Stability of Lie Groups of Nonlinear Hyperbolic Equations

by

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Chapter 0

Introduction

Nonlinear theories, started in the 1960's, provided for the first time in the literature global generalized solutions for arbitrary continuous nonlinear partial differential equations (PDEs). These theories have now been extended to Lie symmetry groups for classical and global generalized solutions of nonlinear PDEs. My work in this thesis is on *stability analysis* of Lie groups of nonlinear PDEs.

Nonlinear algebraic theory of generalized solutions for large classes of nonlinear PDEs was originated by Elemér E. Rosinger[1] who published the first two papers on the subject in 1966 and 1968. He has since developed the theory further, culminating in the publication of four research monographs (1978, 1980, 1984, 1990). In these monographs the algebraic theory, complete with applications in the study of nonlinear PDEs, is well presented. Some of the major results obtained by Rosinger in this line of research include:

· the solution of the celebrated 1954 impossibilty result of L. Schwartz[2] regarding the multiplication of distributions (1966);

· the characterization of all possible nonlinear algebraic theories of generalized functions (1980);

· the global solution of arbitrary nonlinear analytic PDEs (1987);

· the algebraic characterization of the solvability of large classes of nonlinear PDEs (1990).

J.F. Colombeau[3,4] of Lions, France, is the other main protagonist of this field. He independently developed an algebraic nonlinear theory of generalized functions, first published in 1984. Although Colombeau's theory is considered to be the most complete, very powerful and so far the most widely studied of the various possible algebraic methods, it is of narrower applicability due to the fact that it is merely a particular case of the whole class of such possible theories already developed fully and characterized completely by Rosinger in 1980 (see Colombeau's review in Bull. AMS vol. 20, no. 1, January 1989, pp.

96 – 101). A detailed account of Colombeau's version of the theory is given in his two research monographs of 1984 and 1985.

Of late there have been several other researchers, mostly from Europe and Brazil, in this field. But most of them tend to develop either Rosinger's or Colombeau's theory.

Lie group analysis is a mathematical theory that synthesizes symmetry of differential equations. This theory was originated by a great Norwegian mathematician of the nineteenth century called Sophus Lie[5] (1842 – 1899). In his work published in 1884, Lie pioneered the use of continuous groups of transformations – called *Lie groups* – in the study of symmetry properties of differential equations with a view to their solutions. Lie discovered that the known *ad hoc* methods of integration of differential equations could easily be derived by his theory of continuous groups. He further, among other things, gave a classification of differential equations in terms of their symmetry groups, thereby identifying the set of equations which could be integrated or reduced to lower-order equations by group theoretic arguments. Lie's basic idea was to find all the Lie groups for a given PDE such that any solution of this PDE is transformed into another solution by the coordinate transforms of the respective Lie groups; i.e., all the groups with respect to which the set of solutions of the PDE is *invariant*. The procedure for determining the Lie groups admitted by any given family of differential equations is discussed in detail in Chapter 1.

A symmetry group transforms any solution of the equation in question to another solution of the same equation. Thus from a symmetry group of any given system of equations it is possible to construct nontrivial solutions from the known ones. If a group transformation maps a solution into itself, we arrive at what is called a **group invariant solution**. The process of looking for this type of solution reduces the number of independent variables of the equation in question by one. This process can be repeated until the equation can be solved by integration. If the solution is invariant under a one-parameter group, then the differential equation is reduced to an ordinary differential equation. A multi-parameter group results in the reduction of the differential equation to an algebraic relation. This is well known in the literature (see sources in Chapter 1).

Lie symmetry groups for classical solutions of nonlinear PDEs can be extended to symmetry groups for **global generalized solutions**. Nonlinear Lie group theory for global generalized solutions of nonlinear PDEs was started by Rosinger in 1992. In collaboration with Michael Oberguggenberger[6] of Innsbruck, Austria, they have published a research monograph (1994) on the solution of continuous nonlinear PDEs through **order completion**. The Group Invariance of such global solutions has been the object of two recent doctoral theses in this department (Y.E. Walus[7] and M. Rudolph[8], both supervised by E.E. Rosinger, 1993). M. Kunzinger[9] of the University of Vienna, Austria, has recently studied **Lie transformation groups in Colombeau Algebras** for his doctorate under the supervision of M. Oberguggenberger (1996). So far, some of the major results obtained by Rosinger and his collaborates are:

· the first nonlinear Lie group theory of global generalized solutions of nonlinear PDEs;

· three solutions to Hilbert's fifth problem considered in its full generality;

· the first solution of the 1957 H. Lewy problem on the solvability of smooth PDEs.

Independently, in Russia, N.Kh. Ibragimov[10] and his collaborators have recently developed a theory which amounts to the study of **stability of Lie groups** associated with nonlinear PDEs. Their paper, titled *Approximate Lie Groups* (1989) is the first in this line of research. Ibragimov was a student L.V. Ovsyannikov who had mentioned the possibility of such a theory in the early 1960's, but nobody took it up. Lie symmetry group analysis is very useful in determining all the nonlinear solutions of a given differential equation. Unfortunately, any small perturbation of an equation disturbs the group admitted by it, and this in effect reduces the practical use of symmetry group analysis. It was this realization that motivated the Russian group into developing group analysis methods of differential equations with a small perturbation (ϵ). The theory, however, is valid even for large parameters (ϵ) and this makes the term "approximate" appear somewhat unsuitable. We have therefore given it a more appropriate name, *stability of Lie groups*, which is the subject of study in this thesis. In particular, we aim to find stability of Lie groups of the perturbed nonlinear wave equation

$$u_{tt} + \epsilon u_t = [f(x, u)u_x]_x.$$

The choice of the perturbation $\epsilon u_t(t, x)$ is natural. In an equation of the type studied, the function $u(t, x)$ usually denotes the displacement of the object, here one-dimensional, which occurs at time t and spatial coordinate x. In this case, $u_t(t, x)$ is the local velocity of the object and a perturbation of the form $\epsilon u_t(t, x)$ is the usual model of friction. Hence this perturbation is of particular practical importance, since the inclusion of friction is a realistic necessity in various applied problems; the case $\epsilon = 0$ corresponds to frictionless motion.

Chapter 1 contains the general concepts of Lie groups. This is presented in a self-contained manner as it forms an integral background reading for this thesis. The determination of Lie groups admitted by a given PDE is not triavial, so we have included in this chapter many examples and illustrations.

In Chapter 2, the theory of stability of Lie groups is discussed in detail. An algorithm for constructing an approximate group for a given PDE is presented.

The exact Lie groups for the unperturbed and perturbed nonlinear wave equations

$$u_{tt} = [f(x,u)u_x]_x$$

and

$$u_{tt} + \epsilon u_t = [f(x,u)u_x]_x$$

are discussed in Chapter 3. Although properties of Lie groups of the unperturbed nonlinear wave equation had been studied by Torrisi and Valenti (see [3] in Chapter 3), we show that the assumption they made when solving for the infinitesimals were unnecessary since the conditions present themselves naturally. The exact Lie groups for the perturbed nonlinear wave equation is presented here for the first time.

Lastly, in Chapter 4 we present a general criterion for approximate invariance. The concept is then used to find the Stability Groups for the perturbed nonlinear wave equations.

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Chapter 1

Lie Groups – General Concepts

Sources [1] George W. Bluman and Sekeyuki Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.

[2] Peter J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[3] Joseph J. Rotman, *The Theory of Groups*, Allyn and Bacon, 1973.

This chapter will serve as a short introduction to the theory of Lie groups which is useful in order to set up the notation employed in the sequel. Section 1.1 deals with the concept of Lie groups of transformations. Section 1.2 introduces the idea of infinitesimal transformations. Extended transformations (prolongations) are discussed in Section 1.3. In Section 1.4 some applications of Lie Groups of Transformations to the solution of partial differential equations are discussed.

1.1 Lie Groups of Transformations

First we give the standard definition of a group then discuss the Lie groups of transformations.

Definition 1.1-1

A *group* G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

(i) closure property: $\forall a, b \in G, \quad \phi(a, b) \in G.$

(ii) associative property: $\forall a, b, c \in G, \quad \phi(a, \phi(b, c)) = \phi(\phi(a, b), c).$

(iii) identity element: $\exists\,!$ identity element $e \in G$ such that $\forall a \in G$, $\phi(a, e) = \phi(e, a) = a$.

(iv) inverse element: $\forall a \in G$, $\exists\,!$ inverse element $a^{-1} \in G$ such that $\phi(a, a^{-1}) = \phi(a^{-1}, a) = e$.

□

Definition 1.1-2

A *subgroup* of G is a group formed by a subset of elements of G with the same law of composition ϕ. □

Definition 1.1-3

Let $x = (x_1, x_2, \ldots, x_n)$ lie in a region $D \subset \mathbf{R}$. Consider the set of transformations

$$x^* = X(x, \epsilon) \tag{1.1}$$

defined for each $x \in D$, depending on parameter ϵ, where $\epsilon \in S \subset \mathbf{R}$. Also let $\phi(\epsilon, \delta)$ define a law of composition of parameters $\epsilon, \delta \in S$. Then (1.1) forms a *group of transformations* on D if:

(i) For each $\epsilon \in S$, $x^* \in D$.

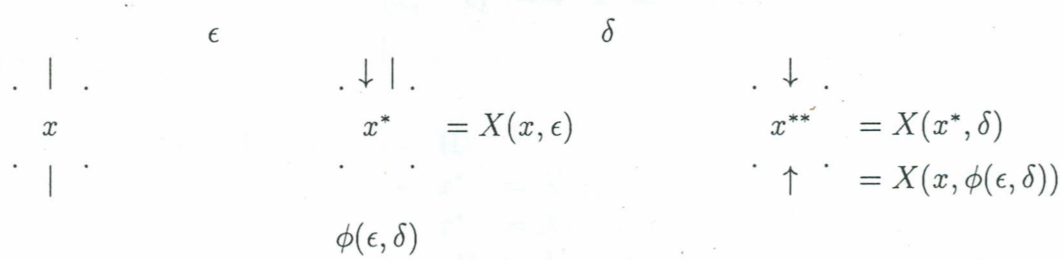
(ii) S with ϕ forms a group G.

(iii) $x^* = x$ when $\epsilon = e$ i.e. $X(x, e) = x$.

(iv) If $x^* = X(x, \epsilon)$ and $x^{**} = X(x^*, \delta)$, then $x^{**} = X(x, \phi(\epsilon, \delta))$.

□

This definition is illustrated in the diagram below.



Here the transformation from x to x^* via ϵ, then from x^* to x^{**} via δ is equivalent to a single transformation from x to x^{**} via $\phi(\epsilon, \delta)$.

The concept of one-parameter Lie group of transformations is introduced formally in the following definition:

Definition 1.1-4

Let (1.1) in Definition 1.1-3 form a group of transformations on D. Then (1.1) defines a *one-parameter* (ϵ) *Lie group of transformations* if:

(v) ϵ is a continuous parameter i.e. $\epsilon \in S$ where S is an interval in $\mathbf{R}$.

(vi) X is infinitely differentiable with respect to x in D and ϵ in S.

(vii) $\phi(\epsilon, \delta)$ is C^∞-continuous.

□

This concept is illustrated in the next four examples.

Example 1.1-1

Show that the transformation

$$x^* = x + \epsilon \tag{1.2}$$

defines a Lie group of transformations.

We note here that

$$D = \mathbf{R} \quad \text{and} \quad S = \mathbf{R},$$

so that the operation

$$\begin{array}{ccccccl} X: & D & \times & S & \longrightarrow & D & \\ & \downarrow & & \downarrow & & \downarrow & \\ & (x, & & \epsilon) & \longrightarrow & x^* = X(x, \epsilon) & \end{array}$$

now becomes

$$\begin{array}{llll} X: & \mathbf{R} \times \mathbf{R} & \longrightarrow & \mathbf{R} \\ & (x, \epsilon) & \longmapsto & x^* = X(x, \epsilon) = x + \epsilon \\ & (x, 0) & \longmapsto & x^* = X(x, 0) = x \\ & (x^*, \delta) & \longmapsto & x^{**} = X(x^*, \delta) = x + \epsilon + \delta \\ & & & \phantom{x^{**}} = X(x, \phi(\epsilon, \delta)) = x + (\epsilon + \delta). \end{array}$$

It can be seen clearly that all the conditions (i) — (vii) in definition 1.1-4 are satisfied, with the identity element $e = 0$ and $\phi(\epsilon, \delta) = \epsilon + \delta$.

Example 1.1-2

The transformations

$$\left.\begin{array}{rcl} x^* &=& x+\epsilon, \\ y^* &=& \dfrac{xy}{x+\epsilon}, \end{array}\right\} \tag{1.3}$$

define a Lie group of transformations.

With $D = \mathbf{R}^2$ and $S = \mathbf{R}$ we have the operation

$$\begin{array}{rcl} X: \ \mathbf{R}^2 \times \mathbf{R} & \longrightarrow & \mathbf{R}^2 \\ ((x,y),\epsilon) & \longmapsto & (x^*,y^*) = (x+\epsilon, \dfrac{xy}{x+\epsilon}). \end{array}$$

This can be illustrated further as in the diagram below.

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\epsilon} \begin{bmatrix} x^* &=& x+\epsilon \\ y^* &=& \dfrac{xy}{x+\epsilon} \end{bmatrix} \xrightarrow{\delta} \begin{bmatrix} x^{**} &=& x^*+\delta &=& x+\epsilon+\delta &=& x+\phi(\epsilon,\delta) \\ y^{**} &=& \dfrac{x^*y^*}{x^*+\delta} &=& \dfrac{xy}{x+\epsilon+\delta} &=& \dfrac{xy}{x+\phi(\epsilon,\delta)} \end{bmatrix}$$

$$\phi(\epsilon,\delta) = \epsilon + \delta$$

With $\phi(\epsilon,\delta) = \epsilon+\delta$ and $e = 0$ it is easy to show that the conditions (i) — (vii) are satisfied. In particular we see that

$$\forall x, y \in D = \mathbf{R}^2 \quad \text{and} \quad \forall \epsilon \in S = \mathbf{R},$$

$$\begin{array}{rcccl} x^{**} &=& x+\phi(\epsilon,\delta) &=& X(x,\phi(\epsilon,\delta)) \quad \text{and} \\ y^{**} &=& \dfrac{xy}{x+\phi(\epsilon,\delta)} &=& Y(y,\phi(\epsilon,\delta)). \end{array}$$

Example 1.1-3

Consider a group of scalings in the plane:

$$\left.\begin{array}{rcl} x^* &=& \epsilon x, \\ y^* &=& \epsilon^2 y, \quad 0 < \epsilon < \infty, \quad (x,y) \in \mathbf{R}^2. \end{array}\right\} \tag{1.4}$$

Here $\phi(\epsilon,\delta) = \epsilon\delta$, the identity element $e = 1$ and we have another example of a one-parameter Lie group of transformations.

Next is a group of transformations which does not define a Lie group of transformations.

Example 1.1-4

A family of transformations

$$\left.\begin{aligned} x^* &= x - \epsilon y, \\ y^* &= y + \epsilon x, \quad \epsilon \in \mathbf{R}, \quad (x, y) \in \mathbf{R}^2, \end{aligned}\right\} \tag{1.5}$$

does not correspond to a Lie group of transformations, since the condition (iv) is not satisfied:

$$x^{**} = X(x^*, \delta) \neq X(x, \phi(\epsilon, \delta)).$$

1.2 Infinitesimal Transformations

Let (1.1) be a one-parameter (ϵ) Lie group of transformations with identity $\epsilon = 0$ and law of composition ϕ. The Taylor expansion of (1.1) about $\epsilon = 0$ gives

$$\begin{aligned} x^* &= X(x,\epsilon) = X(x,0) + \epsilon\left(\frac{\partial X(x,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}\right) + \frac{\epsilon^2}{2}\left(\frac{\partial^2 X(x,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}\right) + \ldots \\ &= x + \epsilon\left(\frac{\partial X(x,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}\right) + 0(\epsilon^2). \end{aligned} \tag{1.6}$$

If we further let

$$\xi(x) = \frac{\partial X(x,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}$$

then (1.6) becomes

$$x^* = x + \epsilon\xi(x) + 0(\epsilon^2) \tag{1.7}$$

and the following definition follows.

Definition 1.2-1

The transformation $x + \epsilon\xi(x)$ in (1.7) is called the *infinitesimal transformation* of the one-parameter Lie group of transformations (1.1). The components of $\xi(x)$ are called the *infinitesimals* of (1.1). □

We wish to remark here that since (1.1) is a one-parameter Lie group of transformations and by Lie's First Fundamental Theorem 1.2-1, the Taylor expansion in (1.7) is completely determined by only having the infinitesimal transformation

$$x^* = x + \epsilon\xi(x).$$

□

The example below illustrates how to get the infinitesimals $\xi(x)$ of a given group of transformations.

Example 1.2-1

For the group of scalings

$$\left.\begin{aligned} x^* &= (1+\epsilon)x, \\ y^* &= (1+\epsilon)^2 y, \quad -1 < \epsilon < \infty, \end{aligned}\right\} \tag{1.8}$$

we see that

$$X((x,y),\epsilon) = (x^*,y^*) = ((1+\epsilon)x,(1+\epsilon)^2y)$$

and the corresponding infinitesimal is

$$\begin{aligned}\xi(x,y) &= \frac{\partial X((x,y),\epsilon)}{\partial\epsilon}\Big|_{\epsilon=0} = \frac{\partial(x^*,y^*)}{\partial\epsilon}\Big|_{\epsilon=0} \\ &= (x, 2(1+\epsilon)y)\,|_{\epsilon=0} \\ &= (x,2y). \end{aligned} \tag{1.9}$$

It is important to note that although the infinitesimal $\xi(x,y)$ appears to be relatively simpler, i.e. it is not a function of ϵ, it is the same infinitesimal, see (1.9), which contains the essential information determining the group of transformations as in (1.8).

The process of getting (1.9) from (1.8) is quite easy. What is not obvious is how to get (1.8) from (1.9). This will be discussed in its full generality later in the chapter.

The next theorem is known as the First Fundamental Theorem of Lie. We shall state it without proof, see [1] for proof.

Theorem 1.2-1

There exists a parameterization $\tau(\epsilon)$ such that the Lie group of transformation (1.1) is equivalent to the solution of the initial value problem (IVP) for the first order differential equations

$$\frac{dx^*}{d\tau} = \xi(x^*) \tag{1.10}$$

with

$$x^* = x \quad \text{when} \quad \tau = 0.$$

In particular

$$\tau(\epsilon) = \int_0^\epsilon \Gamma(\epsilon')d\epsilon', \tag{1.11}$$

where

$$\Gamma(\epsilon) = \frac{\partial\phi(a,b)}{\partial b}\Big|_{(a,b)=(\epsilon^{-1},\epsilon)}$$

and

$$\Gamma(0) = 1.$$

[ϵ^{-1} denotes the inverse element to ϵ.] □

Example 1.2-2

Consider the initial value problem

$$\frac{dx^*(\epsilon)}{d\epsilon} = \xi(x^*) = x^*(\epsilon) \tag{1.12}$$

$$\text{with} \quad x^* = x \quad \text{when} \quad \epsilon = 0. \tag{1.13}$$

The above theorem says that we can find a paramterization $\tau(\epsilon)$ such that the solution of (1.12) is (1.1). To see this we need to solve (1.12) with (1.13), i.e.

$$\frac{dx^*(\epsilon)}{d\epsilon} = x^*(\epsilon)$$

where $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ and $x^*(0) = x$.

$$\begin{aligned} \frac{dx_1^*}{d\epsilon} &= x_1^* \\ \Rightarrow \quad x_1^* &= e^{\epsilon} x_1 \\ &\vdots \\ x_n^* &= e^{\epsilon} x_n \end{aligned}$$

Thus

$$x^* = X(x, \epsilon) = e^{\epsilon}. \tag{1.14}$$

Clearly, there exists a parameterization $\tau(\epsilon) = e^{\epsilon}$ such that (1.14) is the solution of (1.12).

Example 1.2-3

It is easy to see that the group of translations in the plane

$$\left.\begin{aligned} x^* &= x + \epsilon, \\ y^* &= y, \end{aligned}\right\} \tag{1.15}$$

is the solution of the Initial Value Problem (IVP)

$$\left.\begin{aligned} &\frac{dx^*}{d\epsilon} = 1, \quad \frac{dy^*}{d\epsilon} = 0, \\ \text{with} \quad & x^* = x, \quad y^* = y \quad \text{when} \quad \epsilon = 0. \end{aligned}\right\} \tag{1.16}$$

Remark 1.2-1

Arising from Theorem 1.2-1, from now on, without loss of generality, we assume that a one-parameter (ϵ) Lie group of transformations is parameterized such that its law of composition $\phi(a,b) = a + b$ so that $\epsilon^{-1} = -\epsilon$ and $\epsilon = 0$ is the neutral element. Thus in terms of its infinitesimals $\xi(x)$, the one-parameter Lie group of transformations (1.1) becomes

$$\left.\begin{aligned} &\frac{dx^*}{d\epsilon} = \xi(x^*) \\ \text{with} \quad & x^* = x \quad \text{at} \quad \epsilon = 0. \end{aligned}\right\} \tag{1.17}$$

□

Definition 1.2-2

The *infinitesimal generator* of the one-parameter group of transformations (1.1) is the operator

$$\begin{aligned} V = V(x) &= \xi(x) \cdot \nabla \\ &= \xi(x) \cdot \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right) \\ &= \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \end{aligned} \tag{1.18}$$

where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right).$$

For a differentiable function

$$F(x) = F(x_1, x_2, \ldots, x_n)$$

we have

$$VF(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial F(x)}{\partial x_i}.$$

□

The concept of infinitesimal generator may be used to find the explicit solution of IVP (1.12). This follows from the theorem below.

Theorem 1.2-2

The one-parameter Lie group of transformations (1.1) is equivalent to

$$\begin{aligned} x^* &= e^{\epsilon V} x = X(x,\epsilon) \\ &= x + \epsilon V x + \frac{\epsilon^2}{2!} V^2 x + 0(\epsilon^2) x \\ &= [1 + \epsilon V + \frac{\epsilon^2}{2!} V^2 + 0(\epsilon^3)] x \\ &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} V_x^k \end{aligned} \tag{1.19}$$

where the operator $V = V(x)$ is defined by (1.13) and

$$V^k = V V^{k-1}, \quad k = 1, 2, \ldots$$

with $V^0 x \equiv x$. □

For proof, see Bluman and Kumei [1].

The transformation (1.19) above is called a *Lie Series*.

The summary below illustrates the connection between a Lie group, its infinitesimal transformations and its infinitesimal generator.

Start with a Lie Group

$$\begin{aligned} X : D \times \mathbf{R} &\longrightarrow D, \quad D \subseteq \mathbf{R}^n \\ (x,\epsilon) &\longmapsto x^* = X(x,\epsilon) \end{aligned}$$

and

(i) $X(x,0) = x \quad \forall x \in D$,

(ii) $X(X(x,\epsilon),\delta = X(x,\epsilon+\delta)$, $\forall x \in D$, $\epsilon, \delta \in \mathbf{R}$.

whose infinitesimal transformation can be found as

$$\begin{array}{ccc} \xi : D & \longrightarrow & \mathbf{R}^n \\ \downarrow & & \downarrow \\ x & \longmapsto & \xi(x) = \dfrac{\partial X(x,\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} . \end{array}$$

Then for a differentiable function $F(x) = F(x_1, x_2, \ldots, x_n)$,

$$\begin{array}{ccl} F : D & \longrightarrow & \mathbf{R}^m \\ \downarrow & & \downarrow \\ x & \longmapsto & F(x) = (F_1(x), F_2(x), \ldots, F_n(x)), \end{array}$$

we have the infinitesimal generator of the group

$$\begin{array}{lllcl} V: & D \times C^\infty(D, \mathbf{R}^m) & & \longrightarrow & \mathbf{R}^m \\ & \downarrow & \downarrow & & \downarrow \\ & (x, & F) & \longmapsto & VF(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial F(x)}{\partial x_i}. \end{array} \tag{1.20}$$

□

The example below illustrates how to find explicitly a one-parameter Lie group of transformation from its infinitesimal transformation.

Example 1.2-4

Find explicitly a one-parameter Lie group of transformations whose infinitesimal generator is

$$V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

By Theorem 1.2-2, the one-parameter Lie groups will be of the form

$$\left.\begin{aligned} x^* &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} V^k x, \\ y^* &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} V^k y. \end{aligned}\right\} \tag{1.21}$$

Let the infinitesimal for (1.21) be

$$\xi(x) = (\xi_1(x, y), \xi_2(x, y)) = (\xi(x, y), \eta(x, y)).$$

From (1.15) it is clear that

$$\xi(x, y) = y \quad \text{and} \quad \eta(x, y) = -x.$$

We now need to find $V^k x$ and $V^k y$, $k = 1, 2, \ldots$. It can easily be established that

$$V^{4n}x = x,\ V^{4n-1}x = -y,\ V^{4n-2}x = -x,\ V^{4n-3}x = y,\ n = 1, 2, \ldots \tag{1.22}$$

$$V^{4n}y = y,\ V^{4n-1}y = x,\ V^{4n-2}y = -y,\ V^{4n-3}y = -x,\ n = 1, 2, \ldots \tag{1.23}$$

From (1.22) and $(1.21)_1$, we have

$$\begin{aligned} x^* &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} V^k x \\ &= (1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} + \ldots)x + (\epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} + \ldots)y \\ &= x \cos\epsilon + y \sin\epsilon. \end{aligned}$$

Similarly from (1.23) and $(1.21)_2$ we have

$$y^* = -x \sin\epsilon + y \cos\epsilon.$$

□

1.3 Extended Transformations (Prolongations)

Lie groups, and hence their infinitesimal generators, can be naturally extended or "prolonged" to act not only on the space of independent and dependent variables but also derivatives of the dependent variables up to any finite order.

In this section we consider separately, the cases of prolongations of Lie groups of transformations with one independent variable x and one dependent variably y; and that of Lie groups of transformations with n independent variables $x = (x_1, x_2, \ldots, x_n)$ and one dependent variable u, where $u = u(x)$. Key results will be stated for cases of m dependent variables, where $m \geq 2$.

