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On the Norm of a Generalized Derivation

Odero Adhiambo Beatrice

Department of Mathematics, Statistics and Computing
Rongo University, P.O. Box 103-40404 Rongo, Kenya

J. O. Agure

Departmant of Pure and Applied Mathematics
Maseno University, P.O. Box 333, Maseno, Kenya

F. O. Nyamwala

Department of Mathematics and Physics
Moi University, P.O. Box 3900-30100, Eldoret, Kenya

Abstract

Let H be an infinite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For two bounded operators $A, B \in B(H)$, the map $\delta_{AB} : B(H) \to B(H)$ is a generalized inner derivation operator induced by A and B defined by $\delta_{AB}(X) = AX - XB$ (1)
In this paper we show that the norm of a generalized inner derivation operator is given by $\|(\delta_{AB/B(B(H))})\| = \|A\| + \|B\|$ for all $A, B \in B(H)$.

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Introduction

Definition: Generalized derivation
Let H be a separable infinite dimensional complex Hilbert space and let $B(H)$

denote the algebra of all bounded linear operators on H. Let $A, B \in B(H)$. The left and the right multiplication operators induced by A and B is denoted by L_A and R_B respectively and defined by $L_A(X) = AX$ and $R_B(X) = XB$. The generalized derivation $\delta_{AB} : B(H) \to B(H)$ is defined by
$\delta_{AB}(X) = L_A - R_B(X) = AX - XB$ for all $X \in B(H)$.

Definition: Finite rank operator.
A bounded linear operator $T : A \to B$ between Banach spaces is said to be a finite rank operator if its range is finite dimensional. Let E be a complex Banach space and $x, y \in E$ be vectors, then for $(x, f) \in E \times E^*$ the finite rank operator $x \otimes f : E \to \mathbb{C}$ is given by $(x \otimes f)(y) = f(y)x$. If $E = H$ then for all $x, y \in H$ we define the finite rank operator by $(x \otimes y)z = \langle z, y\rangle x$ for all $z \in H$.

Definition: Maximal numerical range
Let $T \in B(H)$. The maximal numerical range of T is defined by the set $W_o(T) = \{\lambda : \langle Tx_n, x_n\rangle \to \lambda \, where \, \|x_n\| = 1 \, and \, \|Tx_n\| \to \|T\|\}$ where x_n is a sequence in H and $\lambda \in \mathbb{C}$.

Main result

Theorem 1
Let $A, B \in B(H)$ and $\delta_{AB} : B(H) \to B(H)$. Then
$\|\delta_{AB/B(H)}\| = \|A\| + \|B\|$.

Proof
By definition,
$$\|\delta_{AB/B(H)}\| = sup\{\|\delta_{AB}(X)\| : X \in B(H), \ \|X\| = 1\}$$
$$= sup\{\|AX - XB\| : X \in B(H), \|X\| = 1\}.$$
Therefore,
$\|\delta_{AB/B(H)}\| \geq \|\delta_{AB}(X)\|$ for all $X \in B(H)$ and $\|X\| = 1$.
Taking an arbitrary $\varepsilon > 0$ we have
$\|\delta_{AB/B(H)}\| - \varepsilon < \|\delta_{AB}(X)\|$ for all $X \in B(H)$ and $\|X\| = 1$. So
$\|\delta_{AB/B(H)}\| - \varepsilon < \|AX - XB\|$.
Since$\|AX - XB\| \leq \|A\| + \|B\|$ and letting $\varepsilon \to 0$, then we have that
$$\|\delta_{AB/B(H)}\| \leq \|A\| + \|B\|. \quad (2)$$
On the other hand, let $s, y, z \in H$ be unit vectors. Let u, v be functionals so that $u \otimes y : H \to \mathbb{C}$ and $v \otimes z : H \to \mathbb{C}$ are finite rank operators defined by $(u \otimes y)s = u(s)y$ and $(v \otimes z)s = v(s)z$ for all $s \in H$ with $\|s\| = 1$.
So
$$\|u \otimes y\| = sup\{\|(u \otimes y)s\| : s \in H, \ \|s\| = 1\}$$
$$= sup\{\|u(s)y\| : s \in H, \ \|s\| = 1\}$$
$$= sup\{|u(s)|\|y\| : s \in H, \ \|s\| = 1\}$$
$$= |u(s)| = \|u\|$$

