + c_1(1+\varepsilon_5 t)}] \quad (5.14.3)$$

Solution (iii)

Inserting invariant solution $u = k\tan\left[\frac{kx}{2\lambda t} + C_2\right]$ into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * \left[k\tan\left[\frac{k(xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2)}{2\lambda(te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1)} + C\right]\right] \quad (5.14.4)$$

Solution (iv)

Inserting invariant solution $u = r\coth\left[C_3 + \frac{rx}{2\lambda}\right]$ into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + \frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t} * r\coth\left[C_3 + \frac{r(xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2)}{2\lambda(1+\varepsilon_5 t)}\right] \quad (5.14.5)$$

Solution (v)

Inserting invariant solution $u = \frac{x}{t} - \frac{k}{t}\sqrt{2}\tanh\left[\frac{\sqrt{2}kx}{2\lambda t} + C_2\right]$ into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] *$$

$$[\frac{xe^{-\varepsilon_4} - (\varepsilon_2\varepsilon_5 + \varepsilon_3)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2}{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1} -$$

$$\frac{k(1+\varepsilon_5 t)}{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}\sqrt{2}\tanh\left[\frac{\sqrt{2}k(xe^{-\varepsilon_4} - (\varepsilon_2\varepsilon_5 + \varepsilon_3)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2)}{2\lambda(te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1)} + C_2\right]] \quad (5.14.6)$$

Solution (vi)

Inserting the invariant solution of the Burgers equation

$u = \frac{x}{t} - \frac{r}{t}\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}rx}{2\lambda\, t}\right]$ into equation (5.14.1) we obtain

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]\times$$

$$[[\ \frac{xe^{-\varepsilon_4} - (\varepsilon_2\varepsilon_5 + \varepsilon_3)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2 - r(1+\varepsilon_5 t)}{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}\ ]\times$$

$$\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}r(xe^{-\varepsilon_4} - (\varepsilon_2\varepsilon_5 + \varepsilon_3)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2)}{2\lambda(te^{-2e_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1)}\right]\ ] \quad (5.14.7)$$

Solution (vii)

Inserting invariant solution $u = [\ \sqrt{t}\,C_3 - \left(2\sqrt{2\lambda}\right)^{-1}\sqrt{t\pi}\,e^{\frac{\alpha^2}{2}} erf(\alpha)\ ]^{-1}$ into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]\times$$

$$[\ \sqrt{\frac{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}{1+\varepsilon_5 t}}\,C_3 - \left(2\sqrt{2\lambda}\right)^{-1}\sqrt{\frac{(te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1)\pi}{1+\varepsilon t}}\,e^{\frac{\overline{\alpha}^2}{2}} erf(\overline{\alpha})\ ]^{-1} \quad (5.14.8)$$

where $\overline{\alpha}(x,t) = \alpha(\hat{x},\hat{t})$

and

$$\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$$

$$\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$$

Solution (viii)

Inserting invariant solution $u = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4\lambda t}}\ [\ e^{-[\frac{4\lambda e^{-\frac{x^2}{4\lambda t}}}{(4\lambda t + x^2)\sqrt{t}}]} + K]$

into equation (5.14.1) we obtain

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]*$$

$$[\ \frac{\sqrt{1+\varepsilon_5 t}}{\sqrt{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}} e^{-\frac{(\hat{x})^2}{4\lambda(1+\varepsilon_4 t)^2 \hat{t}}} * [\ e^{-[\frac{4\lambda(1+\varepsilon_4 t)^2 e^{-\frac{\hat{x}^2}{4\lambda t(1+t\varepsilon)}}}{(4\lambda(1+\varepsilon_4 t)t + \hat{x}^2)\sqrt{\frac{\hat{t}}{1+\varepsilon_4 t}}}]} + K\ ] \quad (5.14.9)$$

where

$$\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$$

$$\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$$

Solution (ix)

Inserting invariant solution $u(x,t) = x(t-c)^{-1}$

into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * [\hat{x}(\hat{t} - c(1+\varepsilon_5 t))^{-1}] \qquad (5.14.10)$$

where

$$\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$$

$$\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$$

Solution (x)