1.3.1 One dependent and one independent variable

In studying the invariance of a $k-$th order ordinary differential equation, with x and y as independent and dependent variables respectively, we will aim to find one-parameter Lie groups of transformations admitted by the ODE. Such groups of transformations will be of the form

$$\left.\begin{aligned} x^* &= X((x,y);\epsilon) \\ y^* &= Y((x,y);\epsilon) \end{aligned}\right\} \qquad (1.24)$$

Let

$$y_k = \frac{d^k y}{dx^k}, \quad k = 1, 2, \ldots.$$

The task here is extending the transformations (1.24) acting on the (x, y)-space to the $(x, y, y_1, \ldots, y_k)$-space. To do this we first demand that (1.24) preserve the contact conditions relating the differentials $dx, dy, dy_1, \ldots, dy_k$, i.e.

$$\begin{aligned} dy &= y_1 dx, \\ dy_1 &= y_2 dx, \\ &\vdots \\ dy_k &= y_{k+1} dx. \end{aligned} \qquad (1.25)$$

Under the action of the group (1.24) the transformed derivatives are defined by

$$
\begin{aligned}
dy^* &= y_1^* dx^* \\
&\vdots \\
dy_k^* &= y_{k+1}^* dx^*.
\end{aligned} \tag{1.26}
$$

Using (1.25) and (1.26) it is easy to show that in particular,

$$
y_1^* = Y_1(x, y, y_1; \epsilon) = \frac{\dfrac{\partial Y(x,\epsilon)}{\partial x} + y_1 \dfrac{\partial Y(x,\epsilon)}{\partial y}}{\dfrac{\partial X(x,\epsilon)}{\partial x} + y_1 \dfrac{\partial X(x,\epsilon)}{\partial y}}. \tag{1.27}
$$

The result in (1.27) is formulated in the following theorem.

Theorem 1.3-1

The Lie group of transformations (1.24) acting on (x, y)-space extends to the following one-parameter Lie group of transformations acting on (x, y, y_1)-space:

$$
\left.
\begin{aligned}
x^* &= X(x, y; \epsilon), \\
y^* &= Y(x, y; \epsilon), \\
y_1^* &= Y_1(x, y, y_1; \epsilon),
\end{aligned}
\right\} \tag{1.28}
$$

where $Y_1(x, y, y_1; \epsilon)$ is given by (1.27). □

The next two theorems are useful in finding the second, and in general, k-th extensions of (1.24) with $k \geq 2$.

Theorem 1.3-2

The Lie group of transformations (1.24) extends to its second extension which is the following one-parameter Lie group of transformations acting on (x, y, y_1, y_2)-space:

$$
\begin{aligned}
x^* &= X(x, y; \epsilon), \\
y^* &= Y(x, y; \epsilon), \\
y_1^* &= Y_1(x, y, y_1; \epsilon),
\end{aligned}
$$

$$
y_2^* = Y_2(x, y, y_1, y_2; \epsilon) = \frac{\dfrac{\partial Y_1}{\partial x} + y_1 \dfrac{\partial Y_1}{\partial y} + y_2 \dfrac{\partial Y_1}{\partial y_1}}{\dfrac{\partial X(x,\epsilon)}{\partial x} + y_1 \dfrac{\partial X(x,\epsilon)}{\partial y}}, \tag{1.29}
$$

where $Y_1 = Y_1(x, y, y_1; \epsilon)$ as defined in (1.27). □

Theorem 1.3-3

The Lie group of transformations (1.24) extends to its k-th extension, $k \geq 2$, which is the following one-parameter Lie group of transformations acting on $(x, y, y_1, y_2, \ldots, y_k)$-space:

$$\begin{aligned} x^* &= X(x, y; \epsilon), \\ y^* &= Y(x, y; \epsilon), \\ y_1^* &= Y_1(x, y, y_1; \epsilon), \end{aligned}$$

$$y_k^* = Y_k(x, y, y_1, \ldots, y_k; \epsilon) = \frac{\dfrac{\partial Y_{k-1}}{\partial x} + y_1 \dfrac{\partial Y_{k-1}}{\partial y} + \ldots + y_k \dfrac{\partial Y_{k-1}}{\partial y_{k-1}}}{\dfrac{\partial X(x,\epsilon)}{\partial x} + y_1 \dfrac{\partial X(x,\epsilon)}{\partial y}}. \tag{1.30}$$

□

Now we consider the following example of prolongations.

Example 1.3-1

A scaling group

$$\left.\begin{aligned} x^* &= X(x, y; \epsilon) = e^{\epsilon} x, \\ y^* &= Y(x, y; \epsilon) = e^{2\epsilon} y, \end{aligned}\right\} \tag{1.31}$$

has as its first extension

$$y_1^* = Y_1(x, y, y_1; \epsilon) = \frac{\dfrac{\partial Y}{\partial x} + y_1 \dfrac{\partial Y}{\partial y}}{\dfrac{\partial X}{\partial x} + y_1 \dfrac{\partial X}{\partial y}} = e^{\epsilon} y_1$$

and the second extension as, see (1.29),

$$y_2^* = Y_2(x, y, y_1, y_2; \epsilon) = \frac{\dfrac{\partial Y_1}{\partial x} + y_1 \dfrac{\partial Y_1}{\partial y} + y_2 \dfrac{\partial Y_1}{\partial y_1}}{\dfrac{\partial X}{\partial x} + y_1 \dfrac{\partial X}{\partial y}} = y_2.$$

By (1.30) the k-th extension is

$$y_i^* = Y_i(x, y, y_1, y_2, \ldots, y_i; \epsilon) = e^{(2-i)\epsilon} y_i, \quad i = 1, 2, \ldots, k. \tag{1.32}$$

By definition 1.1-4, the k–th prolongation (1.32) is also a Lie group of transformations. To verify this we observe that

$$Y_i(x, y_i; 0) = y_i,$$

$$Y_i(Y_i(x, y_i; \epsilon); \delta) = Y_i(y_i^*; \delta) = e^{(2-i)(\epsilon+\delta)}y_i = Y_i(x, y_i; \epsilon + \delta), \ldots$$

and all the other conditions of Definition 1.1-4 are satisfied. Thus the study of extended Lie groups of transformations reduces to the study of infinitesimal transformations. Consequently we need to determine an explicit algorithm to find extended infinitesimal transformations and the corresponding infinitesimal generators.

We consider the one-parameter Lie group of transformations

$$\left.\begin{aligned} x^* &= X(x, y; \epsilon) = x + \epsilon\xi(x, y) + 0(\epsilon^2), \\ y^* &= Y(x, y; \epsilon) = y + \epsilon\eta(x, y) + 0(\epsilon^2), \end{aligned}\right\} \tag{1.33}$$

whose infinitesimal is

$$\xi(x) = (\xi(x, y), \eta(x, y)), \tag{1.34}$$

with corresponding infinitesimal generator

$$V = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{1.35}$$

The k-th extension of (1.33) is given by

$$\left.\begin{aligned} x^* &= X(x, y; \epsilon) = x + \epsilon\xi(x, y) + 0(\epsilon^2), \\ y^* &= Y(x, y; \epsilon) = y + \epsilon\eta(x, y) + 0(\epsilon^2), \\ y_1^* &= Y_1(x, y, y_1; \epsilon) = y_1 + \epsilon\eta^{(1)}(x, y, y_1) + 0(\epsilon^2), \\ &\vdots \\ y_k^* &= Y_k(x, y, y_1, \ldots, y_k; \epsilon) = y_k + \epsilon\eta^{(k)}(x, y, y_1, \ldots, y_k) + 0(\epsilon^2). \end{aligned}\right\} \tag{1.36}$$

The k-th extended infinitesimal of (1.34) will be

$$(\xi(x, y), \eta(x, y), \eta^{(1)}(x, y, y_1), \ldots, \eta^{(k)}(x, y, y_1, \ldots, y_k)),$$

with corresponding k-th extended infinitesimal generator

$$
\begin{aligned}
V^{(k)} &= \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1} + \ldots \\
&\quad + \eta^{(k)}(x,y,y_1,\ldots,y_k)\frac{\partial}{\partial y_k}, \quad k = 1,2,\ldots
\end{aligned}
$$

We use the next theorem to evaluate $\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(k)}$.

Theorem 1.3-4

$$
\eta^{(k)}(x,y,y_1,\ldots,y_k) = \frac{D\eta^{(k-1)}}{Dx} - y_k\frac{D\xi(x,y)}{Dx}, \quad k = 1,2,\ldots \tag{1.37}
$$

where $\eta^{(0)} = \eta(x,y)$ and the total derivative operator

$$
\frac{D}{Dx} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + \ldots + y_{n+1}\frac{\partial}{\partial y_n} + \ldots \tag{1.38}
$$

□

We now find the extended infinitesimal in the next example.

Example 1.3-2

The rotation group

$$
\left.
\begin{aligned}
x^* &= X(x,y;\epsilon) = x\cos\epsilon + y\sin\epsilon, \\
y^* &= Y(x,y;\epsilon) = -x\sin\epsilon + y\cos\epsilon,
\end{aligned}
\right\} \tag{1.39}
$$

whose infinitesimal

$$
\xi(x) = (\xi(x,y), \eta(x,y)) = (y, -x),
$$

has its first extension as

$$
y_1^* = Y_1(x,y,y_1;\epsilon) = y_1 + \epsilon\eta^{(1)}(x,y,y_1). \tag{1.40}
$$

Using (1.37) and (1.38) we have

$$
\begin{aligned}
\eta^{(1)}(x,y,y_1) &= \frac{D\eta^0}{Dx} - y_1\frac{D\xi(x,y)}{Dx} \\
&= -1 - y_1^2.
\end{aligned}
$$

Thus (1.40) now becomes

$$y_1^* = y_1 + \epsilon(-1 - y_1^2).$$

Similarly

$$\begin{aligned} y_2^* &= \frac{D\eta^{(1)}}{Dx} - y_2\frac{D\xi(x,y)}{Dx} \\ &= \frac{\partial\eta^{(1)}}{\partial x} + y_1\frac{\partial\eta^{(1)}}{\partial y} + y_2\frac{\partial\eta^{(1)}}{\partial y_1} - y_2\left[\frac{\partial\xi(x,y)}{\partial x} + y_1\frac{\partial\xi(x,y)}{\partial y}\right] \\ &= y_2 - 3\epsilon y_1 y_2. \end{aligned}$$

□

The case of extended transformations and infinitesimal transformations involving one dependent and n independent variables is considered next.

1.3.2 One dependent and n independent variables

In the case of n independent variables, the one-parameter group of transformations (1.33) takes the form

$$\left.\begin{aligned} x_i^* &= X_i(x,u;\epsilon) = x_i + \epsilon\xi_i(x,u) + 0(\epsilon^2), \\ u^* &= U(x,u;\epsilon) = u + \epsilon\eta(x,u) + 0(\epsilon^2), \end{aligned}\right\} \tag{1.41}$$

$i = 1, 2, \ldots, n.$

The infinitesimal generator in this case is

$$V = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}$$

and the k-th extension of (1.41) is given by

$$\left.\begin{aligned} x_i^* &= X_i(x,u;\epsilon) = x_i + \epsilon\xi_i(x,u) + 0(\epsilon^2), \\ u^* &= U(x,u;\epsilon) = u + \epsilon\eta(x,u) + 0(\epsilon^2), \\ u_i^* &= U_i(x,u,u_1;\epsilon) = u_i + \epsilon\eta_i^{(1)}(x,u,u_1) + 0(\epsilon^2), \\ &\vdots \\ u_{i_1 i_2 \ldots i_k}^* &= U_{i_1 i_2 \ldots i_k}(x,u,u_1,u_2,\ldots,u_k) \\ &= u_{i_1 i_2 \ldots i_k} + \epsilon\eta_{i_1 i_2 \ldots i_k}^{(k)}(x,u,u_1,\ldots,u_k) + 0(\epsilon^2), \end{aligned}\right\} \tag{1.42}$$

where

$$\begin{aligned}
i &= 1,2,\dots,n, \\
i_\lambda &= 1,2,\dots,n \quad \text{for} \quad \lambda = 1,2,\dots,k, \\
u_1 &= \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right), \\
u_2 &= \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \;\middle|\; 1 \le i \le j \le n\right), \\
&\vdots \\
u_k &= \text{the set of coordinates corresponding to all } k\text{-th order} \\
&\quad \text{partial derivatives of } u \text{ w.r.t. } x.
\end{aligned}$$

The k-th extension of the infinitesimal is

$$(\xi(x,u), \eta^{(1)}(x,u,u_1), \dots, \eta^{(k)}(x,u,u_1,\dots,u_k)),$$

and the corresponding k-th extended infinitesimal generator is

$$\begin{aligned}
V^{(k)} &= \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x,u,u_1)\frac{\partial}{\partial u_i} + \dots \\
&\quad + \eta_{i_1 i_2 \dots i_k}^{(k)} \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}}, \quad k = 1,2,\dots
\end{aligned}$$

The theorem below gives explicit formulae for the extended infinitesimals.

Theorem 1.3-5

$$\left.\begin{aligned}
\eta_i^{(1)} &= D_i\eta - (D_i\xi_j)U_j, \quad i = 1,2,\dots,n, \\
\eta_{i_1 i_2 \dots i_k}^{(k)} &= D_{i_k}\eta_{i_1 i_2 \dots i_{k-1}}^{(k-1)} - (D_{i_k}\xi_j)U_{i_1 i_2 \dots i_{k-1} j},
\end{aligned}\right\} \tag{1.43}$$

where

$$i_\lambda = 1,2,\dots,n \quad \text{for} \quad \lambda = 1,2,\dots,k \quad \text{with} \quad k = 2,3,\dots,$$

and

$$D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \dots + u_{i i_1 i_2 \dots i_n}\frac{\partial}{\partial u_{i_1 i_2 \dots i_n}} + \dots, \quad i = 1,2,\dots,n.$$

□

We remark here that the concept of transformations and extended infinitesimal transformations can easily be further extended to the situation of m dependent variables $u = (u^1, u^2, \ldots, u^m)$ and n independent variables $x = (x_1, x_2, \ldots, x_n)$ where $u = u(x)$ and $m \geq 2$.

Details can be found in Bluman and Kumei [1] and P. J. Olver [2].

1.4 Applications of Lie Groups to the Solutions of Partial Differential Equations

Lie pioneered the use of continuous groups of transformations in the study of symmetry properties of ODEs and PDEs with a view to their solutions. The basic idea in Lie's theory is to find groups whose elements transform solutions of a system of differential equations to other solutions of the system. Such are groups with respect to which the set of solutions of the system is invariant. We present, in this section, a brief summary of how to find a one-parameter Lie group of transformations admitted by Partial Differential Equations. For details see [1] and [2].

First we give the following definitions of invariant functions and invariant surfaces then state, without proofs, related theorems. From now on we make the assumption:

$$F(x) = F(x_1, \ldots, x_n)$$

is infinitely differentiable.

Definition 1.4-3

A curve $F(x, y) = 0$ is an *invariant curve* for (1.33) iff

$$F(x^*, y^*) = 0 \quad \text{when} \quad F(x, y) = 0.$$

□

The next two theorems give the definitions of invariant curves and surfaces in relation to a given infinitesimal generator.

Theorem 1.4-1

A surface written in a *solved* form $F(x) = x_n - f(x_1, x_2, \ldots, x_{n-1}) = 0$ is an invariant surface for (1.33) with (1.35) iff

$$VF(x) = 0 \quad \text{when} \quad F(x) = 0. \tag{1.44}$$

□

Theorem 1.4-2

A curve written in a solved form $F(x, y) = y - f(x) = 0$, is an invariant curve for (1.33) iff

$$\left.\begin{aligned} VF(x, y) = \eta(x, y) - \xi(x, y) f'(x) = 0 \\ \text{when} \quad F(x, y) = y - f(x) = 0, \end{aligned}\right\} \tag{1.45}$$

i.e. iff

$$\eta(x, f(x)) - \xi(x, f(x)f'(x)) = 0.$$

□

Definition 1.4-1

A function $F(x)$ is an *invariant function* of the Lie group of transformations (1.1) iff for any group transformation (1.1)

$$F(x^*) \equiv F(x).$$

□

Definition 1.4-2

A surface $F(x) = 0$ is an *invariant surface* for (1.1) iff $F(x^*) = 0$ when $F(x) = 0$. □

Definition 1.4-3

A curve $F(x, y) = 0$ is an *invariant curve* for a one-parameter Lie group of transformations (1.33) iff

$$F(x^*, y^*) = 0 \quad \text{when} \quad F(x, y) = 0.$$

□

As a result of Theorems 1.4-1 and 1.4-2 we can find the invariant surface of a given Lie group of transformations by solving (1.44). This is illustrated in the example below.

Example 1.4-1

Consider the scaling group

$$\left.\begin{aligned} x^* &= e^{\epsilon}x, \\ y^* &= e^{\epsilon}y, \end{aligned}\right\} \qquad (1.46)$$

whose infinitesimal generator is

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

A ray $y - \lambda x = 0,\ x > 0,\ \lambda = \text{constant}$, is an invariant curve for (1.46) since

$$V(y - \lambda x) = y - \lambda x = 0 \quad \text{when} \quad y - \lambda x = 0.$$

But a parabola

$$y - \lambda x^2 = 0, \quad \lambda = \text{constant},$$

is not an invariant curve for (1.46) since

$$V(y - \lambda x^2) = y - 2\lambda x^2 \neq 0 \quad \text{when} \quad y - \lambda x^2 = 0.$$

We now introduce the notion of the *invariance of a PDE.* Let a k-th order PDE be represented by

$$F(x, u, u_1, u_2, \ldots, u_k) = 0, \tag{1.47}$$

where $x = (x_1, x_2, \ldots, x_n)$ denotes n independent variables, u_j denotes the set of coordinates corresponding to all j-th order partial derivatives of u w.r.t. x. Thus

$$\begin{aligned} u_1 &= \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right), \\ u_2 &= \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \;\Big|\; 1 \leq i \leq j \leq n\right), \\ &\;\vdots \\ u_k &= \left(\frac{\partial^j u}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_j}} \Big|_{j=1,2,\ldots,k}^{i_j=1,2,\ldots,n}\right), \end{aligned} \tag{1.48}$$

with $i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, k$.

It is worth pointing out that in terms of the coordinates

$$x, u, u_1, u_2, \ldots, u_k$$

equation (1.47) becomes an algebraic equation which defines a hypersurface in

$$(x, u, u_1, u_2, \ldots, u_k) - \text{space}.$$

The definition of the invariance of a PDE (1.47) follows. We assume that the PDE (1.47) can be written in solved form in terms of some λ-th order partial derivative of u as:

$$F(x, u, u_1, u_2, \ldots, u_k) = u_{i_1 i_2 \ldots i_\lambda} - f(x, u, u_1, u_2, \ldots, u_k) = 0 \tag{1.49}$$

where $f(x, u, u_1, u_2, \ldots, u_k)$ does not depend on $u_{i_1, i_2 \ldots i_\lambda}$.

Definition 1.4-4

The one-parameter Lie group of transformations

$$\left.\begin{aligned} x^* &= X(x,u;\epsilon), \\ u^* &= U(x,u;\epsilon), \end{aligned}\right\} \tag{1.50}$$

leaves PDE (1.47) *invariant* iff its k-th extension, defined by (1.41) and (1.42), leaves the surface (1.47) invariant.

It is generally known, see Bluman and Kumei [1], that for any solution

$$u = \theta(x)$$

of PDE (1.47), the equation

$$(x, u, u_1, u_2, \ldots, u_k) = (x, \theta(x), \theta_1(x), \theta_2(x), \ldots, \theta_k(x))$$

defines a solution surface which lies on the surface (1.47).

The theorem below gives the infinitesimal criterion for invariance of a PDE.

Theorem 1.4-3

Let the infinitesimal generator of (1.50) be

$$V = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} \tag{1.51}$$

and let the k-th extended infinitesimal generator of (1.51) be

$$\begin{aligned} V^{(k)} &= \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x,u,u_1)\frac{\partial}{\partial u_i} + \ldots \\ &\quad + \eta_{i_1 i_2 \ldots i_k}^{(k)}(x,u,u_1,u_2,\ldots,u_k)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}}, \end{aligned}$$

where $\eta_i^{(1)}$ and $\eta_{i_1 i_2 \ldots i_j}^{(j)}$ are given by (1.43) in terms of $(\xi(x,u), \eta(x,u))$. [$\xi(x,u)$ denotes $(\xi_1(x,u), \xi_2(x,u), \ldots, \xi_n(x,u))$].

Then (1.50) is admitted by PDE (1.47) if and only if

$$V^{(k)}F(x,u,u_1,u_2,\ldots,u_k) = 0 \quad \text{when} \quad F(x,u,u_1,\ldots,u_k) = 0. \tag{1.52}$$

□

The next example (1.4-2) illustrates how to find the Lie groups of transformations admitted by Partial Differential Equations.

Example 1.4-2

Let us now find the Lie groups of transformations admitted by the PDE

$$u_{tt} = [f(u)u_x]_x, \tag{1.53}$$

where $u = u(t,x),\ f : \mathbf{R} \to \mathbf{R},\ f \in C^2,\ f' \neq 0,\ f > 0.$

This example has been discussed in detail in [2]. Equation (1.53) can be written as

$$u_{tt} = f'(u)u_x^2 + f(u)u_{xx}. \tag{1.54}$$

Using the notations $u_t = u_1,\ u_x = u_2,\ u_{tt} = u_{11},\ u_{tx} = u_{xt} = u_{12},\ u_{xx} = u_{22}$ and $f(u) = f$ (1.54) now becomes

$$u_{11} = f'(u_2)^2 + fu_{22}. \tag{1.55}$$

The required Lie groups of transformations will be of the form

$$t^* = T(t,x,u;\epsilon), \quad x^* = X(t,x,u;\epsilon), \quad u^* = U(t,x,u;\epsilon), \tag{1.56}$$

with the corresponding infinitesimals

$$\xi(t,x,u) = \frac{\partial T(t,x,u,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0},\ \theta(t,x,u) = \frac{\partial X(t,x,u,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0},\ \eta(t,x,u) = \frac{\partial U(t,x,u,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}.$$

Prolongations of (1.56) with $n = 2$ will be of the form

$$\begin{aligned} u_1^* &= U_1(t,x,u,u_1,u_2,\epsilon),\\ u_2^* &= U_2(t,x,u,u_1,u_2,\epsilon),\\ u_{11}^* &= U_{11}(t,x,u,u_1,u_2,u_{11},u_{12},u_{22},\epsilon),\\ u_{12}^* &= U_{12}(\ldots\ldots),\\ u_{22}^* &= u_{22}(\ldots\ldots). \end{aligned}$$

The infinitesimal generator of (1.56) is

$$V = \xi(t,x,u)\frac{\partial}{\partial t} + \theta(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

with the once and twice extended generators respectively as

$$V^{(1)} = V + \eta_1^{(1)}(t, x, u, u_1, u_2)\frac{\partial}{\partial u_1} + \eta_2^{(1)}(\ldots\ldots)\frac{\partial}{\partial u_2}$$

and

$$\begin{aligned} V^{(2)} &= V^{(1)} + \eta_{11}^{(2)}(t, x, u, u_1, u_2, u_{11}, u_{12}, u_{22})\frac{\partial}{\partial u_{11}} + \eta_{12}^{(2)}(\ldots\ldots)\frac{\partial}{\partial u_{12}} \\ &\quad + \eta_{22}^{(2)}(\ldots\ldots)\frac{\partial}{\partial u_{22}}, \end{aligned}$$

where $\eta_1^{(1)}, \eta_2^{(1)}, \eta_{11}^{(2)}$ and $\eta_{22}^{(2)}$ are known functions of the derivatives of ξ, θ, η and the u_1, u_2, u_{11}, u_{12} and u_{22}.

From (1.55)

$$F = u_{11} - f'(u_2)^2 - fu_{22}.$$

By Theorem 1.4-3

$$V^{(2)}F = V^{(2)}(u_{11} - f'(u_2)^2 - fu_{22}) = 0 \quad \text{when} \quad F = 0. \tag{1.57}$$

From (1.43) with $x_1 = t, x_2 = x, \xi_1 = \xi$ and $\xi_2 = \theta$ we have

$$\begin{aligned} \eta_2^{(1)} &= \eta_x + [\eta_u - \theta_x]u_2 - \xi_x u_1 - \theta_u (u_2)^2 - \xi_u u_1 u_2, \\ \eta_{11}^{(2)} &= \eta_{t^2} + [2\eta_{tu} - \xi_{t^2}]u_1 - \theta_{t^2} u_2 + [\eta_u - 2\xi_t]u_{11} \\ &\quad -2\theta_t u_{12} + [\eta_{u^2} - 2\xi_{tu}](u_1)^2 - 2\theta_{tu} u_1 u_2 - \xi_{u^2}(u_1)^3 \\ &\quad -\theta_{u^2}(u_1)^2 u_2 - 3\xi_u u_1 u_{11} - \theta_u u_2 u_{11} - 2\theta_u u_1 u_{12}, \\ \eta_{22}^{(2)} &= \eta_{x^2} + [2\eta_{xu} - \theta_{x^2}]u_2 - \xi_{x^2} u_1 + [\eta_u - 2\theta_x]u_{22} \\ &\quad -2\xi_x u_{12} + [\eta_{u^2} - 2\theta_{xu}](u_2)^2 - 2\xi_{xu} u_1 u_2 - \theta_{u^2}(u_2)^3 \\ &\quad -\xi_{u^2} u_1 (u_2)^2 - 3\theta_u u_2 u_{22} - \xi_u u_1 u_{22} - 2\xi_u u_2 u_{12}. \end{aligned} \tag{1.58}$$

Substituting (1.58) into (1.57) yields

$$
\begin{aligned}
\eta_{t^2} - f\eta_{x^2} &= 0, && (i)\\
2f'\xi_x + 2\eta_{tu} - \xi_{t^2} + f\xi_{x^2} &= 0, && (ii)\\
-2\xi_t f' + 2f'\theta_x - f\eta u^2 - f''\eta - f'\eta_u - \theta_{t^2} - 2f\theta_{xu} &= 0, && (iii)\\
2f'\xi_u + 2f\xi_{xu} - 2\theta_{tu} &= 0, && (iv)\\
\eta_{u^2} - 2\xi_{tu} &= 0, && (v)\\
\eta_u - 2\xi_t &= 0, && (vi)\\
f'\theta_u + 2f\theta_{x^2} - f\eta_{u^2} &= 0, && (vii)\\
2f\theta_x - f'\eta - f\eta_u &= 0, && (viii)\\
f\xi_x - \theta_t &= 0, && (ix)\\
\theta_u &= 0, && (x)\\
\xi_u &= 0. && (xi)
\end{aligned}
$$

Remark 1.4-1

Since (1.58) has to be identically zero for all values of $u_1, u_2, u_{11}, u_{12}, u_{22}, u_1^2, u_2^2$, $u_1u_2, \ldots$, all coefficients of these terms must vanish and thus we end up with equations (i) - (xi) above. □

We shall only compute the groups which are true for arbitrary f.