Similarly, $\|v \otimes z\| = |v(s)| = \|v\|$
So if we let $A = u \otimes y$ and $B = v \otimes z$, then $\|A\| = |u(s)| = \|u\|$ and $\|B\| = |v(s)| = \|v\|$.
Now,
$\|\delta_{AB/B(H)}\| \geq \|\delta_{AB}(X)\| \geq \|\delta_{AB}(X)s\|$ where $X \in B(H)$ with $\|X\| = 1$.
But,
$$\begin{aligned}\delta_{AB}(X)s = (AX - XB)(s) &= AX(s) - XB(s)\\ &= ((u \otimes y)X(s)) - (X(v \otimes z))(s)\\ &= u(s)yX - Xv(s)z\\ &= u(s)X(y) - X(z)v(s).\end{aligned}$$
Therefore,
$$\begin{aligned}&\|\delta_{AB/B(H)}\|^2 \geq \|(AX - XB)(s)\|^2\\ &= \langle u(s)X(y) - X(z)v(s), u(s)X(y) - X(z)v(s)\rangle\\ &= \langle u(s)X(y), u(s)X(y)\rangle - \langle u(s)X(y), X(z)v(s)\rangle - \langle X(z)v(s), u(s)X(y)\rangle + \langle X(z)v(s), X(z)v(s)\rangle\\ &= \|u(s)X(y)\|^2 - \langle u(s)X(y), X(z)v(s)\rangle - \langle X(z)v(s), u(s)X(y)\rangle + \|X(z)v(s)\|^2\\ &= |u(s)|^2\|X(y)\|^2 - (uX(y)X(z)v)\langle s, s\rangle - (X(z)vuX(y))\langle s, s\rangle + \|X(z)\|^2|v(s)|^2\\ &= |u(s)|^2 - uX(y)vX(z) - vX(z)uX(y) + |v(s)|^2\\ &= \|u\|^2 - uX(y)vX(z) - vX(z)uX(y) + \|v\|^2.\end{aligned}$$
Setting $uX(y) = |uX(y)| = \|A\|$, and
$vX(z) = -|vX(z)| = -\|B\|$ then we have that
$$\begin{aligned}\|u\|^2 - uX(y)vX(z) - vX(z)uX(y) + \|v\|^2 &= \|A\|^2 + 2\|A\|\|B\| + \|B\|^2\\ &= \{\|A\| + \|B\|\}^2.\end{aligned}$$
Thus,
$\|\delta_{AB/B(H)}\|^2 \geq \{\|A\|\|B\|\}^2$.
Taking square root on both sides we obtain
$$\|\delta_{AB/B(H)}\| \geq \|A\| + \|B\|. \quad (3)$$
Equations (2) and (3) together yields,
$\|\delta_{AB/B(H)}\| = \|A\| + \|B\|$. $\square$
We now proceed to show that the equality holds using Stampfli's maximal numerical range.
Let A be a bounded linear operator on $B(H)$. Then the distance $d(A)$ from A to the scalar multiple of the identity is given by
$d(A) = inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$.

Theorem 2
Let $d(A) = inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$ and $d(B) = inf\{\|B - \lambda\| : \lambda \in \mathbb{C}\}$ the distance from A and B respectively to the scalar multiple of the identity. Then $\|\delta_{AB/B(H)}\| = \|A\| + \|B\|$

Proof.
For $\lambda \in \mathbb{C}$ and $X \in B(H)$ with $\|X\| = 1$, we have
$$\begin{aligned}\delta_{AB}(X) &= AX - XB\\ &= (A - \lambda)X - X(B - \lambda)\|\end{aligned}$$
for all $X \in B(H)$ with $A, B \in B(H)$ fixed.

So,

$$\begin{aligned}\|\delta_{AB}(X)\| &= \|(A-\lambda)X - X(B-\lambda)\| \\ &\leq (\|A-\lambda\| + \|B-\lambda\|)\|X\|\end{aligned}$$

Taking supremum with $\|X\| = 1$ we obtain

$$\begin{aligned}\|\delta_{AB}/B(H)\| &\leq \|A-\lambda\| + \|B-\lambda\| \\ &= d(A) + d(B).\end{aligned}$$

To show the reverse inequality we use the maximal numerical range.
For $A \in B(H)$ the maximal numerical range of A is given by
$W_o(A) = \{\lambda \in \mathbb{C} : \langle Ax_n, x_n\rangle\rangle \to \lambda,\ with\ \|x_n\| = 1\ and\ \|Ax_n\| \to \|A\|\}$.
The following lemma shows the relationship between $W_o(A), W_o(B)$ and $\|\delta_{AB}\|$.

Lemma 3
Let $\lambda_1 \in W_o(A)$ and $\lambda_2 \in W_o(B)$. Then
$\|\delta_{AB}\| \geq (\|A\|^2 - |\lambda_1|^2)^{\frac{1}{2}} + (\|B\|^2 - |\lambda_2|^2)^{\frac{1}{2}}$.