Inserting invariant solution $u = \frac{6\lambda}{6c\lambda - x^3}$ into equation (5.14.1) we obtain ,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * [\frac{6\lambda(1+\varepsilon_5 t)^3}{6c\lambda(1+\varepsilon_5 t)^3 - \hat{x}^3}] \qquad (5.14.11)$$

where

$$\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$$

Solution (xi)

Inserting invariant solution $u = \frac{x}{t} - \frac{c}{t}$ into equation (5.14.1) we obtain

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * [\frac{\hat{x} - c(1+\varepsilon_5 t)}{\hat{t}}] \qquad (5.14.12)$$

where

$$\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$$

$$\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$$

Solution (xii)

Inserting invariant solution

$$u = \frac{Ck(x,t)erf\left(l(x,t)xe^{\frac{1}{t}}\right) + L}{C_3 + F(x,t)}$$

into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * \frac{Ck(\hat{x},\hat{t}) erf\left(l(\hat{x},\hat{t})\hat{x}e^{\frac{1}{\hat{t}}} \right) + L}{C_3 + F(\hat{x},\hat{t})} \quad (5.14.13)$$

where

$$\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$$

$$\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$$

Solution (xiii)

Inserting invariant solution

$$u = \frac{K\sqrt{\frac{\pi}{\lambda}} erf\left(x\sqrt{\frac{\lambda}{4t}} \right) + L}{C_3 + F_1(x,t)}$$ into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * \frac{K\sqrt{\frac{\pi}{\lambda}} erf\left(\frac{\hat{x}}{(1+\varepsilon_5 t)}\sqrt{\frac{\lambda(1+\varepsilon_5 t)}{4\hat{t}}} \right) + L}{C_3 + F_1(\hat{x},\hat{t})} \quad (5.14.14)$$

where

$$\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$$

$$\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$$

Solution (xiv)

Inserting invariant solution $u = \frac{x}{t} - \frac{2\lambda}{x + tc_1}$

into equation (5.14.1) we obtain,

$$u_g = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * \left[\frac{\hat{x}}{\hat{t}} - \frac{2\lambda(1+\varepsilon_5 t)}{\hat{x} + \hat{t}c_1} \right] \quad (5.14.15)$$

where

$$\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$$

$$\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$$

CHAPTER 6

RESULTS

6.1 Tabulation of The Solution of The Burgers Equation Results

Invariant and symmetry solutions of the Burgers Equation are given in the tables below. Each generator V_i has the corresponding solution $u_i(x,t)$.

Generators of the generalized heat equation (5.1.1):

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial t}, v_3 = u\frac{\partial}{\partial u}, \quad v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad v_5 = 2t\frac{\partial}{\partial x} - \frac{x}{\lambda}u\frac{\partial}{\partial u},$$

$$v_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - [2ut + x^2\frac{u}{\lambda}]\frac{\partial}{\partial u},,$$

Generators of the Burgers equation (1.1.0):

$$v_1 = \frac{\partial}{\partial t}, \quad v_2 = \frac{\partial}{\partial x}, v_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad v_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

$$v_5 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - [\ ut - x\ ]\frac{\partial}{\partial u}, v_\alpha = \omega\frac{\partial}{\partial u}:\ \omega_{\ t} = \lambda\,\omega_{xx}$$

6.2 INVARIANT SOLUTIONS OF THE BURGERS EQUATION

From section 5.6 we see that the invariant solutions of the Burgers equation are:

Generator (V_i) ↓	Invariant solutions (u) ↓
V_1	$u_1(x,t) = \dfrac{2\lambda}{x + C_3}$, (6.6.1) $u_1(x,t) = k\tan\dfrac{kx + C_1}{2\lambda}$, (6.6.2) $u_1(x.t) = -s\coth\dfrac{sx + C_1}{2\lambda}$ (6.6.3)
V_2	$u_2(x,t) = C$ (6.6.4)
V_3	$u_3(x,t) = \dfrac{x}{t} - \dfrac{c}{t}$ (6.6.5)
V_4	$u_4(x,t) = [\ \sqrt{t}\,C_3 - \left(2\sqrt{2\lambda}\right)^{-1}\sqrt{t\pi}\,e^{\frac{\alpha^2}{2}}erf(\alpha)\ ]^{-1}$ (6.6.6)