From (x) and (xi), it is obvious that

$$\xi = \xi(t, x) \quad \text{and} \quad \theta = \theta(t, x).$$

From (iv) and (ix),

$$\xi = \xi(t) \quad \text{and} \quad \theta = \theta(x).$$

From (v),

$$
\begin{aligned}
& \eta_{u^2} = 0 \\
\Longrightarrow\ & \eta_u \text{ is a function of } t, \text{ from (vi).} \\
\Longrightarrow\ & \eta \text{ is linear in } u \text{ and } \xi.
\end{aligned}
$$

If we let

$$\xi = c_1 t + c_2, \quad c_1, c_2 \text{ constants,} \tag{1.59}$$

then for arbitrary f,

$$\eta = 0. \tag{1.60}$$

From (vii),

$$
\begin{aligned}
& f'\theta_u + 2f\theta_{x^2} - f\eta_{u^2} = 0 \\
\Longrightarrow\ & 2f\theta_{x^2} = 0 \\
\Longrightarrow\ & \theta_{x^2} = 0 \\
\Longrightarrow\ & \theta \text{ is also linear in } x.
\end{aligned}
$$

We can choose

$$\theta = c_4 x + c_5, \quad c_4, c_5 \quad \text{constants.} \tag{1.61}$$

Using (1.59) – (1.61) we can now determine the functions T, X and U in (1.56). Thus

$$\left.\begin{aligned}
\xi(t) &= c_1 t + c_2, \\
\theta(x) &= c_4 x + c_5, \\
\eta(u) &= 0,
\end{aligned}\right\} \tag{1.62}$$

where c_2, c_2, c_3, c_4 and c_5 are arbitrary constants. The infinitesimal generators

$$V = \xi\frac{\partial}{\partial t} + \theta\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}$$

with (1.62) substituted becomes

$$\left.\begin{aligned}
V_1 &= c_1 t\frac{\partial}{\partial t} + 0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial u}, \\
V_2 &= c_2\frac{\partial}{\partial t} + 0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial u}, \\
V_3 &= 0\frac{\partial}{\partial t} + c_4 x\frac{\partial}{\partial x} + 0\frac{\partial}{\partial u}, \\
V_4 &= 0\frac{\partial}{\partial t} + c_5\frac{\partial}{\partial x} + 0\frac{\partial}{\partial u}.
\end{aligned}\right\} \tag{1.63}$$

We use the five generators V_1, V_2, V_3, V_4 and V_5 admitted by the PDE (1.53) to find the Lie groups of transformations admitted by (1.53).

The systems resulting from (1.63) are:

$$
\begin{aligned}
V_1 :\quad & \frac{dt(\epsilon)}{d\epsilon} = c_1 t(\epsilon), && t(0) = t, \\
& \frac{dx(\epsilon)}{d\epsilon} = 0, && x(0) = x, \\
& \frac{du(\epsilon)}{d\epsilon} = 0, && u(0) = u.
\end{aligned}
$$

which gives the group

$$t^* = te^{c_1\epsilon}, \quad x^* = x, \quad u^* = u. \tag{1.64}$$

Similarly

$$\left.\begin{aligned} V_2 : t^* &= t + \epsilon c_2, \quad x^* = x, \quad u^* = u, \\ V_3 : t^* &= t, \quad x^* = xe^{c_4\epsilon}, \quad u^* = u, \\ V_4 : t^* &= t, \quad x^* = x + \epsilon c_5, \quad u^* = u. \end{aligned}\right\} \tag{1.65}$$

□

Remark 1.4-2

Since the constants c_1, c_2, c_4 and c_5 are arbitrary the 4-parameter Lie groups of transformations (1.65) acting on (t, x, u)-space admitted by equation (1.53) are nontrivial. □

Chapter 2

Stability of Lie Groups

Source [1] N.Kh. Ibragimov, V.A. Baikov and R.K. Gazizo, Approximate Symmetries, Matem. Sbornik, TOM 136(178) (1988). Vip. 3 (in Russian).
English translation, Math, USSR Sbornik, Vol. 64 (1989) No. 2 (427-441).

The theory of stability of Lie Groups, the study of which this chapter is devoted to, was started by N.Kh. Ibragimov [1] around 1988. He was a student of L.V. Ovsyannikov who had mentioned it as early as the 1960's. Ibragimov's theory was developed for approximate group analysis of differential equations with a small parameter (ϵ). The theory, however, is valid even for large parameters (ϵ), and this makes the term "approximate" in the title [1] inappropriate. This fact was noticed by E.E. Rosinger and brought to the attention of N.Kh. Ibragimov, who concurred with the appropriateness of the new, better name *"Stability of Lie Groups"* or *"Stability of Symmetries"*.

2.1 Introduction

Families of differential equations (depending on arbitrary parameters or functions) can be classified according to their symmetry groups i.e. given a Lie group of transformations

$$x^* = X(x, \epsilon), \tag{2.1}$$

where (2.1) is as in equation (1.1) of Chapter 1, we can use the infinitesimal methods to determine which types of differential equations admit the group.

Unfortunately, any small pertubation of an equation disturbs the groups admitted by that equation and this reduces the practical value of the equation and of group theoretic methods in general. There is a need, therefore, to work out group analysis methods that are stable under small, or eventually, a class of more arbitrary perturbations of the differential equations involved. This fact is illustrated in example 2.1-1 below.

Example 2.1-1

Consider the linear non-homogeneous O.D.E.

$$y' + p(x)y = g(x) \tag{2.2}$$

which admits the one-parameter (a) Lie group of transformations

$$\left.\begin{aligned} x^* &= x, \\ y^* &= y + au(x), \quad a \in \mathbf{R}, \end{aligned}\right\} \tag{2.3}$$

where $u(x)$ is a particular solution of the associated homogeneous equation

$$u' + p(x)u = 0.$$

If we let

$$p(x,\epsilon) = x^{\epsilon-1}, \quad x \in (0,\infty), \quad \epsilon \in \mathbf{R}, \tag{2.4}$$

with

$$p(x,0) = \frac{1}{x}, \tag{2.5}$$

then

$$u(x,\epsilon) = e^{-x^{\epsilon}/\epsilon}, \quad \epsilon \neq 0,$$

and

$$u(x,0) = \frac{1}{x}.$$

Putting (2.4), (2.5) into (2.2) we have the following unperturbed and perturbed equations respectively:

$$y' + \frac{1}{x}y = g(x), \quad x \in (0,\infty), \tag{2.6}$$

$$y' + x^{\epsilon-1}y = g(x), \quad \epsilon \in \mathbf{R}. \tag{2.7}$$

It is easy to see that the groups admitted by the unperturbed equation (2.6) are:

$$\left.\begin{aligned} x^* &= x, \\ y^* &= y + \frac{a}{x}, \quad a \in \mathbf{R}, \quad x \in (0, \infty). \end{aligned}\right\} \tag{2.8}$$

and those admitted by the perturbed equation (2.7) are:

$$\left.\begin{aligned} x^*_\epsilon &= x, \\ y^*_\epsilon &= y + ae^{-x^\epsilon/\epsilon}, \quad \epsilon \neq 0. \end{aligned}\right\} \tag{2.9}$$

Remark 2.1-1

It is worth pointing out here that the variation between groups (2.9) and (2.8) admitted by equations (2.7) (perturbed) and (2.6) (unperturbed), respectively, is quite large despite the fact that the perturbation may be small. To see this we observe that

$$y^*_\epsilon \not\longrightarrow y^* \quad \text{as} \quad \epsilon \longrightarrow 0$$

although

$$\text{equation (2.7)} \longrightarrow \text{equation (2.6)} \quad \text{as} \quad \epsilon \longrightarrow 0.$$

Notation 2.1-1

The following notation is used throughout this chapter and subsequent chapters, unless otherwise stated:

$z = (z^1, z^2, \ldots, z^N)$ is the independent variable.

ϵ is a parameter.

$\theta_p(z, \epsilon)$ denotes an infinitesimally small function of order ϵ^{p+1}, $p \geq 0$, i.e. $\theta_p(z, \epsilon) = 0(\epsilon^p)$.

Assumptions 2.1-1

We make the following assumptions from now on:

(a) All functions are jointly analytic in their arguments.

(b) $\exists$ constant $c > 0$ such that $|\theta_p(z, \epsilon)| \leq c|\epsilon|^{p+1}$, with c independent of z and ϵ when z is bounded.

(c) The approximate equality $f \approx g$ means the equality $f(z,\epsilon) = g(z,\epsilon) + 0(\epsilon^p)$ for some fixed value of $p \geq 0$.

Theorem 2.1-1

Suppose the functions $f(z,\epsilon)$ and $\tilde{f}(z,\epsilon)$ are analytic in the neighbourhood of the point $(z_0, 0)$ and satisfy the condition

$$\tilde{f}(z,\epsilon) = f(z,\epsilon) + 0(\epsilon^p). \tag{2.10}$$

Let $z = z(t,\epsilon)$ and $\tilde{z} = \tilde{z}(t,\epsilon)$ be the respective solutions of the problems

$$\frac{dz}{dt} = f(z,\epsilon), \quad z|_{t=0} = \alpha(\epsilon),$$

and

$$\frac{d\tilde{z}}{dt} = \tilde{f}(\tilde{z},\epsilon), \quad \tilde{z}|_{t=0} = \tilde{\alpha}(\epsilon),$$

where $\tilde{\alpha}(0) = \alpha(0) = z_0$ and

$$\tilde{\alpha}(\epsilon) = \alpha(\epsilon) + 0(\epsilon^p),$$

then

$$\tilde{z}(t,\epsilon) = z(t,\epsilon) + 0(\epsilon^p), \tag{2.11}$$

i.e. $\tilde{z}(t,\epsilon)$ and $z(t,\epsilon)$ coincide to within $0(\epsilon^p)$. □

We now consider the approximate Cauchy problem

$$\left.\begin{aligned} \frac{dz}{dt} &\approx f(z,\epsilon), \\ z|_{t=0} &\approx \alpha(\epsilon), \end{aligned}\right\} \tag{2.12}$$

where (2.12) is understood to be a family of differential equations

$$\left.\begin{aligned} \frac{dz}{dt} &= \tilde{f}(z,\epsilon) \quad \text{with} \quad \tilde{f}(z,\epsilon) \approx f(z,\epsilon), \\ z|_{t=0} &= \tilde{\alpha}(\epsilon) \quad \text{with} \quad \tilde{\alpha}(\epsilon) \approx \alpha(\epsilon), \end{aligned}\right\} \tag{2.13}$$

i.e. $\tilde{f}(z,\epsilon) = f(z,\epsilon) + 0(\epsilon^p)$ and $\tilde{\alpha}(\epsilon) = \alpha(\epsilon) + 0(\epsilon^p)$.

By Theorem 2.1-1, the solution of the approximate Cauchy problem (2.12) coincides with the solution of (2.13) to within $0(\epsilon^p)$.

2.2 One-Parameter Approximate Groups

The definition of one-parameter approximate groups takes the form of that of one-parameter groups but with the exact equalities replaces by approximate equalities with the meaning of $f \approx g$ as in Notation 2.1-1.

Definition 2.2-1

The transformations

$$z^* \approx f(z, \epsilon, a), \tag{2.14}$$

where

$$\begin{aligned} f : \mathbf{C}^N \times I \times \mathbf{R} &\longrightarrow \mathbf{C}^N, \quad 0 \in I \subseteq \mathbf{R} \\ (z, \epsilon, a) &\longmapsto z^* \approx f(z, \epsilon, a), \end{aligned}$$

form an *approximate one-parameter group w.r.t. a* if

$$f(z, \epsilon, 0) \approx z, \tag{2.15}$$

$$f(f(z, \epsilon, a), \epsilon, b) \approx f(z, \epsilon, a + b), \tag{2.16}$$

and

$$f(z, \epsilon, a) \approx z \ \forall z \implies a = 0. \tag{2.17}$$

□

Next we give an approximate Lie Theorem.

Theorem 2.2-1

Suppose that the transformations (2.14) form an approximate group with the infinitesimal

$$\xi(z, \epsilon) \approx \frac{\partial f(z, \epsilon, a)}{\partial a}\Big|_{a=0}.$$

Then the function $f(z, \epsilon, a)$ satisfies

$$\frac{\partial f(z, \epsilon, a)}{\partial a} \approx \xi(f(z, \epsilon, a), \epsilon).$$

□

The next theorem is a converse of Theorem 2.2-1.

Theorem 2.2-2

For any (smooth) function $\xi(z, \epsilon)$, the solution (2.14) of the approximate Cauchy problem

$$\left.\begin{aligned} \frac{dz^*}{da} &\approx \xi(z^*, \epsilon), \\ z^* \Big|_{a=0} &\approx z, \end{aligned}\right\} \tag{2.18}$$

determines an approximate one-parameter group with group parameter a. □

Remark 2.2-1

Equation $(2.18)_1$ will be called the approximate Lie equation.

2.3 Constructing an Approximate Group

This section contains a brief summary of how to construct an approximate group from a given infinitesimal generator. It is worth pointing out that by Theorem 2.2-2, it suffices to solve the approximate Lie equation (2.18) since its solution (2.14) determines an approximate one-parameter group. We show how to solve the approximate Lie equation (2.18).

First, let us consider the case of $p = 1$.

The task is to find the approximate group of transformations

$$z^* \approx f_0(z,a) + \epsilon f_1(z,a) \tag{2.19}$$

determined by the infinitesimal generator

$$V = (\xi_0(z) + \epsilon\xi_1(z))\frac{\partial}{\partial z}. \tag{2.20}$$

The problem restated is: Can we find (2.19) when (2.20) is known?

From (2.19) we get the corresponding approximate Lie equation

$$\begin{aligned}\frac{dz^*}{da} &\approx \frac{d}{da}(f_0(z,a) + \epsilon f_1(z,a)) \\ &= \xi_0(f_0 + \epsilon f_1) + \epsilon\xi_1(f_0 + \epsilon f_1) + 0(\epsilon).\end{aligned}$$

Thus

$$\frac{df_0}{da} + \epsilon\frac{df_1}{da} = \xi_0(f_0 + \epsilon f_1) + \epsilon\xi_1(f_0 + \epsilon f_1) + 0(\epsilon). \tag{2.21}$$

For $\epsilon = 0$ we have

$$\frac{df_0}{da} = \xi_0(f_0),$$

and equation (2.21) now becomes

$$\epsilon\frac{df_1}{da} = -\xi_0(f_0) + \xi_0(f_0 + \epsilon f_1) + \epsilon\xi_1(f_0 + \epsilon f_1) + 0(\epsilon). \tag{2.22}$$

Taking the Taylor series of $\xi_0(f_0 + \epsilon f_1)$ and $\xi_1(f_0 + \epsilon f_1)$, we write equation (2.22) in the form

$$\epsilon\frac{df_1}{da} = -\xi_0(f_0) + \xi_0(f_0) + \epsilon\xi_0' f_1 + \epsilon\xi_1(f_0) + \epsilon^2\xi_1' f_1 + 0(\epsilon), \tag{2.23}$$

where ξ_0' is the derivative of ξ_0.

Dividing (2.23) by ϵ gives

$$\frac{df_1}{da} = \xi_0' f_1 + \xi_1(f_0) + \epsilon\xi_1' f_1 + \frac{0(\epsilon)}{\epsilon}.$$

Taking the limit as $\epsilon \to 0$, we have

$$\frac{df_1}{da} = \xi_1'(f_0) f_1 + \xi_1(f_0). \tag{2.24}$$

The initial condition $(2.18)_2$ gives

$$f_0|_{a=0} = z, \quad f_1|_{a=0} = 0. \tag{2.25}$$

From equations (2.22), (2.24) and (2.25) we have the exact Cauchy problem

$$\left.\begin{aligned} \frac{df_0}{da} &= \xi_0(f_0), \\ \frac{df_1}{da} &= \xi_0'(f_0) f_1 + \xi_1(f_0), \end{aligned}\right\} \tag{2.26}$$

with the initial conditions

$$f_0|_{a=0} = z, \quad f_1|_{a=0} = 0. \tag{2.27}$$

Thus by Theorem 2.2-2, to construct the approximate group (2.19), to within $0(\epsilon)$ from the given infinitesimal generator (2.20), it suffices to solve the exact Cauchy problem (2.26).

The concept will be understood better after the following examples.

Example 2.3-1

Given that $N = 1$, construct an approximate group of transformations

$$x^* \approx f_0(x, a) + \epsilon f_1(x, a)$$

determined by the infinitesimal generator

$$V = (1 + \epsilon x)\frac{\partial}{\partial x}. \tag{2.28}$$

Comparing (2.28) and (2.20) with $z = x$, we have

$$\begin{aligned}
\xi_0(x,a) &= \frac{\partial f_0(x,a)}{\partial a}\Big|_{a=0} = 1 \\
\xi_1(x,a) &= \frac{\partial f_1(x,a)}{\partial a}\Big|_{a=0} = x.
\end{aligned}$$

From (2.26) and (2.27) we have

$$\begin{aligned}
\frac{df_0}{da} &= \xi_0(f_0) = 1 \\
\Longrightarrow f_0(x,a) &= a + c \quad \text{and by (2.25),} \quad c = x. \\
\Longrightarrow f_0(x,a) &= a + x.
\end{aligned}$$

Also

$$\begin{aligned}
\frac{df_1}{da} &= \xi_0'(f_0)f_1 + \xi_1(f_0), \quad f_0|_{a=0} = x, \quad f_1|_{a=0} = 0 \\
&= 0 + \xi_1(f_0) = f_0 = a + x \\
\Longrightarrow f_1(x,a) &= ax + \frac{1}{2}a^2.
\end{aligned}$$

Example 2.3-2

We now construct the approximate group of transformations

$$\left.\begin{aligned}
x^* &\approx f_0^1(x,y,a) + \epsilon f_1^1(x,y,a), \\
y^* &\approx f_0^2(x,y,a) + \epsilon f_1^2(x,y,a),
\end{aligned}\right\} \tag{2.29}$$

determined by the operator

$$V = (1 + \epsilon x^2)\frac{\partial}{\partial x} + \epsilon xy\frac{\partial}{\partial y} \tag{2.30}$$

in the (x,y)-plane.

Here $N = 2$, $\xi_0 = (1,0)$ and $\xi_1 = (x^2, xy)$. Thus

$$\begin{aligned}\xi_0^1(f_0^1) &= 1, \quad \xi_0^2(f_0^2) = 0,\\ \xi_1^1(f_0^1) &= (f_0^1)^2 \quad \text{and} \quad \xi_1^2(f_0^2) = f_0^1 f_0^2.\end{aligned}$$

Thus the exact Cauchy problem (2.26) now becomes

$$\left.\begin{aligned}\frac{df_0^1}{da} &= 1,\\ \frac{df_0^2}{da} &= 0,\\ \frac{df_1^1}{da} &= \frac{\partial \xi_0^1(f_0^1)}{\partial x} f_1^1 + \xi_1^1(f_0^1) = 0 + \xi_1^1(f_0^1) = (f_0^1)^2,\\ \frac{df_1^2}{da} &= \frac{\partial \xi_0^2(f_0^2)}{\partial y} f_1^2 + \xi_1^2(f_0^2) = 0 + \xi_1^2(f_0^2) = f_0^1 f_0^2,\end{aligned}\right\} \tag{2.31}$$

with the initial conditions

$$f_0^1|_{a=0} = x, \quad f_0^2|_{a=0} = y, \quad f_1^1|_{a=0} = 0, \quad f_1^2|_{a=0} = 0. \tag{2.32}$$

Solving (2.31) we get

$$\begin{aligned}f_0^1 &= a + x,\\ f_0^2 &= y,\\ f_1^1 &= ax^2 + a^2x + \frac{a^3}{3},\\ f_1^2 &= axy + \frac{1}{2}a^2y.\end{aligned}$$

Thus the approximate group of transformations (2.29) now becomes

$$\left.\begin{aligned}x^* &\approx x + a + \epsilon(x^2a + xa^2 + \frac{a^2}{3}),\\ y^* &\approx y + \epsilon(xya + y\frac{a^2}{2}).\end{aligned}\right\} \tag{2.33}$$

The method of constructing an approximate group to within $0(\epsilon)$ (i.e. when $p = 1$) discussed above may easily be extended to the case when p is arbitrary (i.e. to within $0(\epsilon^p)$). We only give a brief summary. See [1] for the detail.

The aim is to construct the approximate group of transformations

$$z^* \approx f_0(z,a) + \epsilon f_1(z,a) + \ldots + \epsilon^p f_p(z,a) \tag{2.34}$$

determined by the infinitesimal generator

$$V = [\xi_0(z) + \epsilon\xi_1(z) + \ldots + \epsilon^p \xi_p(z)]\frac{\partial}{\partial z}.$$

By Theorem 2.2-2, (2.34) is determined by the corresponding approximate Lie equation

$$\frac{d}{da}(f_0 + \epsilon f_1 + \ldots + \epsilon^p f_p) \approx \sum_{i=0}^{p} \epsilon^i \xi_i(f_0 + \epsilon f_1 + \ldots + \epsilon^p f_p),$$

which is equivalent to solving the exact Cauchy problem

$$\frac{df_0}{da} = \xi_0(f_0), \tag{2.35}$$

$$\frac{df_i}{da} = \xi_i(f_0) + \sum_{j=1}^{i} \sum_{|\sigma|=1}^{j} \frac{1}{\sigma!}\xi_{i-j}^{(\sigma)}(f_0) \sum_{|\nu|=j} f_{(\nu)}, \quad i = 1, \ldots, p, \tag{2.36}$$

under the initial conditions

$$f_0\big|_{a=0} = z, \quad f_i\big|_{a=0} = 0, \quad i = 1, 2, \ldots, p, \tag{2.37}$$

where

$$\begin{aligned} \sigma &= (\sigma_1, \sigma_2, \ldots, \sigma_N) \quad \text{is a multi-index,} \\ |\sigma| &= \sigma_1 + \sigma_2 + \ldots + \sigma_N, \\ \sigma! &= \sigma_1!\sigma_2!\ldots\sigma_N!, \\ &\quad \text{and the indices } \sigma_1, \sigma_2, \ldots, \sigma_N \text{ run from 0 to } p. \end{aligned}$$

$\nu = \nu(\sigma) = (\nu_1, \nu_2, \ldots, \nu_N)$ is a multi-index associated with the multi-index σ in such a way that if the index σ_s in σ is equal to zero, then the corresponding index ν_s is absent in ν, and each of the remaining indices ν_k takes values from σ_k to p; for example, for

$$\sigma = (0, \sigma_2, \sigma_3, 0, 0, \ldots, 0) \quad \text{with} \quad \sigma_2, \sigma_3 \neq 0$$

we have that

$$\nu = (\nu_2, \nu_3) \quad \text{so that} \quad y_{(\nu)} = y^2_{(\nu_2)} y^3_{(\nu_3)}.$$

For $i = 1$ equation (2.36) becomes

$$\frac{df_1}{da} = \sum_{k=1}^{N} \frac{\partial \xi_0(f_0)}{\partial z^k} f_1^k + \xi_1(f_0). \tag{2.38}$$

Remark 2.3-1

We remark here that since $i = 1$, we have $j = 1$ and the first summation in (2.36) disappears, leaving

$$\frac{df_1}{da} = \sum_{|\sigma|=1} \frac{1}{\sigma!} \xi_0^{(\sigma)}(f_0) \sum_{|\nu|=1} f_{(\nu)} + \xi_1(f_0). \tag{2.39}$$

The condition $|\sigma| = 1$ implies that the possible values of σ are

$$(1,0,0,\ldots,0),\ (0,1,\ldots,0),\ (0,0,1,\ldots,0),\ldots,(0,0,\ldots,0,1),$$

from whence

$$\nu = (\nu_1,\ldots,\nu_N) \quad \text{with} \quad |\nu| = \nu_1 + \nu_2 + \ldots + \nu_n = 1.$$

Thus

$$\nu = (\nu_1), \quad \sigma_1 = 1 \leq \nu_1 \leq N$$

and

$$\sigma! = \sigma_1!\sigma_2!\ldots\sigma_N! = 1.$$

It is now easy to see that (2.36) becomes (2.38) for $i = 1$. Similarly, for $i = 2$ equation (2.36) becomes

$$\frac{df_2}{da} = \sum_{k=1}^{N} \frac{\partial \xi_0(f_0)}{\partial z^k} f_2^k + \frac{1}{2} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{\partial^2 \xi_0(f_0)}{\partial z^k \partial z^\ell} f_1^k f_1^\ell + \sum_{k=1}^{N} \frac{\partial \xi_1(f_0)}{\partial z^k} f_1^k + \xi_2(f_0). \tag{2.40}$$

Example 2.3-3

We construct the approximate group to within $0(\epsilon^2)$ determined by the operator (2.30) in Example 2.3-2.