Proof.
By definition, $\|\delta_{AB}/B(H)\| = sup\{\|AX - XB\| : X \in B(H)\ and\ \|X\| = 1\}$.
Since $\lambda_1 \in W_o(A)$, there exists $x_n \in H$ such that $\|Ax_n\| \to \|A\|$ and $\langle Ax_n, x_n\rangle \to \lambda_1$.
Also, for $\lambda_2 \in W_o(B)$, there exists $x_n \in H$ such that $\|Bx_n\| \to \|B\|$ and $\langle Bx_n, x_n\rangle \to \lambda_2$.
We set $Ax_n = \alpha_n x_n + \beta_n y_n$ and $Bx_n = \alpha_n x_n + \omega_n y_n$ where $\langle x_n, y_n\rangle = 0$ and $\|y_n\| = 1$. Given that $V_n x_n = x_n$, $V_n y_n = -y_n$ and $V_n = 0$ on $\{x_n, y_n\}$, then

$$\begin{aligned}\|(AV_n - V_nB)x_n\| &= \|AVx_n - V_nBx_n\| \\ &= \|Ax_n - V_n(\alpha_n x_n + \omega_n y_n)\| \\ &= \|Ax_n - V_n\alpha_n x_n - V_n\omega_n y_n\| \\ &= \|\alpha_n x_n + \beta_n y_n - \alpha_n x_n + \omega_n y_n\| \\ &= \|\beta_n y_n + \omega_n y_n\| \\ &= |\beta_n + \omega_n| \\ &\leq |\beta_n| + |\omega_n|.\end{aligned}$$

But
$\|Ax_n\| = \|\alpha_n x_n + \beta_n y_n\| \leq \|\alpha_n x_n\| + \|\beta_n y_n\| = |\alpha_n| + |\beta_n|$.
So $|\beta_n| \geq \|Ax_n\| - |\alpha_n|$ and since $\|Ax_n\| \to \|A\|$, then
$|\beta_n| \geq (\|A\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n$ where $\varepsilon_n \to 0$ and $\alpha_n \to \lambda_1$.
Also,
$\|Bx_n\| = \|\alpha_n x_n + \omega_n y_n\| \leq \|\alpha_n x_n\| + \|\omega_n y_n\| = |\alpha_n| + |\omega_n|$
So $|\omega_n| \geq \|Bx_n\| - |\alpha_n|$ and since $\|Bx_n\| \to \|B\|$ then
$|\omega_n| \geq (\|B\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n$ where $\varepsilon_n \to 0$ and $\alpha_n \to \lambda_2$
Thus

$$\begin{aligned}|\beta_n| + |\omega_n| &\geq (\|A\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n + (\|B\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n \\ &= (\|A\|^2 - |\lambda_1|^2)^{\frac{1}{2}} + (\|B\|^2 - |\lambda_2|^2)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$\|\delta_{AB}\| \geq \|\delta_{AB}(V_n)\| \geq \|(AV_n - V_nB)x_n\| \geq (\|A\|^2 - |\lambda_1|^2)^{\frac{1}{2}} + (\|B\|^2 - |\lambda_2|^2)^{\frac{1}{2}}.\square$
If λ_1 and λ_2 are as defined in lemma 3 and we let $\alpha_n = \langle Ax_n, x_n\rangle \to \lambda_1$ and $\alpha_n = \langle Bx_n, x_n\rangle \to \lambda_2$ so that
$|\alpha_n|^2 + |\beta_n|^2 = \|Ax_n\|^2 \to \|A\|^2$ that is, $|\beta_n| = (\|AX_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}}$ and
$|\alpha_n|^2 + |\omega_n|^2 = \|Bx_n\|^2 \to \|B\|^2$ that is, $|\omega_n| = (\|Bx_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}}$.
Also, let $V_n = x_n \otimes x_n - y_n \otimes y_n$, then $\|V_n\| = 1$ and
$(AV_n - V_nB)x_n = \beta_n y_n + \omega_n y_n$.
Then

$$\begin{aligned}\|\delta_{AB}\| \geq \|(AV_n - V_nB)x_n\| &= |\beta_n| + |\omega_n| \\ &= (\|Ax_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}} + (\|Bx_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}} \\ &= (\|Ax_n\|^2 - |\langle Ax_n, x_n\rangle|^2)^{\frac{1}{2}} + (\|Bx_n\|^2 - |\langle Bx_n, x_n\rangle|^2)^{\frac{1}{2}} \\ &\to (\|A\|^2 - |\lambda_1|^2)^{\frac{1}{2}} + (\|B\|^2 - |\lambda_2|^2)^{\frac{1}{2}}.\end{aligned}$$

Now, if $0 \in W_o(A)$ and $0 \in W_o(B)$ then we have that
$\|\delta_{AB}\| \geq \|A\| + \|B\|$.
Furthermore, $\|A\| + \|B\| \leq \|\delta_{AB}\| \leq d(A) + d(B) \leq \|A\| + \|B\|$.
Thus, $\|\delta_{AB/B(H)}\| = \|A\| + \|B\|.\square$

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