V_5	$u_5(x,t)=\frac{x}{t}-\frac{2\lambda}{tx+t^2c_1}$,	(6.6.7)
	$u_5(x,t)=\frac{x}{t}-\frac{k}{t}\sqrt{2}\tanh\left[\frac{\sqrt{2}kx}{2\lambda t}+C_2\right]$,	(6.6.8)
	$u_5(x,t)=\frac{x}{t}-\frac{r}{t}\sqrt{2}\tan\left[C_3-\frac{\sqrt{2}rx}{2\lambda t}\right]$	(6.6.9)
V_{w1}	$u_{w1}(x,t)=\frac{-2\lambda}{x+c_1}$,	(6.6.10)
	$u_{w1}(x,t)=k\tan\left[\frac{kx}{2\lambda t}+C_2\right]$,	(6.6.11)
	$u_{w1}(x,t)=r\coth\left[C_3+\frac{rx}{2\lambda t}\right]$	(6.6.12)
$V_{w2/3}$	$u_{w2}(x,t)=x(t-c)^{-1}$,	(6.6.13)
	$u_{w3}(x,t)=\frac{6\lambda}{6c\lambda-x^3}$	(6.6.14)
V_{w4}	$u_{w4}(x,t)=\frac{K\sqrt{\frac{\pi}{\lambda}}erf\left(x\sqrt{\frac{\lambda}{4t}}\right)+L}{C_3\ \ +F_1(x,t)}$:,.	(6.6.15)
V_{w5}	$u_{w5}(x,t)=\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4\lambda t}}\left[e^{-\left[\frac{4\lambda e^{-\frac{x^2}{4\lambda t}}}{(4\lambda t=x^2)\sqrt{t}}\right]}+K\right]$	(6.6.16)
V_{w6}	$u_{w6}(x,t)=\frac{Ck(x,t)erf\left(l(x,t)xe^{\frac{1}{t}}\right)+L}{C_3\ \ +F(x,t)}$:,.	(6.6.17)

Table 6.1 INVARIANT SOLUTIONS OF THE BURGERS EQUATION

6.3 SYMMETRY SOLUTIONS OF THE BURGERS EQUATION

From section 5.7, we can extract the symmetry solutions of the Burgers equation to obtain :

Generator (V_i) ↓	Symmetry solutions (u) ↓
V_1	$u_1(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t}\left[\frac{-2\lambda(1+\varepsilon t)}{x+c_1(1+\varepsilon t)}\right]$ (6.7.1)
V_1	$u_1(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \frac{1}{1+\varepsilon t}\left[k\tan\left[\frac{kx}{2\lambda t}+C\right]\right]$ (6.7.2)
V_1	$u_1(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}]\, r\coth\left[C_3 + \frac{rx}{2\lambda(1+\varepsilon t)}\right]$ (6.7.3)
V_2	$u_2(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + \lambda[\frac{1}{1+\varepsilon t}]$ (6.7.4)
V_3	$u_3(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}]\left[\frac{x-c(1+\varepsilon t)}{t}\right]$ (6.7.5)
V_4	$u_4(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}][\sqrt{\frac{t}{1+\varepsilon t}}C_3 - \left(2\sqrt{2\lambda}\right)^{-1}\sqrt{\frac{t\pi}{1+\varepsilon t}}e^{\frac{\bar{\alpha}^2}{2}} erf(\bar{\alpha})]^{-1}$ (6.7.6)
V_5	$u_5(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}][\frac{x}{t} - \frac{k(1+\varepsilon t)}{t}\sqrt{2}\tanh\left[\frac{\sqrt{2}kx}{2\lambda t(1+\varepsilon t)}+C_2\right]]$ (6.7.7)**
V_5	$u_5(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}]\left[\frac{x}{t} - \frac{2\lambda(1+\varepsilon t)}{x+tc_1}\right]$ $\left[\frac{x}{t} - \frac{r(1+\varepsilon t)}{t}\sqrt{2}\tan\left[C_3 - \frac{\sqrt{2}rx}{2\lambda t}\right]\right]$ (6.7.8)**
V_5	$u_5(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}]$ (6.7.9)**
V_{w2}	$u_{w2}(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}][x(t-c(1+\varepsilon t))^{-1}]$ (6.7.10)
V_{w3}	$u_{w3}(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}]\left[\frac{6\lambda(1+\varepsilon t)^3}{6c\lambda(1+\varepsilon t)^3 - x^3}\right]$ (6.7.11)
V_{w4}	$u_{w4}(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}]\frac{K\sqrt{\frac{\pi}{\lambda}}erf\left(\frac{x}{(1+\varepsilon t)}\sqrt{\frac{\lambda(1+\varepsilon t)}{4t}}\right)+L}{C_3 + \overline{F}_1(x,t)}$ (6.7.12)