The approximate group is of the form

$$\left.\begin{aligned} x^* &\approx f_0^1(x,y;a)+\epsilon f_1^1(x,y;a)+\epsilon^2 f_2^1(x,y;a),\\ y^* &\approx f_0^2(x,y,a)+\epsilon f_1^2(x,y,a)+\epsilon^2 f_2^2(x,y,a). \end{aligned}\right\} \tag{2.41}$$

We only need to find f_2^1 and f_2^2 since f_0^1, f_1^1, f_0^2 and f_1^2 are known from Example 2.3-2. Observe here that

$$N=2, \quad z=(z^1,z^2)=(x,y), \quad f_k=(f_k^1,f_k^2), \quad k=0,1,2,$$

$$\xi_0=(1,0), \quad \xi_1=(x^2,xy) \quad \text{and} \quad \xi_\ell=0 \quad \text{for} \quad \ell\geq 2.$$

By (2.40),

$$\begin{aligned} \frac{df_2^1}{da} &= \frac{\partial\xi_0^1(f_0^1)}{\partial x}f_2^1+\frac{\partial\xi_0^1(f_0^1)}{\partial y}f_2^2+\\ &+\frac{1}{2}\left[\frac{\partial^2\xi_0^1(f_0^1)}{\partial x\partial x}f_1^1f_1^1+\frac{\partial^2\xi_0^1(f_0^1)}{\partial x\partial y}f_1^1f_1^2+\frac{\partial^2\xi_0^1(f_0^1)}{\partial y\partial x}f_1^2f_1^1+\frac{\partial^2\xi_0^1(f_0^1)}{\partial y\partial y}f_1^2f_1^2\right]\\ &+\frac{\partial\xi_1^1(f_0^1)}{\partial x}f_1^1+\frac{\partial\xi_1^1(f_0^1)}{\partial y}f_1^2+\xi_2^1(f_0^1)\\ &= 0+0+\frac{1}{2}[0]+2f_0^1f_1^1+0+0\\ &= 2f_0^1f_1^1. \end{aligned} \tag{2.42}$$

Similarly

$$\frac{df_2^2}{da}=f_0^2f_1^1+f_0^1f_1^2. \tag{2.43}$$

The exact Cauchy problem (2.26) now becomes (2.31) and (2.42) – (2.43), with the initial conditions (2.32) and

$$f_2^1|_{a=0}=f_2^2|_{a=0}=0.$$

Solving (2.42) and (2.43) we get

$$\begin{aligned} f_2^1 &= x^3a^2+\frac{2}{3}x^2a^3+\frac{2}{3}xa^4+\frac{2}{3}xa^3+\frac{2}{15}a^5\\ \text{and}\quad f_2^2 &= x^2ya^2+\frac{5}{6}xya^3+\frac{5}{24}ya^4. \end{aligned}$$

Thus (2.41) now becomes

$$\left.\begin{aligned} x^* &\approx x + a + \epsilon(x^2a + xa^2 + \frac{a^3}{a}) + \epsilon^2(x^3a^2 + \frac{2}{3}x(a^4 + a^3) + \frac{2}{15}a^5), \\ y^* &\approx y + \epsilon(xya + y\frac{a^2}{2}) + \epsilon^2(x^2ya^2 + \frac{5}{6}xya^3 + \frac{5}{24}a^4y). \end{aligned}\right\} \tag{2.44}$$

For further clarification of (2.36) we write out the equations for $i = 3$ and $i = 4$:

$$\begin{aligned} \frac{df_3}{da} &= \sum_{k=1}^{N} \frac{\partial \xi_0(f_0)}{\partial z^k} f_3^k + \sum_{k=1}^{N} \frac{\partial \xi_1(f_0)}{\partial z^k} f_1^k + \sum_{k=1}^{N} \frac{\partial \xi_2(f_0)}{\partial z^k} f_2^k + \\ &+ \frac{1}{2} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{\partial^2 \xi_0(f_0)}{\partial z^k \partial z^\ell} f_1^k f_1^\ell + \frac{1}{2} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{\partial^2 \xi_1(f_0)}{\partial z^k \partial z^\ell} f_1^k f_1^\ell + \\ &+ \frac{1}{3!} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \sum_{m=1}^{N} \frac{\partial^3 \xi_0(f_0)}{\partial z^k \partial z^\ell \partial z^m} f_1^k f_1^\ell f_1^m + \xi_3(f_0), \end{aligned} \tag{2.45}$$

$$\begin{aligned} \frac{df_4}{da} &= \sum_{k=1}^{N} \frac{\partial \xi_0(f_0)}{\partial z^k} f_4^k + \sum_{k=1}^{N} \frac{\partial \xi_1(f_0)}{\partial z^k} f_1^k + \frac{1}{2} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{\partial^2 \xi_0(f_0)}{\partial z^k \partial z^\ell} f_1^k f_1^\ell + \\ &+ \sum_{k=1}^{N} \frac{\partial \xi_2(f_0)}{\partial z^k} f_2^k + \frac{1}{2} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{\partial^2 \xi_1(f_0)}{\partial z^k \partial z^\ell} f_1^k f_1^\ell + \sum_{k=1}^{N} \frac{\partial \xi_3(f_0)}{\partial z^k} f_3^k + \\ &+ \frac{1}{3!} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \sum_{m=1}^{N} \frac{\partial^3 \xi_0(f_0)}{\partial z^k \partial z^\ell \partial z^m} f_1^k f_1^\ell f_1^m + \frac{1}{2} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{\partial^2 \xi_2(f_0)}{\partial z^k \partial z^\ell} f_1^k f_1^\ell + \\ &+ \frac{1}{3!} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \sum_{m=1}^{N} \frac{\partial^3 \xi_1(f_0)}{\partial z^k \partial z^\ell \partial z^m} f_1^k f_1^\ell f_1^m + \\ &+ \frac{1}{4!} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \sum_{m=1}^{N} \sum_{r=1}^{N} \frac{\partial^4 \xi_0(f_0)}{\partial z^k \partial z^\ell \partial z^m \partial z^r} f_1^k f_1^\ell f_1^m f_1^r + \xi_4(f_0). \end{aligned} \tag{2.46}$$

We wish to point out here that equations (2.45), (2.46) can be simplified further, depending on the form of the vector ξ. In particular, in Example 2.3-3,

$$\xi_0^{(\sigma)}(f_0) = \xi_\ell^{(\sigma)}(f_0) = 0 \quad \forall \ell \geq 2,$$

so only terms with $i - j = 1$ will be present on the right-hand side of (2.36). Thus

$$\frac{df_i}{da} = \sum_{|\sigma|=1}^{i-1} \frac{1}{\sigma!}\xi_1^{(\sigma)}(f_0) \sum_{|\nu|=i-1} f_{(\nu)}, \quad i = 3, 4, \ldots, p. \tag{2.47}$$

Equation (2.47) above can be simplified even further as illustrated in the following example.

Example 2.3-4

We compute the approximate group of transformations of order ϵ^p generated by the operator (2.28) in Example 2.3-1. Here

$$\xi_0 = 1, \quad \xi_1 = x \quad \text{and} \quad \xi_\ell = 0 \quad \forall \ell \geq 2.$$

By (2.40), (2.45) and (2.46) we have

$$\begin{aligned}
\frac{df_2}{da} &= \frac{\partial \xi_1(f_0)}{\partial x} f_1 = f_1, \\
\frac{df_3}{da} &= \frac{\partial \xi_1(f_0)}{\partial x} f_2 + \sum_{k=1}^{2} \frac{\partial^2 \xi_1(f_0)}{\partial z^k} f_2^k = f_2, \\
\frac{df_4}{da} &= \frac{\partial \xi_1(f_0)}{\partial x} f_3 + \frac{1}{2} \sum_{k=1}^{3} \sum_{\ell=1}^{3} \frac{\partial^2 \xi_1(f_0)}{\partial z^k \partial z^\ell} f_3^k f_3^\ell \\
&\quad + \frac{1}{3!} \sum_{k=1}^{3} \sum_{\ell=1}^{3} \sum_{m=1}^{3} \frac{\partial^3 \xi_1(f_0)}{\partial z^k \partial z^\ell \partial z^m} f_3^k f_3^\ell f_3^m \\
&= f_3.
\end{aligned}$$

In this case we see that (2.36) actually simplifies to

$$\frac{df_i}{da} = f_{i-1}, \quad i = 1, \ldots, p, \tag{2.48}$$

under the initial conditions (2.37). From (2.48) we have

$$f_i = \frac{x a^i}{i!} + \frac{a^{i+1}}{(i+1)!}, \quad i = 0, \ldots, p,$$

and the corresponding approximate group of transformation required is

$$x^* \approx \sum_{i=0}^{p} \frac{a^i}{i!} \left(x + \frac{a}{i+1} \right) \epsilon^i.$$

Chapter 3

The Lie Groups of $u_{tt} + \epsilon u_t = [f(x,u)u_x]_x$

Sources: [1] N.Kh. Ibragimov et al, Approximate Symmetries, Math. USSR Sbornik, Vol. 64 (1989), No. 2, pp. 427–440.

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[4] N.Kh. Ibragimov and E.E. Rosinger, Approximate Groups; a jet based global representation, Proceedings of the VI-th International Conference on Modern Group Analysis, Johannesburg, January 1996.

3.1 Introduction

In this chapter we study the properties of the exact Lie groups and algebras associated with the nonlinear wave equation

$$u_{tt} + \epsilon u_t = [f(x,u)u_x]_x, \quad f \in C^2(\mathbf{R} \times \mathbf{R}),\ f > 0,\ f_u \neq 0. \tag{3.1}$$

First we consider the unperturbed equation $u_{tt} = [f(x,u)u_x]_x$ and then the perturbed equation $u_{tt} + \epsilon u_t = [f(x,u)u_x]_x$.

Notations 3.1

The following notations will be used throughout this chapter unless otherwise stated.

(i) A one-parameter Lie Group of transformations acting on (t, x, u) - space will be denoted by

$$\left.\begin{aligned} t^* &= T(t,x,u;a), \\ x^* &= X(t,x,u;a), \\ u^* &= U(t,x,u;a), \end{aligned}\right\} \tag{3.2}$$

where a is the group parameter.

(ii) The infinitesimal generator

$$V = \xi(t,x,u)\frac{\partial}{\partial t} + \theta(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

will have as its once and twice extended generators

$$\left.\begin{aligned} V^{(1)} &= V + \eta_1^{(1)}(t,x,u,u_1,u_2)\frac{\partial}{\partial u_1} + \eta_2^{(1)}(t,x,u,u_1,u_2)\frac{\partial}{\partial u_2}, \\ V^{(2)} &= V^{(1)} + \eta_{11}^{(2)}(t,x,u,u_1,u_2,u_{11},u_{12},u_{22})\frac{\partial}{\partial u_{11}} \\ &\quad + \eta_{12}^{(2)}(\ldots)\frac{\partial}{\partial u_{12}} + \eta_{22}^{(2)}(\ldots)\frac{\partial}{\partial u_{22}}, \end{aligned}\right\} \tag{3.3}$$

where

$$u_1 = u_t, \; u_2 = u_x, \; u_{11} = u_{tt}, \; u_{12} = u_{tx} = u_{xt}, \; u_{22} = u_{xx}. \tag{3.4}$$

3.2 Properties of the Lie Groups of the Nonlinear Wave Equation $u_{tt}=[f(x,u)u_x]_x$

The properties of Lie groups and algebras associated with the non-linear wave equation of the form

$$F \equiv u_{tt} - [f(x,u)u_x]u_x = 0, \quad f \in C^2(\mathbf{R}\times\mathbf{R}),\ f>0,\ f_u \neq 0, \tag{3.5}$$

have been studied in detail in [3]. It is worth pointing out that some of the assumptions made in [3] when solving for the infinitesimals are unnecessary since the conditions present themselves naturally. We now give a brief discussion of the derivation of the infinitesimal transformation group for equation (3.5). Here we shall present the appropriate general version of the computation of the respective Lie groups and algebras.

Using notations (3.4), equation (3.5) becomes

$$F \equiv u_{11} - f_x u_2 - f_u(u_2)^2 - f u_{22}. \tag{3.6}$$

For the group of (3.5), the infinitesimal generator and consequently the once and twice extended infinitesimal generators will be as in (3.3). The invariance condition

$$V^{(2)}F \equiv 0 \quad \text{when} \quad F=0,$$

where F is given by (3.6), implies

$$\begin{aligned}
0 \;=\; & -\eta_x f_x - \eta_{xx} f + \eta_{tt} \\
& + u_1[\xi_x f_x + \xi_{xx} f + 2\eta_{tu} - \xi_{tt}] \\
& + u_2[-\eta f_{xu} - \theta f_{xx} - 2\eta_x f_u + \theta_x f_x - 2\eta_{xu} f + \theta_{xx} f - \theta_{tt} - 2\xi_t f_x] \\
& + u_1u_2[2\xi_x f_u - 2\xi_u f_x + 2\xi_{xu} f - 2\theta_{tu} f] + (u_1)^2[\eta_{uu} - 2\xi_{tu}] \\
& + (u_2)^2[-\eta f_{uu} - \theta f_{xu} - 2\eta_u f_u + 2\theta_x f_u - \eta_{uu} f + 2\theta_{xu} f + \eta_u f_u - 2\xi_t f_u] \\
& - (u_1)^3\xi_{uu} + (u_2)^3[\theta_u f_u + \theta_{uu} f] - (u_1)^2u_2\theta_{uu} + u_1(u_2)^2[\xi_{uu} f - \xi_u f_u] \\
& + u_{12}[2\xi_x f - 2\theta_t] + u_{22}[\eta f_u + \theta f_x - 2\theta_x f + 2\xi_t f] \\
& - 2u_1u_{12}\theta_u - 2u_1u_{22}\xi_u f + 2u_2u_{12}\xi_u f + 2u_2u_{22}\theta_u f,
\end{aligned} \tag{3.7}$$

where u_{11} has been eliminated by putting $u_{11} = f_x u_2 + f_u(u_2)^2 + f u_{22}$.
The coefficients of $u_1u_{22}, u_2u_{22}, u_1u_{12}, (u_1)^2u_{12}$ and $(u_1)^3$ in (3.7) give the relation

$$\xi_u = \xi_{uu} = \theta_u = \theta_{uu} = 0, \tag{3.8}$$

and then we get

$$\xi_x = \theta_t = 0 \tag{3.9}$$

from equation (3.7) and the coefficients of $u_1 u_2$ and u_{12}. Moreover, (3.7) and (3.8) together with the coeffficients of $(u_1)^2$ give the following:

$$\left.\begin{aligned} \xi(t,x,u) &= \xi(t) = a(t), \\ \theta(t,x,u) &= \theta(x) = b(x), \\ \eta(t,x,u) &= c(t,x)u + d(t,x), \end{aligned}\right\} \tag{3.10}$$

where a, b, c, d are arbitrary functions.

Considering (3.10) and the remaining independent conditions supplied by (3.7), we now have the following:

$$-\eta_x f_x - \eta_{xx} f + \eta_{tt} = 0, \tag{3.11}$$

$$2\eta_{tu} - \xi_{tt} = 0, \tag{3.12}$$

$$\eta f_u + \theta f_x - 2\theta_x f + 2\xi_t f = 0, \tag{3.13}$$

$$-\eta f_{xu} - \theta f_{x^2} - 2\eta_x f_u + \theta_x f_x - 2\eta_{xu} f + \theta_{x^2} f - \theta_{t^2} - 2\xi_t f_x = 0, \tag{3.14}$$

$$-\eta f_{u^2} - \theta f_{xu} - \eta_u f_u + 2\theta_x f_u - \eta_{u^2} f + 2\theta_{xu} f - 2\xi_t f_u = 0. \tag{3.15}$$

Putting (3.10) into (3.11), (3.12) and (3.13) gives, respectively,

$$\begin{aligned} -(c_x u + d_x) f_x - (c_{x^2} u + d_{x^2}) f + c_{t^2} u + d_{t^2} &= 0, \\ 2c_t - a''(t) &= 0, \\ (cu + d) f_u + b f_x - 2b'(x) f + 2a'(t) f &= 0. \end{aligned} \tag{3.16}$$

From (3.16) we have

$$c(t,x) = \frac{1}{2}a'(t) + e(x),$$

which in view of (3.10) implies

$$\eta(t,x,u) = (\frac{1}{2}a'(t) + e(x))u + d(t,x). \tag{3.17}$$

Thus in general the infinitesimals of (3.5) will be of the form

$$\left.\begin{aligned} \xi &= a(t), \\ \theta &= b(x), \\ \eta &= (\frac{1}{2}a'(t) + e(x))u + d(t,x). \end{aligned}\right\} \tag{3.18}$$

For complete analysis of the Lie groups admitted by (3.5) it is necessary to consider the case when f is arbitrary or other particular forms of f.

Case 1: f is arbitrary.
Since f is arbitrary, we can assume for instance that

$$\left.\begin{aligned} &f(x,u) = e^{\lambda x + \mu u}, \quad \lambda, \mu \text{ are constants} \\ &\Rightarrow \; f_u = \mu e^{\lambda x + \mu u} \quad \text{and} \quad f_x = \lambda e^{\lambda x + \mu u}. \end{aligned}\right\} \tag{3.19}$$

Putting (3.19) into (3.13) gives

$$\eta \mu f + b(x)\lambda f + 2(a'(t) - b'(x))f = 0$$

$$\Rightarrow \; \eta\mu + b(x)\lambda + 2(a'(t) - b'(x)) = 0,$$

which is polynomial in μ and λ.
Separating the coefficients of μ, λ and 1, respectively, gives

$$\eta = 0, \quad b(x) = 0 \quad \text{and} \quad 2(a'(t) - b'(x)) = 0$$

$$\Rightarrow \quad a'(t) = 0 \quad \Rightarrow \quad a(t) = a_0 = \text{ constant}.$$

Thus for arbitrary f we have the infinitesimals

$$\xi = a_0, \quad \theta = 0, \quad \eta = 0. \tag{3.20}$$

□

The result in (3.20) suggests that a lot of symmetries may be lost in the case of general f and hence the need to consider paricular forms of the function f, which we now discuss below.
Putting (3.17) into (3.14) and (3.15) gives, respectively,

$$-((\frac{1}{2}a' + e)u + d)f_{xu} - bf_{x^2} - 2(e'u + d_x)f_u + b'f_x - 2e'f + b''f - 2a'f_x = 0, \tag{3.21}$$

$$-((\frac{1}{2}a' + e)u + d)f_{u^2} - bf_{xu} - (\frac{1}{2}a' + e)f_u + 2b'f_u - 2a'f_u = 0. \tag{3.22}$$

Differentiating (3.21) and (3.22) with respect to u and x respectively gives

$$\begin{aligned} &-(\frac{1}{2}a' + e)f_{xu} - ((\frac{1}{2}a' + e)u + d)f_{xu^2} - bf_{x^2u} - 2e'f_u \\ &- 2(e'u + d_x)f_{u^2} + b'f_{xu} - 2e'f_u + b''f_u - 2a'f_{xu} = 0, \end{aligned} \tag{3.23}$$

$$\begin{aligned} &-(e'u + d_x)f_{u^2} - ((\frac{1}{2}a' + e)u + d)f_{xu^2} - b'f_{xu} - b'f_{x^2u} \\ &- (\frac{1}{2}a' + e)f_{xu} + 2b''f_u + 2b'f_{xu} - 2a'f_{xu} = 0. \end{aligned} \tag{3.24}$$

Subtracting (3.24) from (3.23) gives

$$3e'(x)f_u + (e'(x)u + d_x(t,x))f_{u^2} + b''(x)f_u = 0. \tag{3.25}$$

We now consider (3.25) when $f_{u^2} = 0$, and later the case when $f_{u^2} \neq 0$.

Case 2: $f_{u^2} = 0$.
When $f_{u^2} = 0$, then

$$f(x,u) = uf_1(x) + f_2(x). \tag{3.26}$$

Putting (3.26) into (3.25) yields

$$3e'f_u + b''f_u = 0$$

$$\Rightarrow \; b''(x) = -3e'(x), \quad \text{since } f_u \neq 0$$

$$\Rightarrow \; b(x) = -3\int e(x)dx + k_0 x + k_1. \tag{3.27}$$

Substituting (3.26) into (3.27) gives

$$-(e'u + d_x)(uf_1'(x) + f_2'(x)) - (e''u + d_{xx})(uf_1(x) + f_2(x)) = 0$$
$$\Rightarrow \; e'f_1'(x)u^2 + e'f_2'(x)u + d_x f_1'(x)u + d_x f_2'(x)$$
$$+ \; e''f_1(x)u^2 + e''f_2(x)u + d_{xx}f_1(x)u + d_{xx}f_2(x) = 0.$$

The coefficients of 1 give

$$d_x f_2'(x) + d_{xx} f_2(x) = 0. \tag{3.28}$$

From (3.28) we have

$$\frac{d_{xx}(t,x)}{d_x(t,x)} = -\frac{f_2'(x)}{f_2(x)}, \quad d_x \neq 0, \; f_2(x) \neq 0,$$

which on integration with respect to x yields

$$\ln d_x(t,x) = -\ln f_2(x) + \ln k_2(t)$$

$$\Rightarrow \qquad d_x(t,x) = \frac{k_2(t)}{f_2(x)}$$

$$\Rightarrow \qquad d(t,x) = \int \frac{k_2(t)}{f_2(x)}\,dx \tag{3.29}$$

Substituting (3.29) into (3.13) we get

$$\begin{aligned}&[(\frac{1}{2}a' + e)u + \int \frac{k_2(t)}{f_2(x)}\,dx]f_1(x) + b(x)(uf_1'(x) + f_2'(x)) \\ &- 2b'(x)(uf_1(x) + f_2(x)) + 2a'(t)(uf_1(x) + f_2(x)) = 0.\end{aligned} \tag{3.30}$$

From the coefficients of u we get

$$\frac{1}{2}a'(t)+2a'(t)=-e(x)+2b'(x)-b(x)\frac{f_1'(x)}{f_1(x)}.$$

Thus

$$\begin{aligned} & \partial_t(\frac{1}{2}a'(t)+2a'(t))=0 \\ \Rightarrow \quad & a''(t)=0 \\ \Rightarrow \quad & a(t)=k_3t+k_4. \end{aligned} \tag{3.31}$$

Finally in case 2, we have

$$\left.\begin{aligned} f(x,u) &= uf_1(x)+f_2(x), \\ \xi &= a(t)=k_3t+k_4, \\ \theta &= b(x)=3\int e(x)dx+k_0x+k_1, \\ \eta &= (\frac{k_3}{2}+e(x))u+\int\frac{k_2(t)}{f_2(x)}\,dx. \end{aligned}\right\} \tag{3.32}$$

□

Case 3: $f_{u^2}\neq 0$.
When $f_{u^2}\neq 0$, (3.25) becomes

$$(e'u+d_x)\frac{f_{u^2}}{f_u}=-b''(x)-3e'. \tag{3.33}$$

Differentiation with respect to t yields

$$\begin{aligned} & d_{xt}\frac{f_{u^2}}{f_u}=0 \\ \Rightarrow \quad & d_{xt}(t,x)=0 \quad \text{since} \quad \frac{f_{u^2}}{f_u}\neq 0 \\ \Rightarrow \quad & d_x(t,x)=g'(x) \\ \Rightarrow \quad & d(t,x)=g(x)+h(t). \end{aligned} \tag{3.34}$$

Also, differentiation with respect to u yields

$$e' f_{u^2} f_u + (e'u + d_x)(f_{u^3} f_u - (f_{u^2})^2) = 0. \tag{3.35}$$

□

The following cases arise from (3.35).

Case 3.1: $e'(x) \neq 0$.
Then (3.35) can be expressed as

$$e' + (e'u + d_x)\left(\frac{f_{u^3}}{f_{u^2}} - \frac{f_{u^2}}{f_u}\right) = 0$$
$$\Rightarrow \frac{f_{u^2}}{f_u} - \frac{f_{u^3}}{f_{u^2}} = \frac{1}{u + \frac{d_x}{e'}} = \frac{1}{u + \frac{g'(x)}{e'}}.$$

which on integration in u gives

$$\ln f_u - \ln f_{u^2} = \ln\left(u + \frac{g'(x)}{e'}\right) + \ln c_1(x)$$
$$\Rightarrow \frac{f_u}{f_{u^2}} = c_1(x)\left(u + \frac{g'(x)}{e'}\right).$$

Further integration with respect to u yields

$$f_u = c_2(x)\left(u + \frac{g'(x)}{e'}\right)^{1/c_1(x)}.$$

Thus

$$\begin{aligned} f(x,u) &= c_2(x) \int \left(u + \frac{g'(x)}{e'}\right)^{1/c_1(x)} du \\ &= \frac{c_2(x)}{1 + \frac{1}{c_1(x)}}\left(u + \frac{g_1'(x)}{e'}\right)^{1+1/c_1(x)} + c_3(x), \end{aligned}$$

which is of the general form

$$f(x,u) = \alpha(x)\left[u + \frac{g'(x)}{e'(x)}\right]^{\beta(x)} + \lambda(x). \quad \square \tag{3.36}$$

Case 3.2: $e'(x) = 0$.
From (3.35) we have

$$d_x(t)\left(\frac{f_{u^3}}{f_{u^2}} - \frac{f_{u^2}}{f_u}\right) = 0, \tag{3.37}$$

which further implies the following conditions:

Case 3.2.1: $d_x(t,x) = 0$

$$\left.\begin{aligned} d_x(t,x) = 0 \Rightarrow g'(x) = 0 \quad &\Rightarrow g(x) = g_0 = \text{constant.} \\ e'(x) = 0 \quad &\Rightarrow e(x) = e_0 = \text{constant.} \end{aligned}\right\} \tag{3.38}$$

Using (3.18) and (3.38) in (3.10), we have

$$\eta = (\frac{1}{2}a'(t) + e_0)u + g_0 + h(t) = 0$$

$$\Rightarrow \qquad \eta_t = \frac{1}{2}a''(t)u + h'(t) = 0$$

$$\Rightarrow \qquad \eta_{tt} = \frac{1}{2}a'''(t)u + h''(t) = 0$$

$$\Rightarrow \quad a'''(t) = 0 \quad \Rightarrow \quad a(t) = \frac{a_0}{2}t^2 + a_1 t + a_2 \tag{3.39}$$

$$\text{and} \qquad h''(t) = 0 \quad \Rightarrow \quad h(t) = h_0 t + h_1. \tag{3.40}$$

From (3.25) we now have

$$b''(x)f_u = 0 \quad \Rightarrow \quad b(x) = b_0 x + b_1. \tag{3.41}$$

Putting (3.38) – (3.41) into (3.18) gives

$$\left.\begin{aligned} &\xi = a(t) = \frac{a_0}{2}t^2 + a_1 t + a_2, \\ &\theta = b(x) = b_0 x + b_1, \\ &\eta = (\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0)u + g_0 + h_0 t + h_1, \end{aligned}\right\} \tag{3.42}$$

where $a_0, a_1, a_2, b_0, b_1, e_0, g_0, h_0$ and h_1 are constants. Putting (3.42) into (3.13) yields

$$((\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0)u + g_0 + h_0 t + h_1)f_u + (b_0 x + b_1)f_x - 2b_0 f + 2(a_0 t + a_1)f = 0 \quad (3.43)$$

$$\Rightarrow \quad ((\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0)u + g_0 + h_0 t + h_1)\frac{f_u}{f} + (b_0 x + b_1)\frac{f_x}{f} = 2(b_0 - a_0 t - a_1). \quad (3.44)$$

Differentiating (3.44) with respect to u gives

$$(\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0)\frac{f_u f}{f^2} + ((\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0)u + g_0 + h_0 t + h_1)\frac{f f_{u^2} - (f_u)^2}{f^2} + (b_0 x + b_1)\frac{f_{xu} f - f_x f_u}{f^2} = 0.$$

Dividing throughout by f_u/f, we have

$$\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0 + ((\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0)u + g_0 + h_0 t + h_1)(\frac{f_{u^2}}{f_u} - \frac{f_u}{f}) + (b_0 x + b_1)(\frac{f_{xu}}{f_u} - \frac{f_x}{f}) = 0 \quad (3.45)$$

from where the following conditions arise:

Case 3.2.1(i): f arbitrary, with

$$(\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0)u + g_0 + h_0 t + h_1 = 0 \quad \text{and} \quad b_0 x + b_1 = 0. \quad (3.46)$$

Separating the coefficients of u and 1 gives

$$\left.\begin{array}{l} \frac{a_0}{2}t + \frac{1}{2}a_1 + e_0 = 0 \;\Rightarrow\; a_0 = 0, \;\; \frac{1}{2}a_1 = -e_0, \\ g_0 + h_0 t + h_1 = 0 \;\Rightarrow\; h_0 = 0, \;\; g_0 = -h_1, \\ b_0 x + b_1 = 0 \;\Rightarrow\; b_0 = 0, \;\; b_1 = 0. \end{array}\right\} \quad (3.47)$$

We get the infinitesimals (see (3.10))

$$\xi = a_2, \quad \theta = 0, \quad \eta = 0 \quad (3.48)$$

by substituting (3.47) into (3.43), since $a_1 = 0$.

Case 3.2.1(ii):

$$(\frac{a_0}{2}t + \frac{1}{2}a_1 + e_0)u + g_0 + h_0 t + h_1 = 0 \quad \text{and} \quad \frac{f_{xu}}{f_u} - \frac{f_x}{f} = 0. \tag{3.49}$$

It is easy to see that

$$\ln f_u - \ln f = \ln \lambda(u) \quad \Rightarrow \quad \frac{f_u}{f} = \lambda(u).$$

Integrating with respect to u yields

$$\ln f = \int \lambda(u) du + \gamma(x)$$

$$\Rightarrow \quad f(x, u) = \gamma(x) e^{\int \lambda(u) du}. \tag{3.50}$$

Introducing (3.50) into (3.13) and using (3.39) gives

$$(b_0 x + b_1)\frac{\gamma'(x)}{\gamma(x)} = 2b_0 - 2a'(t) = k_0 = \text{ constant.} \tag{3.51}$$

Thus

$$2b_0 - 2a'(t) = k_0 \quad \Rightarrow \quad a(t) = (b_0 - \frac{k_0}{2})t + k_1 \tag{3.52}$$

and

$$(b_0 x + b_1)\frac{\gamma'(x)}{\gamma(x)} = k_0 \quad \Rightarrow \quad \gamma(x) = k_2 (b_0 x + b_1)^{k_0/b_0}. \tag{3.53}$$

In view of (3.53) and equations (3.11) – (3.15), f in (3.50) now takes the form

$$f(x, u) = c(u)(b_0 x + b_1)^{2 + k_0/b_0}. \tag{3.54}$$

The following infinitesimals are immediate from (3.49) – (3.54):

$$\left.\begin{aligned} \xi &= -\frac{k_0}{2}t + k_1, \\ \theta &= b_0 x + b_1, \\ \eta &= 0. \end{aligned}\right\} \tag{3.55}$$

Case 3.2.1(iii):

$$(\frac{a_0}{2}t + \frac{1}{2}t + e_0)u + g_0 + h_0 t + h_1 \neq 0 \tag{3.56}$$

Differentiating (3.45) with respect to t gives

$$\frac{a_0}{2} + (\frac{a_0}{2}u + h_0)(\frac{f_{uu}}{f_u} - \frac{f_u}{f}) = 0$$

$$\Rightarrow \frac{f_u}{f} - \frac{f_{uu}}{f_u} = \frac{1}{u + \frac{2h_0}{a_0}}$$

$$\Rightarrow \ln f - \ln f_u = \ln \left(\frac{u + \frac{2h_0}{a_0}}{c(x)} \right)$$

$$\Rightarrow \frac{f_u}{f} = \frac{c(x)}{u + \frac{2h_0}{a_0}}.$$

Further integration with respect to u yields

$$\ln f = c(x) \ln [d(x)(u + \frac{2h_0}{a_0})]$$

$$\Rightarrow \quad f(x,u) = [d(x)(u + \frac{2h_0}{a_0})]^{c(x)}. \tag{3.57}$$

□

Case 3.2.2:

$$\frac{f_{u^3}}{f_{u^2}} - \frac{f_{u^2}}{f_u} = 0 \tag{3.58}$$

$$\Rightarrow \quad \ln f_{u^2} = \ln f_u + \ln q(x)$$

$$\Rightarrow \quad \frac{f_{u^2}}{f_u} = q(x),$$

which can be integrated further to give

$$\ln f_u = q(x)u + \ln p_1(x)$$

$$\Rightarrow \qquad f_u = p_1(x)e^{q(x)u}$$

$$\Rightarrow \qquad f = \frac{p_1(x)}{q(x)}e^{q(x)u} + w(x)$$

$$\Rightarrow \quad f(x,u) = p(x)e^{q(x)u} + w(x). \tag{3.59}$$

□

We have so far considered exhaustively all the possible forms of the function $f(x,u)$ in view of the exact Lie groups of the non-linear wave equation (3.1). Figure 3.1 gives a summary of this in the form of a tree diagram.

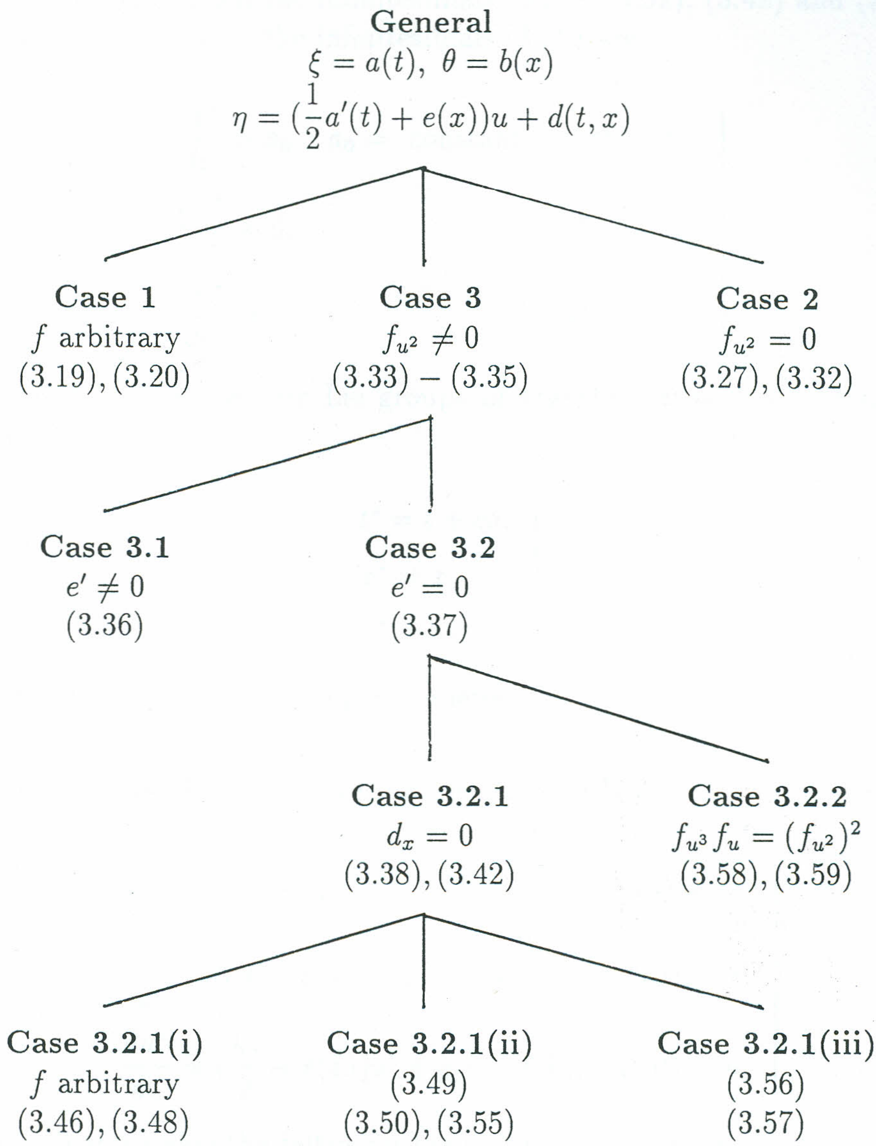

Fig. 3.1 A tree diagram illustrating different cases of the form of the function $f(x,u)$ in the non-linear wave equation $u_{tt} = [f(x,u)u_x]_x$.

The exact Lie groups of transformations for the unperturbed non-linear wave equation (3.5) can easily be found from the infinitesimals (3.20), (3.32), (3.42) and (3.55).
The O.D.E.s corresponding to the infinitesimals (3.20) are

$$\left.\begin{aligned} &\frac{dt^*}{da}=a_0, \quad a_0=\text{ constant}, && t^*(0)=0,\\ &\frac{dx^*}{da}=0, && x^*(0)=0,\\ &\frac{du^*}{da}=0, && u^*(0)=0, \end{aligned}\right\} \tag{3.60}$$

which is easily solved to give the Lie groups of transformations admitted by (3.5) for arbitrary f as

$$\left.\begin{aligned} t^*&=t+ca,\\ x^*&=x,\\ u^*&=u, \end{aligned}\right\} \tag{3.61}$$

where c is a constant and a is a group parameter. □

From the infinitesimals (3.32) we have the system of O.D.E.s

$$\left.\begin{aligned} &\frac{dt^*}{da}=k_3t^*+k_4, && t^*(0)=t,\\ &\frac{dx^*}{da}=3\int e(x)dx+k_0x^*+k_1, && x^*(0)=x,\\ &\frac{du^*}{da}=(\frac{k_3}{2}+e(x))u^*+g_1(x)+k_5, && u^*(0)=u, \end{aligned}\right\} \tag{3.62}$$

which can be solved to give the following Lie groups of transformations:

$$\left.\begin{aligned} t^*&=\frac{k_3t+k_4}{k_3}e^{k_3a}-\frac{k_4}{k_3}, && k_3\neq 0,\\ x^*&=\frac{1}{k_0}[k_0x+3\int e(x)x+k_1]e^{(3e(x)dx+k_0)a}\\ &\quad-\frac{1}{k_0}(3\int e(x)dx+k_1), && k_0\neq 0,\\ u^*&=\frac{1}{B}[(Bu+g_1(x)+k_5)e^{Ba}-g_1(x)-k_5], && B=\frac{k_3}{2}+e(x). \end{aligned}\right\} \tag{3.63}$$

□

Systems of linear O.D.E.s corresponding to the infinitesimals (3.42) and (3.55), respectively, are

$$\left.\begin{aligned} \frac{dt^*}{da} &= \frac{a_0}{2}(t^*)^2 + a_1 t^* + a_2, \\ \frac{dx^*}{da} &= b_0 x^* + b_1, \\ \frac{du^*}{da} &= (\frac{a_0}{2}t^* + \frac{1}{2}a_1 + e_0)u^* + g_0 + h_0 t^* + h_1 \end{aligned}\right\} \tag{3.64}$$

and

$$\left.\begin{aligned} \frac{dt^*}{da} &= -\frac{k_0}{2}t^* + k_1, \\ \frac{dx^*}{da} &= b_0 x^* + b_1, \\ \frac{du^*}{da} &= 0, \end{aligned}\right\} \tag{3.65}$$

where $a_0, a_1, a_2, b_0, b_1, e_0, g_0, h_0$ and h_1 are constants and a is group parameter.

Solving (3.64) for t^*, x^* and u^*, we have

$$\frac{dt^*(a)}{da} = \frac{a_0}{2}(t^*)^2 + a_1 t^* + a_2$$

$$\Rightarrow \quad \frac{dt^*(a)}{\frac{a_0}{2}(t^*)^2 + a_1 t^* + a_2} = da.$$

By completing the squares, and writing t for t^*, we have

$$\begin{aligned} \int \frac{dt}{\frac{a_0}{2}t^2 + a_1 t + a_2} &= \frac{a_0}{2}\int \frac{dt}{(t + \frac{a_1}{a_0})^2 - [\frac{1}{a_0}(a_0^2 - 2a_0 a_2)^{1/2}]^2} \\ = \frac{a_0}{2}\int \frac{T'}{T^2 - A^2}dt &= \frac{a_0}{4A}\ln\left|\frac{T-A}{T+A}\right| = \int da = a + c \end{aligned} \tag{3.66}$$

where

$$T = t + \frac{a_1}{a_0} \quad \text{and} \quad A = \frac{1}{a_0}(a_0^2 - 2a_0 a_2)^{1/2}. \tag{3.67}$$

From (3.66) we get

$$T^* = \frac{A(1+\left[\dfrac{T-A}{T+A}\right] e^{(4A/a_0)a})}{1-\left[\dfrac{T-A}{T+A}\right] e^{(4A/a_0)a}}$$

and thus

$$\left.\begin{aligned} t^* &= \frac{A(1+\left[\dfrac{T-A}{T+A}\right] e^{(4A/a_0)a})}{1-\left[\dfrac{T-A}{T+A}\right] e^{(4A/a_0)a}}, \\ x^* &= \frac{1}{b_0}((b_0x+b_1)e^{b_0a}-b_1), \\ u^* &= \frac{1}{B}((Bu+C)e^{Ba}-C), \end{aligned}\right\} \tag{3.68}$$

where T and A are as in (3.67),

$$B=\frac{a_0}{2}t+\frac{1}{2}a_1+e_0 \quad \text{and} \quad C=g_0+h_0t+h_1.$$

Similarly, from (3.65) we get the Lie groups of transformations admitted by the system of O.D.E.s as:

$$\left.\begin{aligned} t^* &= \frac{2k_1}{k_0}+(t-\frac{2k_1}{k_0})e^{-k_0a/k_1}, \\ x^* &= \frac{1}{b_0}((b_0x+b_1)e^{b_0a}-b_1), \\ u^* &= u. \end{aligned}\right\} \tag{3.69}$$

□

3.3 Lie Groups of the Perturbed Non-linear Wave Equation $u_{tt} + \epsilon u_t = [f(x,u)u_x]_x$

We present a detailed discussion of the properties of Lie groups of the perturbed non-linear wave equation of the form

$$u_{tt} + \epsilon u_t = [f(x,u)u_x]_x, \quad f \in C(\mathbf{R} \times \mathbf{R}), \quad f > 0, \quad f_u \neq 0. \tag{3.70}$$

Using the notations (3.5) we have

$$F = u_{11} + \epsilon u_1 - f_x u_2 - f_u (u_2)^2 - f u_{22}. \tag{3.71}$$

The invariant condition

$$V^{(2)} F = 0 \quad \text{when} \quad F = 0,$$

where F is given by (3.71), gives

$$\begin{aligned}
0 =\ & \epsilon\eta_t - \eta_x f_x + \eta_{tt} - \eta_{xx} f \\
& + u_1[\epsilon\eta_u - \epsilon\xi_t + \xi_x f_x + 2\eta_{tu} - \xi_{tt} - \epsilon\eta_u + 2\epsilon\xi_t + \xi_{xx} f] \\
& + u_2[-\theta f_{xx} - \eta f_{xu} - \epsilon\theta_t + \theta_x f_x - 2\eta_x f_u - \theta_{tt} - 2\xi_t f_x - 2\eta_{xu} f + \theta_{xx} f] \\
& + u_1 u_2[-\epsilon\theta_u + \xi_u f_x + 2\xi_x f_u - 2\theta_{tu} - 3\xi_u f_x + \epsilon\theta_u + 2\xi_{xu} f] \\
& + (u_1)^2[-\epsilon\xi_u + \eta_{uu} - 2\xi_{tu} + 3\epsilon\xi_u] \\
& + (u_2)^2[-\theta f_{xu} - \eta f_{uu} + \theta_u f_x - \eta_u f_u + 2\theta_x f_x - 2\xi_t f_u - \theta_u f_x - \eta_{uu} f + 2\theta_{xu} f] \\
& - (u_1)^3 \xi_{uu} + (u_2)^2[\theta_u f_u - \theta_{uu} f] - (u_1)^2 u_2 \theta_{uu} \\
& + u_1 (u_2)^2 [\xi_{uu} - \xi_u f_u] + u_{12}[2\xi_x f - 2\theta_t] \\
& + u_{22}[-\theta f_x - \eta f_u - 2\xi_t f + 2\theta_x f] - 2u_1 u_{12}\theta_u - 2u_1 u_{22}\xi_u f \\
& + 2u_2 u_{12}\xi_u f + 2u_2 u_{22}\theta_u f.
\end{aligned} \tag{3.72}$$

The coefficients of $u_1 u_{22}$, $u_2 u_{22}$, $u_1 u_{12}$, $(u_1)^2 u_2$, $u_2 u_{12}$ and $(u_1)^3$ in (3.72) give the relation

$$\xi_u = \xi_{uu} = \theta_u = \theta_{uu} = 0, \tag{3.73}$$

and then we get

$$\xi_x=\theta_t=0 \tag{3.74}$$

from (3.72) and the coefficients of u_1u_2 and u_{12}. Also (3.72) and (3.73) together with the coefficients of $(u_1)^2$ give the following:

$$\left.\begin{aligned}\xi(t,x,u)&=\xi(t)=a(t),\\ \theta(t,x,u)&=\theta(x)=b(x),\\ \eta(t,x,u)&=c(t,x)u+d(t,x),\end{aligned}\right\} \tag{3.75}$$

where a,b,c,d are arbitrary functions. Finally, considering (3.72) - (3.75) and the remaining independent coefficients supplied by (3.72), we have the following:

$$\xi\eta_t-\eta_xf_x+\eta_{tt}-\eta_{xx}f=0, \tag{3.76}$$

$$2\eta_{tu}-\xi_{tt}+\epsilon\xi_t=0, \tag{3.77}$$

$$\eta f_u+\theta f_x-2\theta_xf+2\xi_tf=0, \tag{3.78}$$

$$-\eta f_{xu}-\theta f_{x^2}-\epsilon\theta_t+\theta_xf_x-2\eta_xf_u-2\xi_tf_x-2\eta_{xu}f+\theta_{x^2}-\theta_{t^2}=0, \tag{3.79}$$

$$-\eta f_{u^2}-\theta f_{xu}-\eta_uf_u+2\theta_xf_u-2\xi_tf_u-\eta_{u^2}f=0. \tag{3.80}$$

Putting (3.75) into (3.76), (3.77) and (3.78) gives, respectively,

$$\begin{aligned}\epsilon(c_tu+d_t)-(c_xu+d_x)f_x+c_{t^2}u+d_{t^2}-(c_{x^2}+d_{x^2})f&=0,\\ 2c_t-a''(t)+\epsilon a'(t)&=0,\\ (cu+d)f_u+b(x)f_x-2b'(x)f+2a'(t)f&=0.\end{aligned} \tag{3.81}$$

From (3.81) we have

$$c(t,x)=\frac{1}{2}(a'(t)-\epsilon a(t))+e(x), \tag{3.82}$$

which in view of (3.75) implies

$$\eta(t,x,u) = [\frac{1}{2}a'(t) - \frac{1}{2}\epsilon a(t) + e(x)]u + d(t,x). \tag{3.83}$$

In view of (3.83), (3.75) now takes the form

$$\left.\begin{aligned} &\xi = a(t), \\ &\theta = b(x), \\ &\eta = [\frac{1}{2}a'(t) - \frac{1}{2}\epsilon a(t) + e(x)]u + d(t,x). \end{aligned}\right\} \tag{3.84}$$

To be able to give a complete analysis of the exact Lie groups admitted by (3.70), it is only fitting that we consider the case when f is arbitrary. When needed or useful, we shall consider other particular forms of f a well.

Case 1: f arbitrary.
Let us, for instance, take the form of f as in (3.19). Since (3.13) and (3.78) are the same, we see that the infinitesimals

$$\xi = a_0, \quad \theta = 0, \quad \eta = 0, \tag{3.85}$$

follow as in the case of (3.5). □

A discussion of particular forms of the function f follows.

Putting (3.82) into (3.79) and (3.80) gives, respectively,

$$\begin{aligned} &-\{[\frac{1}{2}((a'(t) - \epsilon a(t)) + e(x)]u + d\}f_{xu} - b(x)f_{x^2} - b'(x)f_x \\ &- 2(e'(x)u + d_x)f_u - 2a'(t)f_x - 2e'(x)f + b''(x)f = 0, \end{aligned} \tag{3.86}$$

$$\begin{aligned} &[(-\frac{1}{2}a'(t) + \frac{1}{2}\epsilon a(t) - e(x))u - d]f_{u^2} - b(x)f_{xu} \\ &+ [-\frac{1}{2}a'(t) + \frac{1}{2}\epsilon a(t) - e(x)]f_u + 2b'(x)f_u - 2a'(t)f_u = 0. \end{aligned} \tag{3.87}$$

Differentiating (3.86) and (3.87) with respect to u and x respectively gives

$$\begin{aligned} &[-\frac{1}{2}a'(t) + \frac{1}{2}\epsilon a(t) - e(x)]f_{xu} + [(-\frac{1}{2}a'(t) + \frac{1}{2}\epsilon a(t) - e(x))u - d]f_{u^2x} \\ &- 2e'(x)f_u - 2(e'(x)u + d_x)f_{u^2} - 2a'(t)f_{xu} - 2e'(x)f_u \\ &- b(x)f_{x^2u} - b'(x)f_{xu} + b''(x)f_u = 0, \end{aligned} \tag{3.88}$$

$$
\begin{aligned}
&[(-\frac{1}{2}a'(t)+\frac{1}{2}\epsilon a(t)-e(x))u-d]f_{u^2x}+(-e'(x)u-d_x)f_{u^2}\\
&-b'(x)f_{xu}-b(x)f_{x^2u}+[-\frac{1}{2}a'(t)+\frac{1}{2}\epsilon a(t)-e(x)]f_{ux}\\
&-e'(x)f_u+2b''(x)f_u+2b'(x)f_{xu}-2a'(t)f_{xu}=0.
\end{aligned} \tag{3.89}
$$

Subtracting (3.88) from (3.89) yields

$$3e'(x)f_u+(e'(x)u+d_x)f_{u^2}+b''(x)f_u=0. \tag{3.90}$$

Equation (3.90) can be considered when $f_{u^2}\neq 0$ and when $f_{u^2}=0$ since $f_u\neq 0$ from (3.70).

Case 2: $f_{u^2}=0$.
When $f_{u^2}=0$, we have the form of f as

$$f(x,u)=uf_1(x)+f_2(x), \tag{3.91}$$

and

$$b(x)=-3\int e(x)dx+k_0x+k_1 \tag{3.92}$$

is immediate from (3.91) and (3.90); see (3.27). Substituting (3.91) into (3.76) gives

$$
\begin{aligned}
&[\frac{1}{2}\epsilon a''(t)-\frac{1}{2}\epsilon^2a'(t)]u+\epsilon d_{tt}-(e'(x)u+d_x)(uf_1'(x)+f_2'(x))\\
&+[\frac{1}{2}a'''(t)-\frac{1}{2}\epsilon a''(t)]u+d_{tt}-(e''(x)u+d_{x^2})(uf_1(x)+f_2(x))=0.
\end{aligned} \tag{3.93}
$$

Separating the coefficients of $1,u$ and u^2 in (3.93) gives, respectively, the following equations:

$$\epsilon d_t+d_{tt}-d_xf_2'(x)-d_{x^2}f_2(x) = 0, \tag{3.94}$$

$$
\begin{aligned}
-\frac{1}{2}\epsilon^2a'(t)-\frac{1}{2}a'''(t)-e'(x)f_2'(x)-e''(x)f_2(x)&\\
-d_xf_1'(x)-d_{x^2}f_1(x) &= 0,
\end{aligned} \tag{3.95}
$$

$$-e'(x)f_1'(x)-e''(x)f_1(x) = 0. \tag{3.96}$$

Substituting (3.91) into (3.78) gives

$$[(\frac{1}{2}a'(t) - \frac{1}{2}\epsilon a(t) + e(x))u + d]f_1(x) + b(x)(uf_1'(x) + f_2'(x))$$

$$- 2b'(x)(uf_1(x) + f_2(x)) + 2a'(t)(uf_1(x) + f_2(x)) = 0.$$

Separating the coefficients of 1 and u respectively gives

$$df_1(x) + b(x)f_2'(x) - 2b'(x)f_2(x) + 2a'(t)f_2(x) = 0, \tag{3.97}$$

$$(\frac{1}{2}a'(t) - \frac{1}{2}\epsilon a(t) + e(x))f_1(x) + b(x)f_1'(x) - 2b'(x)f_1(x) + 2a'(t)f_1(x) = 0. \tag{3.98}$$

Note that we can assume $f_1(x) \neq 0$, since otherwise (3.91) would give $f_u \neq 0$, contradicting the initial hypothesis.