V_{w5}	$u_{w5}(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}][\ \frac{1}{\sqrt{\frac{t}{1+\varepsilon t}}} e^{-\frac{x^2}{4\lambda(1+\varepsilon t)^2 t}} *$ $[\ e^{-[\frac{4\lambda(1+\varepsilon t)^2 e^{-\frac{x^2}{4\lambda t(1+t\varepsilon)}}}{(4\lambda(1+\varepsilon t)t + x^2)\sqrt{\frac{t}{1+\varepsilon t}}}]} + K \quad]\ ]$ (6.7.13)
V_{w6}	$u_{w6}(x,t) = \frac{\varepsilon x}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}] \frac{C\bar{k}(x,t) erf\left(\bar{l}(x,t)\bar{x} e^{\frac{1}{\bar{i}}}\right) + L}{C_3 + \bar{F}(x,t)}$ (6.7.14)

Table 6.2 SYMMETRY SOLUTIONS OF THE BURGERS EQUATION

Remark. The solutions marked **, are regarded as non pure symmetry solutions of the Burgers equation since these solutions are generated from invariant solutions corresponding to generator V_5 and are merely transformed by the same transformation G_5.
This concept may be generalized.

6.4 GENERAL SYMMETRY SOLUTIONS OF THE BURGERS EQUATION

From section 5.8, we can extract the general symmetry solutions of the Burgers equation to get :

Generator (V_i) ↓	Symmetry solutions (u) ↓
V_1	$u_1(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ] * [\ \frac{-2\lambda(1+\varepsilon t)}{x - \varepsilon^2 t^2 - \varepsilon t + c_1(1+\varepsilon t)}\ ]$ (6.8.1)
V_1	$u_1(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ] * \left[k \tan\left[\frac{k(x - \varepsilon^2 t^2 - \varepsilon t)}{2\lambda t} + C_2\right]\right]$ (6.8.2)
V_1	$u_1(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ] * r \coth\left[C_3 + \frac{r(x - \varepsilon^2 t^2 - \varepsilon t)}{2\lambda(1+\varepsilon t)}\right]$ (6.8.3)
V_2	$u_2(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon + \lambda}{1+\varepsilon t}$ (6.8.4)
V_3	$u_3(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\ \frac{1}{1+\varepsilon t}\ ] * [(x - \varepsilon^2 t^2 - \varepsilon t)(t - c(1+\varepsilon t))^{-1}]$ (6.8.5)