From (3.98) we have

$$\frac{5}{2}a'(t) - \frac{1}{2}\epsilon a(t) = 2b'(x) - e(x) - b(x)\frac{f_1'(x)}{f_1(x)} = K = \text{constant}, \tag{3.99}$$

from whence we get the first order linear O.D.E.

$$5a'(t) - \epsilon a(t) = 2K \tag{3.100}$$

whose solution in $a(t)$ is

$$a(t) = a_0 e^{\epsilon t/5} - \frac{2K}{\epsilon}, \quad a_0, K \text{ constants}. \tag{3.101}$$

To be able to get $\eta(t,x,u)$ in (3.84), we shall use both (3.95) and (3.96) to find $e(x)$ and $d(t,x)$. From (3.96) we have

$$e'(x) = e''(x) = 0 \quad \Rightarrow \quad e(x) = e_0 = \text{ constant}. \tag{3.102}$$

From (3.95) and in view of (3.102), we have

$$d_x f_1'(x) + d_{x^2} f_1'(x) = -\frac{1}{2}\epsilon^2 a'(t) - \frac{1}{2}a'''(t). \tag{3.103}$$

Differentiating (3.103) with respect to x yields

$$d_x f_1''(x) + d_{x^2} f_1'(x) + d_{x^2} f_1''(x) + d_{x^3} f_1''(x) = 0,$$

which is a polynomial in $f_1'(x)$ and $f_1''(x)$. Thus

$$d_{x^3} = d_{x^2} = d_x = 0 \quad \Rightarrow \quad d(t,x) = d(t). \tag{3.104}$$

Putting (3.104) into (3.94) gives the O.D.E.

$$d_{tt} + \epsilon d_t = 0,$$

which can easily be solved to get

$$d(t) = d_0 e^{-\epsilon t} + d_1, \tag{3.105}$$

where d_0 and d_1 are constants.
We further note that in view of (3.102), (3.92) now becomes

$$b(x) = (k_0 - 3e_0)x + k_1 + e_1.$$

Thus for case 2 the infinitesimals (3.84) are

$$\left.\begin{aligned} \xi &= a(t) = a_0^{\epsilon t/5} - \frac{2K}{\epsilon}, \\ \theta &= b(x) = (k_0 - 3e_0)x + k_1 + e_1, \\ \eta &= (K + e_0 - \frac{2\epsilon a_0}{5} e^{\epsilon t/5})u + d_0 e^{-\epsilon t} + d_1. \end{aligned}\right\} \tag{3.106}$$

□

Case 3:

$$f_{u^2} \neq 0. \tag{3.107}$$

When f takes the form (3.107), we may express (3.90) as

$$(e'(x)u + d_x)\frac{f_{u^2}}{f_u} = -b''(x) - 3e'(x),$$

which is the same expression which yields (3.100) – (3.105). Thus

$$d(t,x) = g(x) + h(t) \quad \text{and} \quad g(x) = g_0, \quad e(x) = e_0, \tag{3.108}$$

where g_0, e_0 are constants.
Considering (3.76), and in view of (3.84) and (3.108), we have

$$(\frac{1}{2}\epsilon a''(t) - \frac{1}{2}\epsilon^2 a'(t))u + \epsilon h'(t) + (\frac{1}{2}a'''(t) - \frac{1}{2}\epsilon a''(t))u + h''(t) = 0. \tag{3.109}$$

From the coefficients of 1 and u, respectively, we get the homogeneous O.D.E.s

$$h''(t) + \epsilon h'(t) = 0,$$
$$a'''(t) - \epsilon^2 a'(t) = 0,$$

which are easily solved to give

$$h(t) = h_0 e^{-\epsilon t} + h_1 \tag{3.110}$$

$$\text{and} \quad a(t) = \frac{a_0}{\epsilon^2} + a_1 e^{\epsilon t} + a_2 e^{-\epsilon t}, \tag{3.111}$$

where h_0, h_1, a_0, a_1, a_2 are constants.
Putting (3.108), (3.110) and (3.111) into (3.78) we get

$$\begin{aligned} &[(e_0 - \frac{a_0}{2\epsilon})u + h_0 e^{-\epsilon t} + g_0 + h_1]f_u + 2\epsilon(a_1 e^{\epsilon t} - a_2 e^{-\epsilon t})f \\ &+ b(x)f_x - 2b'(x)f = 0. \end{aligned} \tag{3.112}$$

From the coefficients of u and 1 we have, respectively,

$$e_0 - \frac{a_0}{2\epsilon} = 0, \tag{3.113}$$

$$(h_0 e^{-\epsilon t} + g_0 + h_1)\frac{f_u}{f} + b(x)\frac{f_x}{f} = 2b'(x) - 2\epsilon a_1 e^{\epsilon t} + 2\epsilon a_2 e^{-\epsilon t}. \tag{3.114}$$

Differentiating (3.114) with respect to u yields

$$(h_0 e^{-\epsilon t} + g_0 + h_1)\frac{f_{u^2} f - (f_u)^2}{f^2} + b(x)\frac{f_{xu} f - f_x f_u}{f^2} = 0. \tag{3.115}$$

The following cases arise from (3.115):

Case 3.1:

$$h_0 e^{-\epsilon t} + g_0 + h_1 = 0 \quad \text{and} \quad b(x) = 0.$$

Case 3.2:

$$h_0e^{-\epsilon t}+g_0+h_1=0 \quad \text{and} \quad f_{xu}f-f_xf_u=0.$$

Case 3.3:

$$f_{u^2}f-(f_u)^2=0 \quad \text{and} \quad b(x)=0.$$

Case 3.4:

$$f_{u^2}f-(f_u)^2=0 \quad \text{and} \quad f_{xu}f-f_xf_u=0.$$

Case 3.1:
$h_0e^{-\epsilon t}+g_0+h_1=0$ and $b(x)=0$, together with (3.78), (3.84), (3.108) and (3.109), yield the infinitesimals

$$\xi=a(t)=a_0, \quad \theta=b(x)=0, \quad \eta=0. \tag{3.116}$$

□

Case 3.2:
When $h_0e^{-\epsilon t}+g_0+h_1=0$ and $f_{xu}f-f_xf_u=0$, we have

$$f(x,u)=q(u)e^{\int p(x)dx} \tag{3.117}$$

$$\Rightarrow \quad f_x=p(x)q(u)e^{\int p(x)dx} \quad \text{and} \quad f_u=q'(u)e^{\int p(x)dx}.$$

Thus (3.112) now becomes

$$2\epsilon a_1e^{\epsilon t}-2\epsilon a_2e^{-\epsilon t}=2b'(x)-b(x)p(x)=k= \text{ constant}.$$

Solving $2b'(x)-b(x)p(x)=k$ gives

$$b(x)=\frac{k}{2}e^{\int p(x)dx/2}\int e^{-\int p(x)dx/2}dx. \tag{3.118}$$

Putting (3.117) and (3.118) into (3.78) and dividing by $q(u)$, we obtain

$$(e_0-\epsilon a_2e^{-\epsilon t})u\frac{q'(u)}{q(u)}-k+2\epsilon a_1e^{\epsilon t}-2\epsilon a_1e^{-\epsilon t}=0.$$

From the coefficients of 1 and u we get, respectively,

$$2\epsilon a_1 e^{\epsilon t} - 2\epsilon a_1 e^{-\epsilon t} = k \quad \Rightarrow \quad a_1 = k = 0, \tag{3.119}$$

$$\text{and} \quad e_0 - \epsilon a_2 e^{-\epsilon t} = 0 \quad \Rightarrow \quad e_0 = \epsilon a_2 e^{-\epsilon t}. \tag{3.120}$$

Putting (3.119) and (3.120) into (3.111) gives

$$a(t) = \frac{a_0 + \epsilon e_0}{\epsilon^2} = \frac{a_0}{\epsilon^2}. \tag{3.121}$$

From (3.108), (3.110), (3.121) and (3.84) we get the following infinitesimals:

$$\left.\begin{aligned} \xi &= a(t) = \frac{a_0}{\epsilon^2}, \\ \theta &= b(x) = 0, \\ \eta &= 0. \end{aligned}\right\} \tag{3.122}$$

□

Case 3.3:
When $b(x) = 0$ and $f_{u^2} f - (f_u)^2 = 0$, we have

$$\frac{f_{u^2}}{f_u} = \frac{f_u}{f},$$

which on repeated integration yields

$$f(x,u) = q(x)e^{p(x)u}. \tag{3.123}$$

Putting this into (3.78) gives

$$\eta \frac{f_u}{f} + 2a'(t) = 0 \quad \Rightarrow \quad \eta \frac{p(x)q(x)e^{p(x)u}}{q(x)e^{p(x)u}} + 2a'(t) = 0$$

$$\Rightarrow \quad (-\epsilon a_2 e^{-\epsilon t} u + g_0 + h_0 e^{-\epsilon t} + h_1)p(x) = -2\epsilon a_1 e^{\epsilon t} + 2\epsilon a_2 e^{-\epsilon t}. \tag{3.124}$$

Differentiating (3.124) with respect to x yields

$$(-\epsilon a_2 e^{-\epsilon t} u + g_0 + h_0 e^{-\epsilon t} + h_1)p'(x) = 0,$$

which results further into the following cases:

Case 3.3.1:

$$-\epsilon a_2 e^{-\epsilon t} u + g_0 + h_0 e^{-\epsilon t} + h_1 = 0 \tag{3.125}$$

$$\Rightarrow \quad -\epsilon a_2 e^{-\epsilon t} = 0 \quad \text{and} \quad g_0 + h_0 e^{-\epsilon t} + h_1 = 0.$$

Thus (3.124) now becomes

$$-2\epsilon a_1 e^{\epsilon t} + 2\epsilon a_2 e^{-\epsilon t} = 0$$

$$\Rightarrow \quad a_1 = a_2 e^{-2\epsilon t}. \tag{3.126}$$

In view of (3.84), (3.111), (3.125) and (3.126) we get the following infinitesimals:

$$\xi = \frac{a_0}{\epsilon^2}, \quad \theta = 0, \quad \eta = 0. \tag{3.127}$$

□

Case 3.3.2:
$p'(x) = 0 \quad \Rightarrow \quad p(x) = p_0 = \text{constant}.$ Thus

$$f(x,u) = q(x)e^{p_0 u}. \tag{3.128}$$

Putting (3.128) into (3.79) gives

$$(-\epsilon a_2 e^{-\epsilon t} u + h_0 e^{-\epsilon t} + h_1 + g_0)p_0 = -2\epsilon a_1 e^{\epsilon t} + 2\epsilon a_2 e^{-\epsilon t},$$

which on differentiation with respect to u yields

$$\epsilon a_2 e^{-\epsilon t} = 0 \quad \Rightarrow \quad \eta = h_0 e^{-\epsilon t} + h_1 + g_0.$$

Thus

$$(h_0 e^{-\epsilon t} + h_1 + g_0)p_0 = -2\epsilon a_1 e^{\epsilon t} = 2a'(t)$$

$$\Rightarrow \quad a(t) = -\frac{1}{2}p_o(\frac{h_0}{\epsilon}e^{-\epsilon t} - h_1 t - g_0 t) + h_2.$$

Finally we have the infinitesimals as

$$\left.\begin{aligned} &\xi = -\frac{1}{2}p_o(\frac{h_0}{\epsilon}e^{-\epsilon t} - h_1 t - g_0 t) + h_2, \\ &\theta = 0, \\ &\eta = h_0 e^{-\epsilon t} + h_1 + g_0, \end{aligned}\right\} \tag{3.129}$$

where h_0, h_1, h_2, g_0 and p_0 are constants. □

Case 3.4:

$$f_{u^2}-(f_u)^2=0 \quad \text{and} \quad f_{xu}f-f_xf_u=0. \tag{3.130}$$

The above equations, when integrated, yield respectively

$$f(x,u) = q(x)e^{p(x)u}, \tag{3.131}$$

$$f(x,u) = q(u)e^{\int p(x)dx}=f_1(x)f_2(u). \tag{3.132}$$

From (3.131) and (3.132) we see that

$$p(x)=p_0= \text{ constant}.$$

Hence f will be of the form

$$f(x,u)=q(x)e^{p_0u} \tag{3.133}$$

or

$$f(x,u)=q(u)e^{p_0x}. \tag{3.134}$$

Case 3.4.1:

$$f(x,u)=q(x)e^{p_0u} \quad \text{as in (3.133)}.$$

Putting (3.133) into (3.80) and simplifying, we get

$$-[(e_0-\frac{a_0}{2\epsilon}-\epsilon a_2e^{-\epsilon t})u+h_0e^{-\epsilon t}+h_1+g_0]p_0^2-b(x)p_0\frac{q'(x)}{q(x)},$$
$$-(e_0-\frac{a_0}{2\epsilon}-\epsilon a_2e^{-\epsilon t})+2b'(x)-2a'(t)=0.$$

Separating the coefficients of u and 1 gives, respectively,

$$(e_0-\frac{a_0}{2\epsilon}-\epsilon a_2e^{-\epsilon t})p_0^2=0$$
$$\Rightarrow \quad e_0-\frac{a_0}{2\epsilon}-\epsilon a_2e^{-\epsilon t}=0 \quad \text{since} \quad p_0\neq 0, \tag{3.135}$$

$$-(h_0e^{-\epsilon t}+h_1+g_0)p_0^2-2a'(t)=-2b'(x)+p_0b(x)\frac{q'(x)}{q(x)}=k_1=\text{ constant.} \tag{3.136}$$

Hence

$$2a'(t)=-(h_0e^{-\epsilon t}+h_1+g_0)p_0^2+k_1$$

$$\Rightarrow\quad a(t)=\frac{p_0^2}{2}(\frac{h_0}{\epsilon}e^{-\epsilon t}-h_1t-g_0t)+\frac{1}{2}k_1t+k_2. \tag{3.137}$$

Also, from (3.136) we have

$$b'(x)-\frac{p_0q'(x)}{2q(x)}b(x)=-\frac{k_1}{2}.$$

Solving the above first order linear O.D.E gives

$$b(x)=[q(x)]^{p_0/2}[k_3-\frac{k_1}{2}\int\frac{dx}{[q(x)]^{p_0/2}}]. \tag{3.138}$$

Thus we have the following infinitesimals:

$$\left.\begin{aligned}\xi&=\frac{p_0^2}{2}(\frac{h_0}{\epsilon}e^{-\epsilon t}-h_1t-g_0t)+\frac{1}{2}k_1t+k_2,\\ \theta&=[q(x)]^{p_0/2}(k_3-\frac{k_1}{2}\int\frac{dx}{[q(x)]^{p_0/2}}),\\ \eta&=h_0e^{-\epsilon t}+h_1+g_0,\end{aligned}\right\} \tag{3.139}$$

where p_0,k_1,k_2,k_3,h_0,h_1 and g_0 are constants. □

Case 3.4.2:
If we now let

$$f(x,u)=q(u)e^{p_0x},$$

then from (3.80) we get

$$\eta\frac{q''(u)}{q'(u)}=p_0b(x)-2b'(x)+2\epsilon a_1e^{-\epsilon t}-3\epsilon a_2e^{-\epsilon t}. \tag{3.140}$$

Differentiating the right hand side of (3.140) with respect to x yields

$$2b''(x) = p_0 b'(x)$$
$$\Rightarrow \quad \frac{b''(x)}{b'(x)} = \frac{p_0}{2}.$$

which upon two successive integrations with respect to x gives

$$b(x) = \frac{2p_1}{p_0} e^{p_0 x/2} + p_2,$$

where p_0, p_1, p_2 are constants.

The resultant infinitesimals in this case are

$$\left.\begin{aligned} \xi &= \frac{a_0}{\epsilon^2} + a_1 e^{\epsilon t} + a_2 e^{-\epsilon t}, \\ \theta &= \frac{2p_1}{p_0} e^{p_0 x/2} + p_2, \\ \eta &= -\epsilon a_2 e^{-\epsilon t} u + h_0 e^{-\epsilon t} + h_1 + g_0. \end{aligned}\right\} \quad (3.141)$$

□

Case 3.5:

$$h_0 e^{-\epsilon t} + h_1 + g_0 = 0 \quad \text{and} \quad f_{u^2} f - (f_u)^2 = 0. \quad (3.142)$$

The three conditions from (3.115) with (3.142) are:

Case 3.5.1:

$$b(x) = 0 \quad \text{and} \quad f_{xu} f - f_x f_u = 0,$$

which implies (see (3.132))

$$f(x,u) = q(u) e^{\int p(x)dx}.$$

Putting these into (3.78) gives

$$\eta \frac{f_u}{f} + 2a'(t) = 0. \quad (3.143)$$

Differentiating (3.143) with respect to x yields

$$\eta_x \frac{f_u f}{f^2} + \eta \frac{f_{xu} f - f_x f_u}{f^2} = 0.$$

Since $f_{xu}f - f_x f_u = 0$, we now have

$$\begin{aligned} &\eta_x f_u f = 0 \\ \Rightarrow \quad &\eta_x = 0 \qquad \text{since } f_u f \neq 0 \\ \Rightarrow \quad &\eta = \eta(t). \end{aligned}$$

Thus (3.143) now implies

$$a(t) = a_0 = \text{ constant.}$$

Finally we have the following infinitesimals:

$$\xi = a_0, \quad \theta = 0, \quad \eta = \eta(t). \tag{3.144}$$

□

Case 3.5.2:

$$b(x) = 0 \quad \text{and} \quad f_{xu}f - f_x f_u \neq 0. \tag{3.145}$$

Putting this into (3.78) gives

$$\begin{aligned} &\eta \frac{f_u}{f} + 2a'(t) = 0 \\ \Rightarrow \quad &\eta_x f_u f + \eta(f_{xu}f - f_x f_u) = 0; \end{aligned}$$

see (3.143). Since $f_{xu}f - f_x f_u \neq 0$ and $f_u f \neq 0$, we have from the above equation

$$\eta = \eta(t).$$

Hence we have the same infinitesimals as in case 3.5.1 above, i.e. (3.144):

$$\xi = a_0, \quad \theta = 0, \quad \eta = \eta(t).$$

□

Case 3.5.3:

$$b(x) \neq 0 \quad \text{and} \quad f_{xu}f - f_x f_u = 0. \tag{3.146}$$

In this case we have

$$f(x,u) = q(u)e^{\int p(x)dx}.$$

Hence (3.78) now becomes

$$[(e_0 - \frac{a_0}{2\epsilon})u + h_0 e^{-\epsilon t} + h_1 + g_0]\frac{q'(u)}{q(u)} + b(x)p(x) - 2b'(x) + 2a'(t) = 0.$$

Separating the coefficients of u and 1 gives, respectively,

$$(e_0 - \frac{a_0}{2\epsilon})\frac{q'(u)}{q(u)} = 0 \quad \Rightarrow \quad e_0 - \frac{a_0}{2\epsilon} = 0, \quad \text{since } q'(u) \neq 0, \tag{3.147}$$

$$(h_0 e^{-\epsilon t} + h_1 + g_0)\frac{q'(u)}{q(u)} + b(x)p(x) - 2b'(x) + 2a'(t) = 0. \tag{3.148}$$

Differentiating (3.148) with respect to t yields

$$-\epsilon h_0 e^{-\epsilon t}\frac{q'(u)}{q(u)} + 2a''(t) = 0$$

$$\Rightarrow \quad \frac{q'(u)}{q(u)} = \frac{2a''(t)}{\epsilon h_0}e^{\epsilon t} = \lambda_0 = \text{constant}. \tag{3.149}$$

From (3.149) it is easy to see that

$$q(u) = \lambda_1 e^{\lambda_0 u}$$

$$\Rightarrow \quad f(x,u) = \lambda_1 e^{\lambda_0 u} e^{\int p(x)dx} \tag{3.150}$$

and

$$a''(t) = \frac{\epsilon h_0 \lambda_0}{2}e^{-\epsilon t},$$

which on being integrated twice yields

$$a(t) = \frac{h_0 \lambda_0}{2\epsilon}e^{-\epsilon t} + \lambda_1 t + \lambda_2, \quad \lambda_1, \lambda_2 \text{ constants}. \tag{3.151}$$

In order to determine $b(x)$, we introduce (3.149) into (3.148) to get

$$(h_0 e^{-\epsilon t} + h_1 + g_0)\lambda_0 + 2a'(t) = 2b'(x) - b(x)p(x) = \beta = \text{ constant.}$$

Hence we have the first order linear O.D.E.

$$b'(x) - \frac{1}{2}p(x)b(x) = \frac{\beta}{2},$$

which can easily be solved in $b(x)$ to give

$$b(x) = e^{\int p(x)dx/2}[\frac{\beta}{2}\int e^{-\int p(x)dx/2}dx + \alpha], \quad \alpha, \beta \text{ constants.} \tag{3.152}$$

From (3.147) we have

$$\eta = h_0 e^{-\epsilon t} + h_1 + g_0. \tag{3.153}$$

The infinitesimals

$$\left.\begin{aligned} \xi &= \frac{h_0\lambda_0}{2\epsilon}e^{-\epsilon t} + \lambda_1 t + \lambda_2, \\ \theta &= e^{\int p(x)dx/2}[\frac{\beta}{2}\int e^{-\int p(x)dx/2}dx + \alpha], \\ \eta &= h_0 e^{-\epsilon t} + h_1 + g_0, \end{aligned}\right\} \tag{3.154}$$

are immediate from (3.151), (3.152) and (3.153). □

Case 3.6:

$$b(x) = 0 \quad \text{and} \quad f_{xu}f - f_x f_u = 0. \tag{3.155}$$

In view of (3.115) the above condition (3.155) will lead to the following cases:

Case 3.6.1:

$$h_0 e^{-\epsilon t} + h_1 + g_0 \neq 0 \quad \text{and} \quad f_{u^2}f - (f_u)^2 = 0. \tag{3.156}$$

In this case f is of the form

$$f(x,u) = q(u)e^{\int p(x)dx}. \tag{3.157}$$

Hence (3.114) now becomes

$$(h_0 e^{-\epsilon t} + h_1 + g_0)\frac{q'(u)}{q(u)} + b(x)p(x) - 2b'(x) = -2\epsilon a_1 e^{\epsilon t} + 2\epsilon a_2 e^{-\epsilon t}, \tag{3.158}$$

which on differentiation with respect to t and after some calculations yields

$$\frac{q'(u)}{q(u)} = \frac{2\epsilon}{h_0}(a_1 e^{2\epsilon t} - a_2) = q_0 = \text{ constant} \tag{3.159}$$

$$\Rightarrow \qquad q(u) = q_1 e^{q_0 u}$$

$$\Rightarrow \quad f(x,u) = q_1 e^{q_0 u} e^{\int p(x)dx}. \tag{3.160}$$

From (3.159) we get

$$a_2 = a_1 e^{2\epsilon t} - \frac{q_0 h_0}{2\epsilon}$$

$$\Rightarrow \quad a(t) = \frac{a_0}{\epsilon^2} + 2a_1 e^{\epsilon t} - \frac{q_0 h_0}{2\epsilon} e^{-\epsilon t}. \tag{3.161}$$

Introducing (3.160) into (3.158) gives

$$(h_0 e^{-\epsilon t} + h_1 + g_0)q_0 + 2\epsilon a_1 e^{\epsilon t} - 2\epsilon a_2 e^{-\epsilon t} = 2b'(x) - b(x)p(x) = b_0 = \text{ constant},$$

from whence we have the O.D.E.

$$b'(x) - \frac{1}{2}p(x)b(x) = \frac{b_0}{2}$$

which can easily be solved in $b(x)$ to give

$$b(x) = e^{\int p(x)dx/2}[\frac{b_0}{2}\int e^{-\int p(x)dx/2}dx + C].$$

In order to determine η, we use (3.108) – (3.111) and (3.161). Thus

$$\eta = \frac{q_0 h_0}{2} e^{-\epsilon t} u + h_0 e^{-\epsilon t} + h_1 + g_0.$$

Hence we have the following infinitesimals:

$$\left.\begin{aligned} \xi &= \frac{a_0}{\epsilon^2} + 2a_1 e^{\epsilon t} - \frac{q_0 h_0}{2\epsilon} e^{-\epsilon t}, \\ \theta &= e^{\int p(x)dx/2}[\frac{b_0}{2}\int e^{-\int p(x)dx/2}dx + C], \\ \eta &= (\frac{q_0}{2}u + 1)h_0 e^{-\epsilon t} + h_1 + g_0. \end{aligned}\right\} \quad (3.162)$$

□

Case 3.6.2:

$$h_0 e^{-\epsilon t} + h_1 + g_0 = 0 \quad \text{and} \quad f_{u^2} f - (f_u)^2 \neq 0. \quad (3.163)$$

Introducing (3.163) into (3.114) gives

$$2b'(x) - b(x)\frac{f_x}{f} = 2\epsilon a_1 e^{\epsilon t} - 2\epsilon a_2 e^{-\epsilon t}. \quad (3.164)$$

Differentiating (3.164) with respect to u yields

$$b(x)(f_{xu}f - f_x f_u) = 0, \quad (3.165)$$

which in view of (3.163) has been considered in cases 3.1 and 3.2. □

Case 3.7:

$$(h_0 e^{-\epsilon t} + h_1 + g_0)(f_{u^2} f - (f_u)^2) \neq 0, \quad (3.166)$$

which in view of (3.115) implies

$$b(x)(f_{xu}f - f_x f_u) \neq 0. \quad (3.167)$$

Further we see from (3.166) and (3.167) that

$$h_0 e^{-\epsilon t} + h_1 + g_0 \neq 0, \quad f_{u^2} f - f_u^2 \neq 0, \quad b(x) \neq 0$$

and

$$f_{xu}f - f_x f_u \neq 0. \quad (3.168)$$

Putting (3.168) into (3.115) yields

$$b(x) = (h_0 e^{-\epsilon t} + h_1 + g_0)\frac{f_{u^2} f - f_u^2}{f_{xu} f - f_x f_u}. \quad (3.169)$$

The infinitesimals

$$\left.\begin{aligned}
\xi &= \frac{a_0}{\epsilon^2} + a_1 e^{\epsilon t} + a_2 e^{-\epsilon t}, \\
\theta &= (h_0 e^{-\epsilon t} + h_1 + g_0)\frac{f_{u^2} f - f_u^2}{f_{xu} f - f_x f_u}, \\
\eta &= -\epsilon a_2 e^{-\epsilon t} u + h_0 e^{-\epsilon t} + h_1 + g_0,
\end{aligned}\right\} \qquad (3.170)$$

are immediate from (3.84), (3.108), (3.110), (3.111), (3.113) and (3.169). □

NOTE

The diagram in Figure 3.2 on the next page illustrates the different cases of the form of $f(x,u)$ in (3.70).

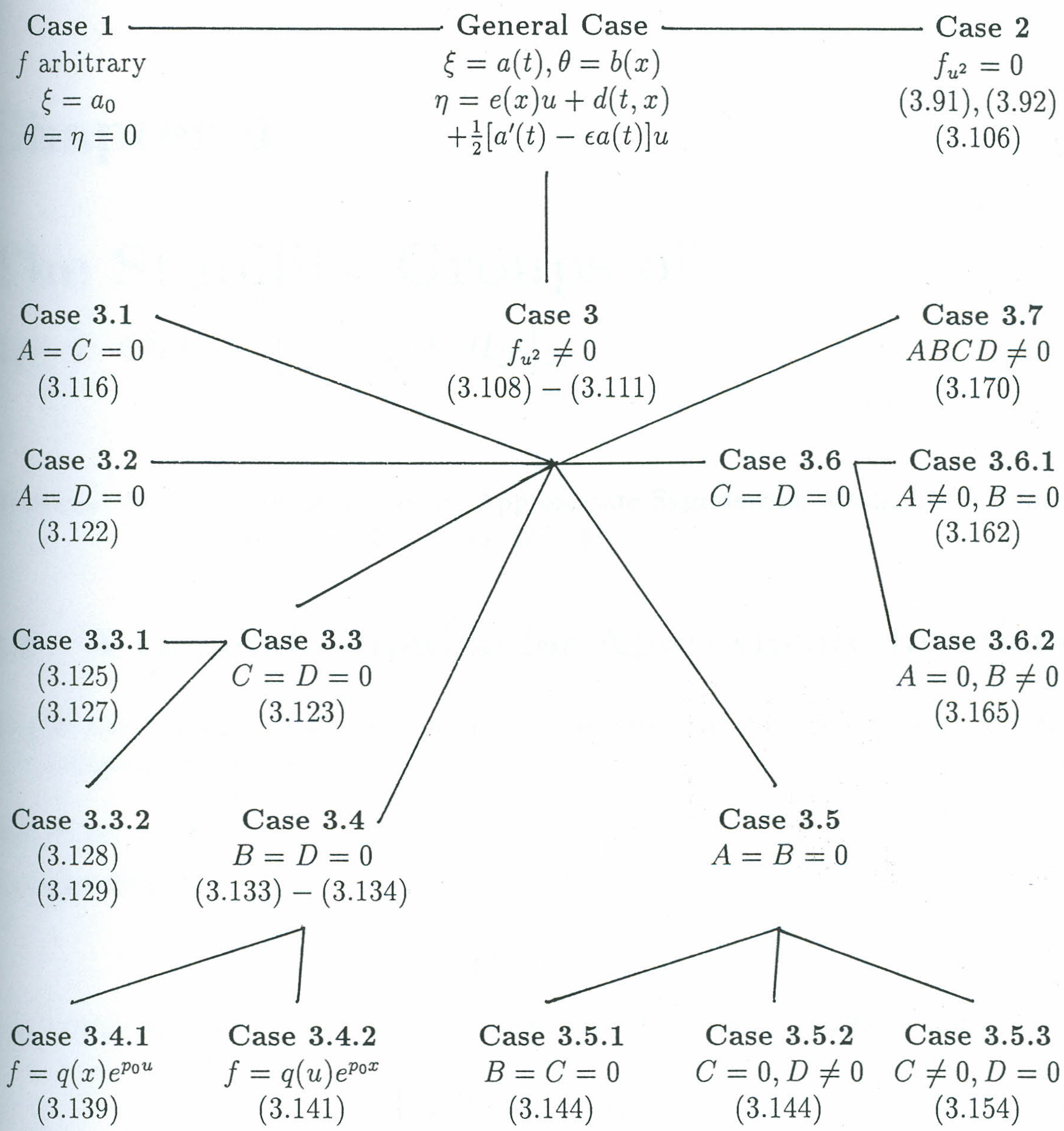

Figure 3.2

$$A = h_0 e^{-\epsilon t} + h_1 + g_0, \quad B = f_{u^2} f - f_u^2, \quad C = b(x), \quad D = f_{xu} f - f_x f_u$$

Chapter 4

The Stability Groups of $u_{tt} + \epsilon u_t = [f(x,u)u_x]_x$

Sources: [1] N.Kh. Ibragimov et al, Approximate Symmetries, Math. USSR Sbornik, Vol. 64 (1989), No. 2, pp. 427-440.

4.1 A General Criterion for Approximate Invariance

We give the definition of the invariance of an approximate equation with respect to an approximate group of transformations.

Definition 4.1-1

The approximate equation

$$F(z, \epsilon) \approx 0 \tag{4.1}$$

is said to be *invariant with respect to the approximate group of transformations*

$$z^* \approx f(z, \epsilon, a) \tag{4.2}$$

if

$$F(f(z, \epsilon, a), \epsilon) \approx 0 \tag{4.3}$$

for all $z = (z^1, z^2, \ldots, z^N)$ satisfying (4.1).

The following theorem gives the necessary and sufficient conditions for (4.1) to be invariant under (4.2).

Theorem 4.1-1

Suppose that the function

$$F(z,\epsilon) = (F'(z,\epsilon), F^2(z,\epsilon), \ldots, F^n(z,\epsilon), \quad n < N, \tag{4.4}$$

which is jointly analytic in the variables z and ϵ, satisfies the condition

$$\text{rank } F'(z,0)|_{F(z,0)=0} = n, \tag{4.5}$$

where

$$F'(z,\epsilon) = \|\partial F^\nu(z,\epsilon)/\partial z^i\| \quad \text{for} \quad \nu = 1,\ldots,n \quad \text{and} \quad i = 1,\ldots,N.$$

Then, for the approximate equation (4.1), written as

$$F(z,\epsilon) = 0(\epsilon^p), \tag{4.6}$$

to be invariant under the approximate group of transformations

$$x^* = f(z,\epsilon,a) + 0(\epsilon^p)$$

with infinitesimal generator

$$V = \xi(z,\epsilon)\frac{\partial}{\partial z}, \quad \xi = \frac{\partial f}{\partial a}|_{a=0} + 0(\epsilon^p), \tag{4.7}$$

it is necessary and sufficient that

$$VF(z,\epsilon)|_{(4.6)} = 0(\epsilon^p). \tag{4.8}$$

For proof of Theorem 4.1-1, see [1].

The following example illustrates the above theorem further.

Example 4.1-1

In Example 2.2-2 we constructed the approximate group of transformations

$$\left.\begin{aligned} x^* &\approx x + a + \epsilon(x^2 a + x a^2 + \frac{a^3}{3}), \\ y^* &\approx y + \epsilon(xya + y\frac{a^2}{2}) \end{aligned}\right\} \tag{4.9}$$

determined by the infinitesimal generator

$$V = (1 + \epsilon x^2)\frac{\partial}{\partial x} + \epsilon xy\frac{\partial}{\partial y}. \tag{4.10}$$

Let $N = 2,\ z = (x, y)$ and $p = 1$.

We want to show that the approximate equation

$$F(x, y, \epsilon) \equiv y^{2+\epsilon} - \epsilon x^2 - 1 = 0(\epsilon) \tag{4.11}$$

is invariant with respect to the transformations (4.9).

First we verify the invariance of (4.11) according to Definition 4.1-1. To do this we shall use equation

$$\overline{F}(x, y, \epsilon) \equiv y^2 - \epsilon(x^2 - y^2 \ln y) - 1 \approx 0 \tag{4.12}$$

which is equivalent to (4.11).

Equating (4.11) and (4.12), we have

$$\begin{aligned} y^{2+\epsilon} - \epsilon x^2 - 1 + 0_1(\epsilon) &= y^2 - \epsilon(x^2 - y^2 \ln y) - 1 \\ \Longrightarrow y^{2+\epsilon} + 0_1(\epsilon) &= y^2 + \epsilon y^2 \ln y. \end{aligned} \tag{4.13}$$

Taylor's expansion of $y^{2+\epsilon}$ about $\epsilon = 0$ gives

$$y^{2+\epsilon} = y^2 + \epsilon y^2 \ln y + 0(\epsilon^2).$$

Inserting this into (4.13) gives

$$y^2 + \epsilon y^2 \ln y + 0(\epsilon^2) + 0_1(\epsilon) = y^2 + \epsilon y^2 \ln y,$$

which implies that (4.11) is equivalent to (4.13) within $0_1(\epsilon)$.

We now proceed to establish the invariance of (4.11) using the equivalent equation (4.13). The transformation (4.9) implies

$$\begin{aligned} \overline{F}(x^*, y^*; \epsilon) &= {y^*}^2 - \epsilon({x^*}^2 - {y^*}^2 \ln y^*) - 1 \\ &\approx y^2 - \epsilon(x^2 - y^2 \ln y) - 1 + \epsilon(2xa + a^2)(y^2 - 1) \\ &= \overline{F}(x, y, \epsilon) + \epsilon(2xa + a^2)[\overline{F}(x, y, \epsilon) + \epsilon(x^2 - y^2 \ln y)] \\ &= [1 + \epsilon(2ax + a^2)]\overline{F}(x, y, \epsilon) + 0(\epsilon), \end{aligned}$$

which implies (4.3). Thus

$$\overline{F}(x^*, y^*, \epsilon) \approx 0.$$

Obviously (4.11) satisfies condition (4.5) of Theorem 4.1-1, since $n = 1$.

By (4.8), we have that

$$\begin{aligned} VF &= (2+\epsilon)\epsilon x y^{2+\epsilon} - 2\epsilon x(1+\epsilon x^2) \\ &= 2\epsilon x y^{2+\epsilon} - 2x\epsilon \\ &= 2\epsilon x(y^{2+\epsilon} - 1) + 0(\epsilon) \\ &= 2\epsilon x F + 0(\epsilon) = 0(\epsilon) \end{aligned}$$

in view of (4.11). Thus

$$VF(x, y; \epsilon) = 0(\epsilon)$$

and the invariance of (4.11) with respect to (4.10) is established.

According to Theorem 4.1-1, the construction of the approximate group leaving equation (4.1) invariant reduces to the solution of the defining equation (4.8) for the coordinates $\xi^k(z,\epsilon)$ of the infinitesimal operator (4.7).

We now discuss how to solve the defining equation (4.8) to within $0(\epsilon^p)$. The following representations will be useful:

Let

$$\left.\begin{aligned} z &\approx y_0 + \epsilon y_1 + \ldots + \epsilon^p y_p, \\ F(z,\epsilon) &\approx \sum_{i=0}^{p} \epsilon^i F_i(z) \\ \text{and} \quad \xi^k(z,\epsilon) &\approx \sum_{i=0}^{p} \epsilon^i \xi_i^k(z). \end{aligned}\right\} \qquad (4.14)$$

Substituting (4.14) into (4.8) and singling out the principal terms we have

$$\begin{aligned} VF &= \xi^k \frac{\partial F}{\partial z^k} \\ &= \left[\sum_{i=0}^{p} \epsilon^i \xi_i^k (y_0 + \epsilon y_1 + \ldots + \epsilon^p y_p)\right] \cdot \left[\sum_{j=0}^{p} \epsilon^j \frac{\partial}{\partial z^k} F_j(y_0 + \ldots + \epsilon^{p_y} p)\right]. \end{aligned} \qquad (4.15)$$

It is known from [1] that, with the index notation as on page 46,

$$\sum_{i=0}^{p}\epsilon^i F_i(y_0+\epsilon y_1+\ldots+\epsilon^{p_y}p)$$
$$\approx F_0(y_0)+\sum_{i=1}^{p}\epsilon^i\left[F_i(y_0)+\sum_{j=1}^{i}\sum_{|\sigma|=1}^{j}\frac{1}{\sigma!}F_{i-j}^{(\sigma)}(y_0)\sum_{|\nu|=j}y_{(\nu)}\right]. \tag{4.16}$$

By (4.16), we have

$$\sum_{i=0}^{p}\epsilon^i\xi_i^k(y_0)$$
$$\approx \xi_0^k(y_0)+\sum_{i=1}^{p}\epsilon^i\left[\xi_i^k(y_0)+\sum_{j=1}^{i}\sum_{|\sigma|=1}^{j}\frac{1}{\sigma!}\xi_{i-j}^{(\sigma)}(y_0)\sum_{|\nu|=j}y_{(\nu)}\right] \tag{4.17}$$

and

$$\sum_{j=0}^{p}\epsilon^j\frac{\partial F_j(y_0)}{\partial z^k}$$
$$\approx\frac{\partial F_0(y_0)}{\partial z^k}+\sum_{j=1}^{p}\epsilon^j\left[\frac{\partial F_j(y_0)}{\partial z^k}+\sum_{i=1}^{j}\sum_{|\omega|=1}^{i}\frac{1}{\omega!}\left(\frac{\partial F_{j-i}^{(\omega)}}{\partial z^k}\right)(y_0)\sum_{|\mu|=1}y_{(\mu)}\right]. \tag{4.18}$$

Inserting (4.17) and (4.18) into (4.15) gives

$$VF=\left[\xi_0^k(y_0)+\sum_{i=1}^{p}\epsilon^iA_i^k\right]\cdot\left[\frac{\partial F_0(y_0)}{\partial z^k}+\sum_{j=1}^{p}\epsilon^jB_{j,k}\right], \tag{4.19}$$

where

$$A_i^k=\xi_i^k(y_0)+\sum_{j=1}^{i}\sum_{|\sigma|=1}^{j}\frac{1}{\sigma!}\left(\xi_{i-j}^{(\sigma)}(y_0)\right)\sum_{|\nu|=j}y_{(\nu)} \tag{4.20}$$

and

$$B_{j,k}=\frac{\partial F_j(y_0)}{\partial z^k}+\sum_{i=1}^{j}\sum_{|\omega|=1}^{i}\frac{1}{\omega!}\left(\frac{\partial F_{j-i}^{(\omega)}}{\partial z^k}\right)(y_0)\sum_{|\mu|=j}y_{(\mu)}. \tag{4.21}$$

Expanding (4.19) we get

$$\begin{aligned} VF &= \xi_o^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} + \sum_{j=1}^{p}\epsilon^j \xi_0^k(y_0)B_{j,k} + \sum_{i=1}^{p}\epsilon^i A_i^k\frac{\partial f_0(y_0)}{\partial z^k} \\ &+ \sum_{i+j=2}\epsilon^{i+j}\sum_{i+j=l}A_i^k B_{j,k}. \end{aligned}$$

For i = j = 1, we have

$$\begin{aligned} VF &= \xi_0^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} + \epsilon\xi_0^k(y_0)B_{1,k} + \epsilon A_1^k\frac{\partial F_0(y_0)}{\partial z^k} + \sum_{l=2}^{p}\epsilon^l\xi_0^k(y_0)B_{l,k} \\ &+ \sum_{l=2}^{p}\epsilon^l A_l^k\frac{\partial F_0(y_0)}{\partial z^k} + \sum_{l=2}\epsilon^l\sum_{i+j=l}A_i^k B_{j,k} \\ &= \xi_0^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} + \epsilon\left(\xi_0^k(y_0)B_{1,k} + A_1^k\frac{\partial F_0(y_0)}{\partial z^k}\right) \\ &+ \sum_{l=2}\epsilon^l\left(\xi_0^k B_{l,k} + A_l^k\frac{\partial F_0(y_0)}{\partial z^k} + \sum_{i+j=l}A_i^k B_{j,k}\right). \end{aligned} \tag{4.22}$$

Equating the coefficients of $1, \epsilon$ and $\sum \epsilon^l$ in (4.22) to zero (since $VF = 0$) we get the following form of the defining equation:

$$\xi_0^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} = 0, \quad \xi_0^k(y_0)B_{1,k} + A_1^k\frac{\partial F_0(y_0)}{\partial z^k} = 0,$$

$$\xi_0^k(y_0)B_{l,k} + A_l^k\frac{\partial F_0(y_0)}{\partial z^k} + \sum_{i+j=l}A_j^k B_{j,k} = 0, \quad l = 2, 3, \ldots, p. \tag{4.23}$$

Equations (4.23) hold on the set of all $y_0, y_1, \ldots, y_p$ satisfying the system

$$F_0(y_0) = 0,$$

$$F_i(y_0) + \sum_{j=1}^{i}\sum_{|\sigma|=1}^{j}\frac{1}{\sigma!}F_{i-j}^{(\sigma)}(y_0)\sum_{|\nu|=j}y_{(\nu)} = 0, \quad i = 1, 2, \ldots, p, \tag{4.24}$$

which is equivalent to the approximate equation (4.1).

It is now clear that the problem of solving the approximate defining equation (4.8) has been reduced to the solution of the system of exact equations (4.23) and (4.24).

For $p = 1$, (4.23) and (4.24) implies

$$\xi_0^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} = 0, \tag{4.25}$$

$$\xi_1^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} + \xi_0^k(y_0)\frac{\partial F_1(y_0)}{\partial z^k} + \sum_{l=1}^{N}\sum_{k=1}^{N} y_1^l \frac{\partial}{\partial z^l}\left(\xi_0^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k}\right) = 0, \tag{4.26}$$

under the conditions

$$\left.\begin{aligned} F_0(y_0) &= 0, \\ F_1(y_0) + \sum_{l=1}^{N} y_1^l \frac{\partial F_0(y_0)}{\partial z^l} &= 0. \end{aligned}\right\} \tag{4.27}$$

Example 4.1-2

Let the approximate equation

$$F(x, y, \epsilon) = y^{2+\epsilon} - \epsilon x^2 - 1 = 0(\epsilon)$$

be as (4.11) in Example 4.1-1.

Using (4.12) we have

$$F_0(x, y) = y^2 - 1, \quad F_1(x, y) = y^2 \ln y - x^2.$$

We now need to find y_0 and y_1 from (3.95). It is easy to see that $y_0 = 1$ since $y > 0$.

$$\begin{aligned} F_1(y_0) + \sum_{l=1}^{N} y_1^l \frac{\partial F_0(y_0)}{\partial z^l} &= y^2 \ln y - x^2 + y_1^1 \frac{\partial(y^2-1)}{\partial x} + y_1^2 \frac{\partial(y^2-1)}{\partial y} = 0 \\ \Longrightarrow \quad & 1^2 \ln 1 - x_0^2 + x_1 \cdot 0 + 2y_1 \cdot y_0 = 0 \\ \Longrightarrow \quad & y_1 = \frac{x_0^2}{2}. \end{aligned}$$

Thus

$$y_0 = (x_0, 1) \quad \text{and} \quad y_1 = (x_1, \frac{x_0^2}{2}), \tag{4.28}$$

where x_1 is a free variable.

Substituting (4.28) into (4.25) and (4.26) gives

$$\left.\begin{aligned}
\xi_0^2(x_0, y_0) &= 0, \\
\frac{\partial \xi_0^2(x_0, y_0)}{\partial x} &= 0, \\
\xi_1^2(x_0, y_0) - x_0\xi_0^1(x_0, y_0) + \frac{(1 + x_0^2)}{2}\xi_0^2(x_0, y_0) + \frac{x_0^2}{2}\frac{\partial \xi_0^2(x_0, y_0)}{\partial y} &= 0.
\end{aligned}\right\} \quad (4.29)$$

□

4.2 Stability Groups of $u_{tt} + \epsilon u_t = [f(x, u)u_x]_x$

In this section we aim to construct approximate symmetries of first order, i.e. $p = 1$, for the perturbed nonlinear wave equation

$$u_{tt} + \epsilon u_t = [f(x, u)u_x]_x, \ f \in C^2(R \times R), \ f > 0, \ f_x \neq 0, \ f_u \neq 0. \tag{4.30}$$

An algorithm for constructing approximate groups of the form

$$z^* \approx f_0(z, a) + \epsilon f_1(z, a) \tag{4.31}$$

determined by the infinitesimal generator

$$V = [\xi_0(z) + \epsilon \xi_1(z)] \frac{\partial}{\partial z} \tag{4.32}$$

has been discussed in detail in section 3.2 (see [1] for more details).

By theorem 4.1-1, the construction of (4.31) leaving the equation

$$F(z, \epsilon) \approx 0$$

invariant reduces to the solution of

$$VF(z, \epsilon)|_{F \approx 0} \approx 0. \tag{4.33}$$

For equation (4.30) with $p = 1$, (4.33) yields (4.25) and (4.16) under the conditions (4.27), where from (4.30)

$$\left.\begin{aligned} F_0 &= u_{11} - f_x u_2 - f_u (u_2)^2 - f u_{22}, \\ F_1 &= u_1, \\ z &= (t, x, u, u_1, u_2, u_{11}, u_{12}, u_{22}). \end{aligned}\right\} \tag{4.34}$$

Differentiating F_0 and F_1 with respect to $t, x, u, u_1, u_2, u_{11}, u_{12}$ and u_{22} yields

$$\left.\begin{aligned}
\frac{\partial F_0}{\partial t} &= \frac{\partial F_0}{\partial u_1} = \frac{\partial F_0}{\partial u_{12}} = 0,\\
\frac{\partial F_0}{\partial x} &= -f_{x^2}u_2 - f_{ux}(u_2)^2 - f_x u_{22},\\
\frac{\partial F_0}{\partial u} &= -f_{xu}u_2 - f_{u^2}(u_2)^2 - f_u u_{22},\\
\frac{\partial F_0}{\partial u_2} &= -f_x - 2f_u u_2,\\
\frac{\partial F_0}{\partial u_{11}} &= 1,\\
\frac{\partial F_0}{\partial u_{22}} &= -f,\\
\frac{\partial F_1}{\partial u_1} &= 1,\\
\frac{\partial F_1}{\partial t} &= \frac{\partial F_1}{\partial x} = \frac{\partial F_1}{\partial u} = \frac{\partial F_1}{\partial u_2} = \frac{\partial F_1}{\partial u_{11}} = \frac{\partial F_1}{\partial u_{12}} = \frac{\partial F_1}{\partial u_{22}} = 0.
\end{aligned}\right\} \tag{4.35}$$

Using (4.35) it is easy to see that

$$\begin{aligned}
\sum_{k=1}^{8} \xi_1^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} &= -(f_{x^2}u_2 + f_{xu}(u_2)^2 + f_x u_{22})\xi_1^2(y_0)\\
&\quad - (f_{xu}u_2 + f_{u^2}(u_2)^2 + f_u u_{22})\xi_1^3(y_0)\\
&\quad - (f_x + 2f_u u_2)\xi_1^5(y_0) + \xi_1^6(y_0) - f\xi_1^8(y_0),
\end{aligned} \tag{4.36}$$

$$\sum_{k=1}^{8} \xi_0^k(y_0)\frac{\partial F_1(y_0)}{\partial z^k} = \xi_0^4(y_0) = \eta_1^{(1)}, \tag{4.37}$$

where

$$\eta_1^{(1)} = \eta_t + [\eta_u - \xi_t]u_1 - \theta_t u_2,$$

$$\begin{aligned}\sum_{k=1}^{8}\xi_0^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k} &= \theta(-f_{x^2}u_2-f_{xu}(u_2)^2-f_xu_{22})\\ &+\eta(-f_{xu}u_2-f_{u^2}(u_2)^2-f_uu_{22})\\ &+\eta_2^{(1)}(-f_x-2f_uu_2)+\eta_{11}^{(2)}-\eta_{22}^{(2)}f,\end{aligned} \tag{4.38}$$

where $\eta_2^{(1)}$, $\eta_{11}^{(2)}$ and $\eta_{22}^{(2)}$ are as in (1.58), and

$$\begin{aligned}\sum_{l=1}^{8}\sum_{k=1}^{8}y_1^l\frac{\partial}{\partial z^l}\left(\xi_0^k(y_0)\frac{\partial F_0(y_0)}{\partial z^k}\right) &= y_1^1\frac{\partial}{\partial t}[\,\theta(-f_{x^2}u_2-f_{xu}(u_2)^2-f_xu_{22})\\ &+\eta(-f_{xu}u_2-f_{u^2}(u_2)^2-f_{xu}u_{22})\\ &+\eta_2^{(1)}(-f_x-2f_uu_2)+\eta_{11}^{(2)}-\eta_{22}^{(2)}f\,]\\ &+y_1^2\frac{\partial}{\partial x}[\ldots\ldots]\\ &+y_1^3\frac{\partial}{\partial u}[\ldots\ldots]\\ &+y_1^4\frac{\partial}{\partial u_1}[\ldots\ldots]\\ &+y_1^5\frac{\partial}{\partial u_2}[\ldots\ldots]\\ &+y_1^6\frac{\partial}{\partial u_{11}}[\ldots\ldots]\\ &+y_1^7\frac{\partial}{\partial u_{12}}[\ldots\ldots]\\ &+y_1^8\frac{\partial}{\partial u_{22}}[\ldots\ldots].\end{aligned} \tag{4.39}$$

Putting (4.36), (4.37) and (4.39) into (4.26) gives the equation

$$0=(4.36)+(4.37)+(4.39). \tag{4.40}$$

As we did for the exact groups, we shall first consider the case for the arbitrary function $f(x,u)$, then other particular cases of the function f.

The case when f is arbitrary:
We have the infinitesimals (see (3.20))

$$\xi = a_0, \quad \theta = 0, \quad \eta = 0,$$

which correspond to (4.25) and $(4.27)_1$. Thus

$$\left.\begin{aligned}
\xi_0^1(z) &= \xi &&= a_0,\\
\xi_0^2(z) &= \theta &&= 0,\\
\xi_0^3(z) &= \eta &&= 0,\\
\xi_0^4(z) &= \eta_1^{(1)} &&= 0,\\
\xi_0^5(z) &= \eta_2^{(1)} &&= 0,\\
\xi_0^6(z) &= \eta_{11}^{(2)} &&= 0,\\
\xi_0^7(z) &= \eta_{12}^{(2)} &&= 0,\\
\xi_0^8(z) &= \eta_{22}^{(2)} &&= 0.
\end{aligned}\right\} \tag{4.41}$$

We now turn to solving (4.26) and $(4.27)_2$. In view of (4.41), equation (4.40) now becomes

$$\begin{aligned}
0 \;=\; & [f_{x^2}u_2 + f_{xu}(u_2)^2 + f_x u_{22}]\xi_1^2(y_0)\\
& + [f_{xu}u_2 + f_{u^2}(u_2)^2 + f_u u_{22}]\xi_1^3(y_0)\\
& + [f_x + 2f_u u_2]\xi_1^5(y_0) + \xi_1^6(y_0) - f\xi_1^8(y_0).
\end{aligned} \tag{4.42}$$

By identifying with zero the coefficients of $1, u_2, (u_2)^2$ and u_{22} in (4.42), we obtain

$$\left.\begin{aligned}
&\text{(i)} \quad f_x\xi_1^5(y_0) + \xi_1^6(y_0) - f\xi_1^8(y_0) = 0,\\
&\text{(ii)} \quad f_{x^2}\xi_1^2(y_0) + f_{xu}\xi_1^3(y_0) + 2f_u\xi_1^5(y_0) = 0,\\
&\text{(iii)} \quad f_{xu}\xi_1^2(y_0) + f_{u^2}\xi_1^3(y_0) = 0,\\
&\text{(iv)} \quad f_x\xi_1^2(y_0) + f_u\xi_1^3(y_0) = 0.
\end{aligned}\right\} \tag{4.43}$$

From equations (iii) and (iv) in (4.43) we have

$$\xi_1^2(y_0)=\xi_1^3(y_0)=0 \tag{4.44}$$

unless

$$f_u f_{xu}=f_x f_{u^2}. \tag{4.45}$$

Since f is arbitrary, it is obvious that (4.45) does not apply at all.
Putting (4.44) in (4.43ii) we obtain

$$\xi_1^5(y_0)=0 \tag{4.46}$$

since $f_u\neq 0$. Consequently, from (4.43i) we have

$$\xi_1^6(y_0)=f\xi_1^8(y_0). \tag{4.47}$$

Thus $\xi_1^8(y_0)$ may be assumed to be arbitrary. We also note that in view of (4.34) and (4.26), $\xi_1^1(y_0),\xi_1^4(y_0)$ and $\xi_1^7(y_0)$ will also remain arbitrary since their coefficients in (4.26) vanish.

Finally we now find $f_1=(f_1^1,f_1^2,\ldots,f_1^8)$. For this we solve the system (see (2.24) and (2.25))

$$\frac{df_1(z,a)}{da}=\sum_{k=1}^{8}\frac{\partial\xi_0(f_0)}{\partial z^k}f_1^k(z,a)+\xi_1(f_0) \tag{4.48}$$

under the initial conditions

$$f_1^k(z,a)|_{a=0}=0,$$

where $f_0=(f_0^1,f_0^2,\ldots,f_0^8)$ is known from (3.20).