V_4	$u_4(x,t)=\dfrac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\dfrac{1}{1+\varepsilon t}]*$ $[\sqrt{\dfrac{t}{1+\varepsilon t}}C_3-\left(2\sqrt{2\lambda}\right)^{-1}\sqrt{\dfrac{t\pi}{1+\varepsilon t}}e^{\frac{\bar{\alpha}^2}{2}}erf(\bar{\alpha})]^{-1}$	(6.8.6)
V_5	$u_5(x,t)=\dfrac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\dfrac{1}{1+\varepsilon t}]*$ $[\dfrac{x-\varepsilon^2t^2-\varepsilon t-k(1+\varepsilon t)}{t}\sqrt{2}\tanh\left[\dfrac{\sqrt{2}k(x-\varepsilon^2t^2-\varepsilon t)}{2\lambda t}+C_2\right]]$	(6.8.7)
V_5	$u_5(x,t)=\dfrac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\dfrac{1}{1+\varepsilon t}]*$ $[\dfrac{x-\varepsilon^2t^2-\varepsilon t-r(1+\varepsilon t)}{t}\sqrt{2}\tan\left[C_3-\dfrac{\sqrt{2}r(x-\varepsilon^2t^2-\varepsilon t)}{2\lambda t}\right]]$	(6.8.8)
V_5	$u_5(x,t)=\dfrac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\dfrac{1}{1+\varepsilon t}]*[\dfrac{x-\varepsilon^2t^2-\varepsilon t}{t}-\dfrac{2\lambda(1+\varepsilon t)}{x-\varepsilon^2t^2-\varepsilon t+tc_1}]$	(6.8.9)
V_{w2}	$u_{w2}(x,t)=\dfrac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\dfrac{1}{1+\varepsilon t}]*[\dfrac{x-\varepsilon^2t^2-\varepsilon t-c(1+\varepsilon t)}{t}]$	(6.8.10)
V_{w3}	$u_{w3}(x,t)=\dfrac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\dfrac{1}{1+\varepsilon t}]*$ $[\dfrac{6\lambda(1+\varepsilon t)^3}{6c\lambda(1+\varepsilon t)^3-(x-\varepsilon^2t^2-\varepsilon t)^3}]$	(6.8.11)
V_{w4}	$u_{w4}(x,t)=\dfrac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+\dfrac{K\sqrt{\dfrac{\pi}{\lambda}}erf\left(\dfrac{\hat{x}}{(1+\varepsilon t)}\sqrt{\dfrac{\lambda(1+\varepsilon t)}{4t}}\right)+L}{(1+\varepsilon t)(C_3+F_1(\hat{x},t))}$	(6.8.12)
V_{w5}	$u_{w5}(x,t)=\dfrac{\varepsilon x-2\varepsilon^2 t-\varepsilon}{1+\varepsilon t}+[\dfrac{1}{1+\varepsilon t}]*$	

	$[\frac{1}{\sqrt{\frac{t}{1+\varepsilon t}}} e^{-\frac{\hat{x}^2}{4\lambda(1+\varepsilon t)^2 t}} [e^{-[\frac{4\lambda(1+\varepsilon t)^2 e^{-\frac{\hat{x}^2}{4\lambda t(1+t\varepsilon)}}}{(4\lambda(1+\varepsilon t)t+\hat{x}^2)\sqrt{\frac{t}{1+\varepsilon t}}}]} + K]$ where $\hat{x} = x - \varepsilon^2 t^2 - \varepsilon t$	(6.8.13)
V_{w6}	$u_{w6}(x,t) = \frac{\varepsilon x - 2\varepsilon^2 t - \varepsilon}{1+\varepsilon t} + [\frac{1}{1+\varepsilon t}] * \frac{Ck(\hat{x},t) erf\left(l(\hat{x},t)\hat{x} e^{\frac{1}{t}} \right) + L}{C_3 + F(\hat{x},t)}$ where $\hat{x} = x - \varepsilon^2 t^2 - \varepsilon t$	(6.8.14)

Table 6.3 GENERAL SYMMETRY SOLUTIONS OF THE BURGERS EQUATION

6.5 GLOBAL SYMMETRY SOLUTION OF THE BURGERS EQUATION

From section 5.9, we can extract the global symmetry solutions of the Burgers equation to get :

Generator (V_i) ↓	Symmetry solutions (u) ↓	
V_1	$u_{g1}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] *$ $[\frac{-2\lambda(1+\varepsilon_5 t)}{xe^{-\varepsilon_4} - (\varepsilon_2\varepsilon_5 + \varepsilon_3)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2 + c_1(1+\varepsilon_5 t)}]$	(6.9.1)
V_1	$u_{g1}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] *$ $\left[k \tan\left[\frac{k(xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2)}{2\lambda(te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1)} + C \right] \right]$	(6.9.2)
V_1	$u_{g1}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + (\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}) *$ $r \coth\left[C_3 + \frac{r(xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2)}{2\lambda(1+\varepsilon_5 t)} \right]$	(6.9.3)
V_2	$u_{g2}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3 \varepsilon_5 t - \varepsilon_3 + \lambda e^{-\varepsilon_4}}{1+\varepsilon_5 t}$	(6.9.4)

V_3	$u_{g3}(x,t)=\dfrac{\varepsilon_5 x-\varepsilon_3\varepsilon_5 t-\varepsilon_3}{1+\varepsilon_5 t}+[\dfrac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]*[\hat{x}\left(\hat{t}-c(1+\varepsilon_5 t)\right)^{-1}]$ (6.9.5)
V_4	$u_{g4}(x,t)=\dfrac{\varepsilon_5 x-\varepsilon_3\varepsilon_5 t-\varepsilon_3}{1+\varepsilon_5 t}+[\dfrac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]*$ $[\sqrt{\dfrac{te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1}{1+\varepsilon_5 t}}C_3-\left(2\sqrt{2\lambda}\right)^{-1}\sqrt{\dfrac{(te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1)\pi}{1+\varepsilon t}}e^{\frac{\overline{\alpha}^2}{2}}erf(\overline{\alpha})]^{-1}$ (6.9.6) where $\overline{\alpha}(x,t)=\alpha(\hat{x},\hat{t})$, $\hat{x}=xe^{-\varepsilon_4}-(\varepsilon_3+\varepsilon_2\varepsilon_5)t-\varepsilon_3\varepsilon_5 t^2-\varepsilon_2$ $\hat{t}=te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1$
V_5	$u_{g5}(x,t)=\dfrac{\varepsilon_5 x-\varepsilon_3\varepsilon_5 t-\varepsilon_3}{1+\varepsilon_5 t}+[\dfrac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]*$ $[\dfrac{xe^{-\varepsilon_4}-(\varepsilon_2\varepsilon_5+\varepsilon_3)t-\varepsilon_3\varepsilon_5 t^2-\varepsilon_2}{te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1}-$ $\dfrac{k(1+\varepsilon_5 t)}{te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1}\sqrt{2}\tanh\left[\dfrac{\sqrt{2}k(xe^{-\varepsilon_4}-(\varepsilon_2\varepsilon_5+\varepsilon_3)t-\varepsilon_3\varepsilon_5 t^2-\varepsilon_2)}{2\lambda(te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1)}+C_2\right]]$ (6.9.7)
V_5	$u_{g5}(x,t)=\dfrac{\varepsilon_5 x-\varepsilon_3\varepsilon_5 t-\varepsilon_3}{1+\varepsilon_5 t}+[\dfrac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]*$ $[\dfrac{xe^{-\varepsilon_4}-(\varepsilon_2\varepsilon_5+\varepsilon_3)t-\varepsilon_3\varepsilon_5 t^2-\varepsilon_2-r(1+\varepsilon_5 t)}{te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1}*$ $\sqrt{2}\tan\left[C_3-\dfrac{\sqrt{2}r(xe^{-\varepsilon_4}-(\varepsilon_2\varepsilon_5+\varepsilon_3)t-\varepsilon_3\varepsilon_5 t^2-\varepsilon_2)}{2\lambda(te^{-2e_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1)}\right]]$ (6.9.8)
V_5	$u_{g5}(x,t)=\dfrac{\varepsilon_5 x-\varepsilon_3\varepsilon_5 t-\varepsilon_3}{1+\varepsilon_5 t}+[\dfrac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]*[\dfrac{\hat{x}}{\hat{t}}-\dfrac{2\lambda(1+\varepsilon_5 t)}{\hat{x}+\hat{t}c_1}]$ (6.9.9) where $\hat{x}=xe^{-\varepsilon_4}-(\varepsilon_3+\varepsilon_2\varepsilon_5)t-\varepsilon_3\varepsilon_5 t^2-\varepsilon_2$ $\hat{t}=te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1$
V_{w2}	$u_g=\dfrac{\varepsilon_5 x-\varepsilon_3\varepsilon_5 t-\varepsilon_3}{1+\varepsilon_5 t}+[\dfrac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}]*[\dfrac{\hat{x}-c(1+\varepsilon_5 t)}{\hat{t}}]$ (6.9.10) where $\hat{x}=xe^{-\varepsilon_4}-(\varepsilon_3+\varepsilon_2\varepsilon_5)t-\varepsilon_3\varepsilon_5 t^2-\varepsilon_2$ $\hat{t}=te^{-2\varepsilon_4}-\varepsilon_1\varepsilon_5 t-\varepsilon_1$