First we note that in view of (4.34), the functions $\xi_0(z)$ and $\xi_1(z)$ in (4.32) are only supposed to depend on the first three variables in z, namely t,x and u. Consequently in (4.31) only the first three components of f_0 and f_1, that is f_0^1,f_0^2,f_0^3 and f_1^1,f_1^2,f_1^3, have to be determined. Furthermore, these components can also be assumed to depend on t,x and u only. In this way, in the system (4.48), we are only interested in the first three equations and the respective initial conditions. Now, (3.20) will give

$$\left.\begin{aligned} f_0^1(t,x,u,a)&=t^*=t+ca,\\ f_0^2(t,x,u,a)&=x^*=x,\\ f_0^3(t,x,u,a)&=u^*=u,\end{aligned}\right\} \tag{4.49}$$

where c is an arbitrary constant and a is the group parameter.
In view of (4.41), (4.44), (4.46), (4.47) and (4.49), the system (4.48) now becomes

$$\frac{df_1(z,a)}{da} = \xi_1(f_0(z,a)) \tag{4.50}$$

and the first three equations are

$$\left.\begin{aligned} \frac{df_1^1(t,x,u,a)}{da} &= h(t+ca,x,u,a), \\ \frac{df_1^2(t,x,u,a)}{da} &= 0, \\ \frac{df_1^3(t,x,u,a)}{da} &= 0, \end{aligned}\right\} \tag{4.51}$$

where h is an arbitrary function while $a \in R$ is the group parameter. In view of (4.48) the system (4.51) has to be solved for the initial conditions

$$f_1^1(t,x,u,0) = f_1^2(t,x,u,0) = f_1^3(t,x,u,0) = 0. \tag{4.52}$$

Thus (4.51) and (4.52) give

$$\left.\begin{aligned} f_1^1(t,x,u,a) &= \int_0^a h(t+c\alpha,x,u,\alpha)d\alpha, \\ f_1^2(t,x,u,a) &= 0, \\ f_1^3(t,x,u,a) &= 0. \end{aligned}\right\} \tag{4.53}$$

Putting (4.53) into (4.31) we get the first three components in the approximate Lie groups of equation (4.30) for arbitrary f as

$$\left.\begin{aligned} t^* &\approx t + ca + \epsilon\int_0^a h(t+c\alpha,x,u,\alpha)d\alpha, \\ x^* &\approx x, \\ u^* &\approx u. \end{aligned}\right\} \tag{4.54}$$

□

Next we consider **a particular case** when f is of the form

$$f(x,u) = f_1(x)f_2(u). \tag{4.55}$$

It will suffice to consider the case when

$$f(x,u) = c(u)(b_0 x + b_1)^{2+k_0/b_0} \tag{4.56}$$

(see (3.54)) with the infinitesimals

$$\left.\begin{aligned} \xi_0^1(z) &= \xi = -\frac{k_0}{2}t + k_1, \\ \xi_0^2(z) &= \theta = b_0 x + b_1, \\ \xi_0^3(z) &= \eta = 0 \end{aligned}\right\} \tag{4.57}$$

and the corresponding groups of transformations

$$\left.\begin{aligned} f_0^1 &= t^* = \frac{2k_1}{k_0} + (t - \frac{2k_1}{k_0})e^{-k_0 a/k_1}, \\ f_0^2 &= x^* = \frac{(b_0 x + b_1)e^{b_0 a} - b_1}{b_0}, \\ f_0^3 &= u^* = u \end{aligned}\right\} \tag{4.58}$$

(see (3.55) and (3.69)). On prolongation of (4.57) we further obtain the infinitesimals

$$\left.\begin{aligned} \xi_0^4(z) &= \eta_1^{(1)} = \frac{k_0}{2}u_1, \\ \xi_0^5(z) &= \eta_2^{(1)} = -b_0 u_2, \\ \xi_0^6(z) &= \eta_{11}^{(2)} = k_0 u_{11}, \\ \xi_0^7(z) &= \eta_{12}^{(2)} = (\frac{k_0}{2} - b_0)u_{12}, \\ \xi_0^8(z) &= \eta_{22}^{(2)} = -2b_0 u_{22} \end{aligned}\right\} \tag{4.59}$$

and the prolonged Lie group of transformations

$$\left.\begin{aligned} f_0^4 &= u_1^* = (1 + \frac{1}{2}k_0 a)u_1, \\ f_0^5 &= u_2^* = (1 - b_0 a)u_2, \\ f_0^6 &= u_{11}^* = (1 + k_0 a)u_{11}, \\ f_0^7 &= u_{12}^* = [1 + (\frac{k_0}{2} - b_0)a]u_{12}, \\ f_0^8 &= u_{22}^* = (1 - 2b_0 a)u_{22}. \end{aligned}\right\} \tag{4.60}$$

Introducing (4.57) – (4.60) into (4.26), we obtain

$$\begin{aligned} 0 &= -[f_{x^2}u_2 + f_{xu}(u_2)^2 + f_x u_{22}]\xi_1^2(y_0) \\ &- [f_{xu}u_2 + f_{u^2}(u_2)^2 + f_u u_{22}]\xi_1^3(y_0) \\ &- [f_x + 2f_u u_2]\xi_1^5(y_0) + \xi_1^6(y_0) - f\xi_1^8(y_0) + \frac{k_0}{2}u_1 \\ &- y_1^5 f_{x^2}(b_0x + b_1) - y_1^5 b_0 f_x(b_0 x - b_1) \\ &+ y_1^6 k_0 - y_1^8(b_0x + b_1) + 2y_1^8 b_0 f \\ &- y_1^2 f_{x^3}(b_0x + b_1)u_2 - y_1^3 f_{x^2u}(b_0x + b_1)u_2 \\ &+ [y_1^3 b_0 f_{xu} - 2y_1^5 f_{xu}(b_0x + b_1)]u_2 + 2y_1^5 b_0 f_u(b_0x + b_0 + 2b_1)u_2 \\ &+ [y_1^2 b_0 f_{xu} - y_1^2 b_0 x f_{x^2} - y_1^2 b_1 f_{x^2u}](u_2)^2 + [2y_1^3 f_{u^2} - y_1^3(b_0x + b_1)](u_2)^2 \\ &- y_1^3 f_{xu}(b_0x + b_1)u_{22}. \end{aligned} \tag{4.61}$$

From the coefficients of $1, u_1, u_2, (u_2)^2$ and u_{22} in (4.61) we obtain, respectively, the following equations:

$$\begin{aligned} -y_1^5 f_{x^2}(b_0x + b_1) - y_1^5 b_0 f_x(b_0x - b_1) + y_1^6 k_0 & \\ - y_1^8 f_x(b_0x + b_1) + 2y_1^8 b_0 f - f_x\xi_1^5(y_0) + \xi_1^6(y_0) - f\xi_1^8(y_0) &= 0, \quad (4.62) \\ k_0 &= 0, \quad (4.63) \\ -f_{x^2}\xi_1^2(y_0) - f_{xu}\xi_1^3(y_0) - 2f_u\xi_1^5(y_0) - (y_1^2 f_{x^3} + y_1^3 f_{x^2u})(b_0x + b_1) & \\ + y_1^3 f_{xu}b_0 - 2y_1^5 f_{xu}(b_0x + b_1) + 2y_1^5 b_0 f_u(b_0x + b_0 + 2b_1) &= 0, \quad (4.64) \\ -f_{xu}\xi_1^2(y_0) - f_{u^2}\xi_1^3(y_0) + y_1^2 b_0 f_{xu} - y_1^2 b_0 x f_{x^2} - y_1^2 b_1 f_{x^2u} & \end{aligned}$$

$$- y_1^3 f_{xu^2}(b_0x+b_1) + 2y_1^3 f_{u^2} \quad = \quad 0, \qquad (4.65)$$

$$-f_x\xi_1^2(y_0) - f_u\xi_1^3(y_0) + y_1^2 f_x b_0(1+x) - y_1^2 b_1 f_{x^2} - y_1^3 f_{xu}(b_0x+b_1) \quad = \quad 0. \qquad (4.66)$$

From the condition (4.27) we have

$$\begin{aligned} 0 \quad = \quad & u_1 + y_1^2(-f_{x^2}u_2 - f_{u^2}(u_2)^2 - f_x u_{22}) \\ & + y_1^3(-f_{xu}u_2 - f_{u^2}(u_2)^2 - f_u u_{22}) + y_1^5(-f_x - 2f_u u_2) \\ & + y_1^6 - y_1^8 f. \end{aligned} \qquad (4.67)$$

The following equations arise from the coefficients of $u_2, (u_2)^2, u_{22}$ and 1, respectively:

$$\left.\begin{aligned} y_1^2 f_{x^2} + y_1^3 f_{xu} + 2y_1^5 f_u = 0, \\ y_1^2 f_{u^2} + y_1^3 f_{u^2} = 0, \\ y_1^2 f_u + y_1^3 f_u = 0, \\ y_1^6 + y_1^8 f = 0. \end{aligned}\right\} \qquad (4.68)$$

From (4.68) we get

$$\left.\begin{aligned} y_1^2 = -y_1^3, \\ y_1^6 = -y_1^8 f, \end{aligned}\right\} \qquad (4.69)$$

where y_1^8 is arbitrary.

Differentiating (4.66) with respect to u and subtracting the result from (4.65) yields

$$\begin{aligned} 3y_1^2 b_0 f_{xu} + 2y_1^3 f_{u^2} = 0 \\ \Rightarrow \quad y_1^2 = y_1^3 = 0 \end{aligned} \qquad (4.70)$$

since $f_{xu} \neq 0$, $f_{u^2} \neq 0$ from (4.56).

From (4.70) and (4.68) we get

$$2y_1^5 f_u = 0 \quad \Rightarrow \quad y_1^5 = 0. \qquad (4.71)$$

Putting (4.70) and (4.71) into (4.64) and (4.66), respectively, yields

$$f_{x^2}\xi_1^2(y_0) + f_{xu}\xi_1^3(y_0) + 2f_u\xi_1^5(y_0) = 0, \tag{4.72}$$

$$f_x\xi_1^2(y_0) + f_u\xi_1^3(y_0) = 0. \tag{4.73}$$

Differentiating (4.73) with respect to x and subtracting the result from (4.72) gives

$$-2f_u\xi_1^5(y_0) = 0 \quad \Rightarrow \quad \xi_1^5(y_0) = 0. \tag{4.74}$$

From (4.73) we have

$$\xi_1^2(y_0) = \xi_1^3(y_0)\frac{f_u}{f_x}, \quad f_x \neq 0, \tag{4.75}$$

where $\xi_1^3(y_0)$ is arbitrary.

We also have from (4.61) that y_1^1, y_1^4 and y_1^7 are arbitrary since their coefficients vanish. Introducing (4.63), (4.70), (4.71) and (4.75) into (4.62), we obtain

$$\xi_1^6(y_0) = (\xi_1^8(y_0) - 2b_0y_1^8)f + y_1^8(b_0x + b_1)f_x, \tag{4.76}$$

where $\xi_1^8(y_0)$ is arbitrary.

In view of (4.34), we note that $\xi_1^1(y_0), \xi_1^4(y_0)$ and $\xi_1^7(y_0)$ are arbitrary since their coefficients in (4.26) vanish. Thus we so far have established the following:

$$\left.\begin{aligned}
\xi_1^1(y_0) &= \text{ arbitrary},\\
\xi_1^2(y_0) &= -\xi_1^3(y_0)\frac{f_u}{f_x},\\
\xi_1^3(y_0) &= \text{ arbitrary},\\
\xi_1^4(y_0) &= \text{ arbitrary},\\
\xi_1^5(y_0) &= 0,\\
\xi_1^6(y_0) &= \xi_1^8(y_0)f + y_1^8(b_0x + b_1)f_x - 2b_0y_1^8f,\\
\xi_1^7(y_0) &= \text{ arbitrary},\\
\xi_1^8(y_0) &= \text{ arbitrary}.
\end{aligned}\right\} \tag{4.77}$$

Finally we now find $f_1 = (f_1^1, f_1^2, \ldots, f_1^8)$ by solving the following system of O.D.E.s (see (4.48)):

$$\frac{df_1^1(z,a)}{da} = \frac{\partial \xi_0^1(f_0)}{\partial t} f_1^1(z,a) + \xi_1^1(f_0), \tag{4.78}$$

$$\frac{df_1^2(z,a)}{da} = \frac{\partial \xi_0^2(f_0)}{\partial t} f_1^2(z,a) + \xi_1^2(f_0), \tag{4.79}$$

$$\frac{df_1^3(z,a)}{da} = \frac{\partial \xi_0^3(f_0)}{\partial t} f_1^3(z,a) + \xi_1^3(f_0), \tag{4.80}$$

$$\frac{df_1^4(z,a)}{da} = \frac{\partial \xi_0^4(f_0)}{\partial t} f_1^4(z,a) + \xi_1^4(f_0), \tag{4.81}$$

$$\frac{df_1^5(z,a)}{da} = \frac{\partial \xi_0^5(f_0)}{\partial t} f_1^5(z,a) + \xi_1^5(f_0), \tag{4.82}$$

$$\frac{df_1^6(z,a)}{da} = \frac{\partial \xi_0^6(f_0)}{\partial t} f_1^6(z,a) + \xi_1^6(f_0), \tag{4.83}$$

$$\frac{df_1^7(z,a)}{da} = \frac{\partial \xi_0^7(f_0)}{\partial t} f_1^7(z,a) + \xi_1^7(f_0), \tag{4.84}$$

$$\frac{df_1^8(z,a)}{da} = \frac{\partial \xi_0^8(f_0)}{\partial t} f_1^8(z,a) + \xi_1^8(f_0). \tag{4.85}$$

First we solve (4.78). From (4.57) and (4.58) we see that

$$\xi_0^1(z) = -\frac{k_0}{2}t + k_1 \quad \text{and} \quad f_0^1 = t^* = \frac{2k_1}{k_0} + (t - \frac{2k_1}{k_0})e^{-k_0 a/k_1},$$

which implies

$$\xi_0^1(f_0^1) = (k_1 t - \frac{k_0}{2}t^2)e^{-k_0 a/k_1} + k_1 - k_1 t. \tag{4.86}$$

In view of (4.77) we may let

$$\xi_1^1(f_0^1) = h_1(t^*, x, u, a) \quad \text{since it is arbitrary.}$$

Differentiating (4.86) with respect to t yields

$$\frac{\partial \xi_0^1(f_0^1)}{\partial t} = (k_1 - k_0 t)e^{-k_0 a/k_1} - k_1,$$

which, when put into (4.78), gives the O.D.E.

$$\frac{df_1^1(z,a)}{da} = [(k_1 - k_0 t)e^{-k_0 a/k_1} - k_1]f_1^1(z,a) + h_1(t^*, x, u, a). \qquad (4.87)$$

We observe that (4.87) is a first order linear O.D.E. of the form

$$\frac{df_1^1}{da} + P(a)f_1^1 = Q(a)$$

with

$$P(a) = (k_0 t - k_1)e^{-k_0 a/k_1} + k_1.$$

Thus the integrating factor of (4.87) is $e^{\int P(a)da}$, where

$$\int P(a)da = (\frac{k_1^2}{k_0} - k_1 t)e^{-k_0 a/k_1} + k_1 a,$$

and hence

$$f_1^1(z,a) = e^{-A_a}\int_0^a e^{A_s} h_1(t^*, x, u, s)ds + c(z)e^{-A_a}, \qquad (4.88)$$

where

$$A_a = (\frac{k_1^2}{k_0} - k_1 t)e^{-k_0 a/k_1} + k_1 a, \quad A_s = (\frac{k_1^2}{k_0} - k_1 t)e^{-k_0 s/k_1} + k_1 s. \qquad (4.89)$$

From the condition $f_1^1(z,a)|_{a=0} = 0$ we have

$$f_1^1(z,0) = c(z)e^{-A_a} = 0 \quad \Rightarrow \quad c(z) = 0.$$

Finally (4.88) now becomes

$$f_1^1(z,a) = e^{-A_a}\int_0^a e^{A_s} h_1(t^*, x, u, s)ds. \qquad (4.90)$$

Putting (4.58) and (4.90) into (4.31) we get the first component in the approximate Lie groups of equation (4.30) for

$$f(x,u) = c(u)(b_0 x + b_1)^{2+k_0/b_0}$$

as

$$t^* \approx \frac{2k_1}{k_0} + (t - \frac{2k_1}{k_0})e^{-k_0 a/k_1} + \epsilon[e^{-A_a}\int_0^a e^{A_s} h_1(t^*, x, u, s)ds], \qquad (4.91)$$

where A_a, A_s are as in (4.89) and t^* is as in (4.58).

For the solution of (4.79) we observe from (4.57) and (4.58) that

$$\begin{aligned} \xi_0^2(f_0^2) &= \frac{1}{b_0}[(b_0x+b_1)e^{b_0a} - b_1]x + b_1 \\ \Rightarrow \quad \frac{\partial \xi_0^2(f_0^2)}{\partial x} &= \frac{1}{b_0}[(2b_0x+b_1)e^{b_0a} - b_1]. \end{aligned} \tag{4.92}$$

Furthermore, from (4.58) and (4.75) we have

$$\begin{aligned} \xi_1^3(f_0^3) &= h_3(u^*,t,x,a) = h_3(t,x,u,a), \quad \text{since } u^* = u \\ \Rightarrow \quad \xi_1^2(f_0^2) &= -h_3(t,x,u,a)\frac{f_u}{f_x}. \end{aligned} \tag{4.93}$$

But $\dfrac{f_u}{f_x} = \dfrac{c'(u)(b_0x+b_1)}{c(u)(2b_0+k_0)}$ from (4.56), hence

$$\xi_1^2(f_0^2) = -\frac{c'(u)(b_0x+b_1)}{c(u)(2b_0+k_0)}h_3(t,x,u,a). \tag{4.94}$$

Putting (4.92) and (4.94) in (4.79), we have the O.D.E.

$$\begin{aligned} &\frac{df_1^2(z,a)}{da} - \frac{1}{b_0}[(2b_0x+b_1)e^{b_0a} - b_1]f_1^2(z,a) \\ &= -\frac{c'(u)(b_0x+b_1)}{c(u)(2b_0+k_0)}h_3(t,x,u,a), \end{aligned}$$

which is easily solved to give

$$f_1^2(z,a) = -e^{-B_a}\int_0^a e^{B_s}\left[\frac{c'(u)(b_0x+b_1)}{c(u)(2b_0+k_0)}\right]h_3(t,x,u,s)ds + c(z)e^{-B_a} \tag{4.95}$$

where

$$B_a = \frac{1}{b_0^2}[-(2b_0x+b_1)e^{b_0a} + b_0b_1a], \quad B_s = \frac{1}{b_0^2}[-(2b_0x+b_1)e^{b_0s} + b_0b_1s]. \tag{4.96}$$

From the condition $f_1^2(z,a)|_{a=0} = 0$ we have

$$f_1^2(z,0)=c(z)e^{-B_a}=0 \quad\Rightarrow\quad c(z)=0,$$

and finally we obtain

$$f_1^2(z,a)=-e^{-B_a}\int_0^a e^{B_s}\left\{\frac{c'(u)(b_0x+b_1)}{c(u)(2b_0+k_0)}\right\}h_3(t,x,u,s)ds. \tag{4.97}$$

Thus the second component in the approximate Lie groups of (4.30) is

$$x^*\approx\frac{1}{b_0}[(b_0x+b_1)e^{b_0a}-b_1]+\epsilon\left[-e^{-B_a}\int_0^a e^{B_s}\left\{\frac{c'(u)(b_0x+b_1)}{c(u)(2b_0+k_0)}\right\}h_3(t,x,u,s)ds\right]. \tag{4.98}$$

Similarly, it can easily be shown that the solutions of equations (4.80) – (4.85) yield, respectively,

$$f_1^3(z,a)=\int_0^a h_3(t,x,u,s)ds,$$

$$f_1^4(z,a)=e^{k_0a(4+k_0a)/8}\int_0^a e^{-k_0s(4+k_0s)/8}h_4(u_1+\frac{k_0}{2}u_1s,t,x,u,u_1,s)ds,$$

$$f_1^5(z,a)=e^{-b_0a(1-b_0a)}\int_0^a e^{b_0s(1-b_0s)}h_5(u_2-b_0u_2s,t,x,u,u_1,u_2,s)ds,$$

$$f_1^6(z,a)=e^{C_a}\int_0^a e^{-C_s}[h_8(u_{11}^*,t,x,u,u_1,u_2,u_{11},s)+Df_x-Ef]ds,$$

where $C_a=k_0a+\frac{1}{2}k_0^2a^2,\quad C_s=k_0s+\frac{1}{2}k_0^2s^2,$

$D=y_1^8(b_0a+b_1),\qquad E=2b_0y_1^8,$

$u_{11}^*=u_{11}+k_0au_{11},\quad y_1^8=$ arbitrary constant,

$$f_1^7(z,a)=e^{-G_a}\int_0^a e^{G_s}h_7(u_{12}^*,t,x,u,u_1,u_2,u_{11},u_{12},s)ds,$$

where $G_a=(\frac{k_0}{2}-b_0)a+\frac{1}{2}(\frac{k_0}{2}-b_0)a^2,\quad G_s=(\frac{k_0}{2}-b_0)s+\frac{1}{2}(\frac{k_0}{2}-b_0)s^2,$

$u_{12}^*=(\frac{k_0}{2}-b_0)au_{12},$

$$f_1^8(z,a)=e^{-J_a}\int_0^a e^{J_s}h_8(u_{22}-2b_0au_{22},t,x,u,u_1,u_2,u_{11},u_{12},u_{22},s)ds.$$

Finally we get the last six components in the approximate Lie groups of (4.30), respectively, as

$$u^* \approx \epsilon \int_0^a h_3(t,x,u,s)ds, \tag{4.99}$$

$$u_1^* \approx u_1 + \frac{k_0}{2} u_1 a + \epsilon \left[e^{k_0 a(4+k_0 a)/8} \int_0^a e^{-k_0 s(4+k_0 s)/8} h_4(u_1^*, t, x, u, u_1, s) ds \right], \tag{4.100}$$

$$u_2^* \approx u_2 - b_0 u_2 a + \epsilon \left[e^{-b_0 a(1-b_0 a)} \int_0^a e^{b_0 s(1-b_0 s)} h_5(u_2^*, t, x, u, u_1, u_2, s) ds \right], \tag{4.101}$$

$$u_{11}^* \approx u_{11}(1 + k_0 a)$$

$$+ \epsilon \left[e^{C_a} \int_0^a e^{-C_s} [h_8(u_{11}^*, t, x, u, u_1, u_2, u_{11}, s) + Df_x - Ef] ds \right], \tag{4.102}$$

$$u_{12}^* \approx u_{12} + (\frac{k_0}{2} - b_0) u_{12} a \tag{4.103}$$

$$+ \epsilon \left[e^{-G_a} \int_0^a e^{G_s} h_7(u_{12}^*, t, x, u, u_1, u_2, u_{11}, u_{12}, s) ds \right], \tag{4.104}$$

$$u_{22}^* \approx u_{22} - 2b_0 u_{22} a \tag{4.105}$$

$$+ \epsilon \left[e^{-J_a} \int_0^a e^{J_s} h_8(u_{22}^*, t, x, u, u_1, u_2, u_{11}, u_{12}, u_{22}, s) ds \right]. \tag{4.106}$$

□

Chapter 5

Conclusion

In this thesis we have extended the study of the stability of Lie groups of nonlinear wave equations of the form

$$u_{tt} = [f(u)u_x]_x, \tag{5.1}$$

where f is an arbitrary function, to the significantly larger variable coefficient class of the type

$$u_{tt} = [f(x,u)u_x]_x \tag{5.2}$$

under arbitrary large perturbations of the form

$$u_{tt} + \epsilon u_t = [f(x,u)u_x]_x \tag{5.3}$$

As motivated in the Introduction (Chapter 0), the perturbation $\epsilon u_t(t,x)$ is a standard model of friction.

The main results obtained in this thesis include:

· the derivation of the exact Lie groups of the nonlinear wave equation (5.2) without the complicated assumptions made in the study of Torrisi and Valenti (1985).

· the determination of exact Lie groups of the perturbed nonlinear wave equation (5.3) for both arbitrary and specific functions $f(x,u)$. These are new results and hence an improvement on the existing literature on the subject.

· the derivation of the stability of Lie groups of the perturbed nonlinear wave equation (5.3) for various forms of $f(x, u)$. This problem had not been studied before.

The thesis is also of scientific significance because it opens the way for further research in still larger classes of nonlinear hyperbolic variable coefficient equations and their stability.

Stability of Lie Groups of Nonlinear Hyperbolic Equations

by

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Degree: Ph.D.

Summary

In this thesis the stability of the Lie group invariance of classical solutions of large classes of nonlinear partial differential equations is studied. We give a theoretical framework for the construction of approximate groups for nonlinear partial differential equations. In particular, stability symmetries for the perturbed nonlinear wave equation

$$u_{tt} + \epsilon u_t = [f(x,u)u_x]_x$$

are presented here for the first time.

This research is a particularly important stability study, since it applies to large classes of – earlier unknown – classical solutions of nonlinear partial differential equations as well as to their symmetries. These equations and solutions model, amongst others, important laws of nature.

Chapter 1 is devoted to the general concepts of Lie group theory. A detailed account is given of the applications of Lie groups to both ordinary and partial differential equations. To date the only known method of obtaining particular solutions to complicated systems of differential equations is by Lie group symmetry analysis. This is now well known in the literature.

The Lie group symmetry analysis, however, has some limitations. Any small perturbation of an equation disturbs the group admitted by it and this reduces the practical value of group theoretic methods in general. The theory of stability analysis presented in chapter 2 overcomes this problem. This technique, originated by N.Kh. Ibragimov around

1988, generates groups that are stable under small, or even classes of more arbitrary, perturbations of the differential equations involved.

The exact Lie groups admitted by the nonlinear wave equation

$$u_{tt} = [f(x,u)u_x]_x$$

and the corresponding perturbed equation are discussed in chapter 3. Finally, in chapter 4, the construction of stability groups admitted by the perturbed nonlinear wave equation are set out in detail.

Stabiliteit van Lie-groepe van Nie-lineêre Hiperboliese Vergelykings

deur

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Graad: Ph.D.

Opsomming

In hierdie proefskrif word die stabiliteit van die Lie-groep invariansie van klassieke oplossings van groot klasse nie-lineêre parsiële differensiaalvergelykings bestudeer. Ons gee 'n teoretiese raamwerk vir die konstruksie van benaderde groepe vir nie-lineêre parsiële differensiaalvergelykings. In besonder, stabiliteitsimmetrieë vir die geperturbeerde nie-lineêre golfvergelyking

$$u_{tt} + \epsilon u_t = [f(x,u)u_x]_x$$

word hier vir die eerste keer gegee.

Hierdie navorsing is veral belangrik as 'n stabiliteitstudie, want dit is van toepassing op groot klasse – voorheen onbekende – klassieke oplossings van nie-lineêre parsiële differensiaalvergelykings sowel as op hulle simmetrieë. Hierdie vergelykings en oplossings modelleer, onder andere, belangrike natuurwette.

Hoofstuk 1 word gewy aan die algemene begrippe van Lie-groep teorie. Die toepassings van Lie-groepe op beide gewone en parsiële differensiaalvergelykings word in detail behandel. Tot op hede was die enigste bekende metode om partikuliere oplossings van ingewikkelde stelsels differensiaalvergelykings te vind deur middel van Lie-groep simmetrie-analise. Dit is nou goed bekend in die literatuur.

Die Lie-groep simmetrie-analise het egter sekere beperkings. Enige klein perturbasie van 'n vergelyking versteur die ooreenstemmende groep, en dit verminder die praktiese waarde van groep-teoretiese metodes in die algemeen. Die teorie van stabiliteitsanalise wat in hoofstuk 2 uiteengesit word, oorbrug hierdie probleem. Hierdie tegniek, wat rondom 1988

deur N.Kh. Ibragimov ontwikkel is, genereer groepe wat stabiel is onder klein, of selfs klasse van meer arbitrêre, perturbasies van die betrokke differensiaalvergelykings.

Die eksakte Lie-groepe van die nie-lineêre golfvergelyking

$$u_{tt} = [f(x,u)u_x]_x$$

en die ooreenstemmende geperturbeerde vergelyking word in hoofstuk 3 bespreek. Laastens, in hoofstuk 4, word die konstruksie van die stabiliteitsgroepe van die geperturbeerde nie-lineêr golfvergelyking in besonderhede uiteengesit.