V_{w3}	$u_{gw3}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * [\frac{6\lambda(1+\varepsilon_5 t)^3}{6c\lambda(1+\varepsilon_5 t)^3 - \hat{x}^3}]$ (6.9.11) where, $\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$
V_{w4}	$u_{gw4}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + \frac{e^{-\varepsilon_4} K\sqrt{\frac{\pi}{\lambda}} erf\left(\frac{\hat{x}}{(1+\varepsilon_5 t)}\sqrt{\frac{\lambda(1+\varepsilon_5 t)}{4\hat{t}}}\right) + L}{(1+\varepsilon_5 T)C_3 + F_1(\hat{x},\hat{t})}$ (6.9.12) where, $\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$ $\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$
V_{w5}	$u_{gw5}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] *$ $[\frac{\sqrt{1+\varepsilon_5 t}}{\sqrt{te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1}} e^{-\frac{(\hat{x})^2}{4\lambda(1+\varepsilon_4 t)^2 \hat{t}}} * [e^{-[\frac{4\lambda(1+\varepsilon_4 t)^2 e^{-\frac{\hat{x}^2}{4\lambda t(1+t\varepsilon)}}}{(4\lambda(1+\varepsilon_4 t)t + \hat{x}^2)\sqrt{\frac{\hat{t}}{1+\varepsilon_4 t}}}]} + K]$ (6.9.13) where, $\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$ $\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$
V_{w6}	$u_{gw6}(x,t) = \frac{\varepsilon_5 x - \varepsilon_3\varepsilon_5 t - \varepsilon_3}{1+\varepsilon_5 t} + [\frac{e^{-\varepsilon_4}}{1+\varepsilon_5 t}] * \frac{Ck(\hat{x},\hat{t}) erf\left(l(\hat{x},\hat{t})\hat{x}e^{\frac{1}{\hat{t}}}\right) + L}{C_3 + F(\hat{x},\hat{t})}$ (6.9.14) where, $\hat{x} = xe^{-\varepsilon_4} - (\varepsilon_3 + \varepsilon_2\varepsilon_5)t - \varepsilon_3\varepsilon_5 t^2 - \varepsilon_2$ $\hat{t} = te^{-2\varepsilon_4} - \varepsilon_1\varepsilon_5 t - \varepsilon_1$

Table 6.4 GLOBAL SYMMETRY SOLUTIONS OF THE BURGERS EQUATION

6.6 CONCLUSIONS

In this thesis , we have managed to find global solutions of the Burgers Equation using the Lie symmetry approach. The solutions (6.9.1 - 6.9.14) are appearing here in literature for the first time. Other previous attempts to solve this very important equation only managed to find solutions when the coefficient λ is restricted to $\lambda \in [0,1]$. These solutions have been presented in terms of the infinitesimal generators V_i $(i = 1,2.,.......,5,\alpha)$ and their corresponding symmetry solutions $u_i(x,t)$, $u_{gij}(x,t)$ $(i = 1,2,......., j = 1,2,..........)$.

We have verified the validity of these global solutions taking cases where $\lambda = 1$.

In this case the global symmetry solutions obtained compare accurately with some of the solutions obtained by Gandarias [6] for specific values of the arbitrary constants and real parameters ε , ε_j with value of $\lambda = 1$.

In particular

(i) solution (6.9.1) reduces to $u = \dfrac{-2}{x+\beta}$,

(ii) solution (6.9.2) reduces $u = -2\cot x$,

(iii) solution (6.9.11) reduces to $u = \dfrac{x - x^3 + 6(1+t)^3}{(1+t)^3 - x^3}$,

which are in agreement with Gandarias [6] solutions of the form

$$u_* = -\frac{2(-2k_4 \cos x - 2k_1 e^t x - 2k_2 e^t)}{-2k_4 \sin x - k_1 e^t x^2 - 2k_2 e^t x - 2(2k_1 t + 2k)e^t} ,$$

$$u_{**} = -\frac{2(12k_2 x^3 + 2(36k_2 t - 12k_3)x + 8k_1)}{3k_2 x^4 + (36k_2 t - 12k_3)x^2 + 8k_1 x + 36k_2 t^2 - 24k_3 t + 24k_4} .$$

The solutions obtained is a contribution to knowledge in mathematics.

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