ON THE ESSENTIAL NUMERICAL RANGE

BY

OWEGO DANCUN OKESO

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

SCHOOL OF MATHEMATICS, APPLIED STATISTICS AND ACTUARIAL SCIENCE

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Abstract

The convexity, closure and compactness of the numerical range among other properties constitute a considerable literature in operator theory. The properties of the essential numerical range and how they are related to the familiar numerical range are studied. The study underpins the role in operator theory. An outline of the basic concepts and defined terms are provided. Similarly, proofs for simple propositions and theorems used in the sequel are made in *Chapter One*. *Chapter Two* of this work is devoted to the properties of the numerical range. Properties, for instance, convexity, unitary invariance and the projection property have been looked at. The connection between the spectrum of an operator and the numerical range of the operator has also been given. The properties of the essential numerical range have been examined in *Chapter Three*. The proofs of various properties studied, for instance, convexity have been outlined. In *Chapter Four*, the relationships between the numerical range and the essential numerical range have been studied. The proofs of the theorems by J. Christophe and J. Lancaster have been shown. Finally, the roles of the essential numerical range in operator theory have been discussed. Conclusions and recommendations for further research have also been given.

Chapter 1

Introduction

In this Chapter, we begin by setting up the basic terms needed to discuss the numerical range and the essential numerical range. We give the definitions of the numerical range, essential numerical range, spectrum, essential spectrum as well as the spectral radius and the the essential numerical radius.

1.1 Basic concepts

Definition 1.1.1 (Hilbert Space)

A Hilbert space, $\mathcal{H}$, is a vector space endowed with an inner product and associated norm and metric, such that every Cauchy sequence in $\mathcal{H}$ has a limit in $\mathcal{H}$. A Hilbert space is also a Banach space. A Hilbert space has an inner product structure.

Definition 1.1.2 (Algebra)

If $\mathfrak{A}$ is a vector space over the set of complex numbers, $\mathbb{C}$, then $\mathfrak{A}$ is called an associative algebra iff it is equipped with a multiplication operator satisfying;

- $a(bc) = (ab)c$
- $a(b+c) = ab + ac$, $(b+c)a = ba + ca$
- $(\alpha\beta)(ab) = (\alpha a)(\beta b)$

(We simply say an algebra instead of an associative algebra) $\mathfrak{A}$ is called a commutative algebra iff $a, b \in \mathfrak{A}$ implies $ab = ba$.

$\mathfrak{B} \subset \mathfrak{A}$ is called a subalgebra if it is a linear subspace and $a, b \in \mathfrak{B} \Rightarrow ab \in \mathfrak{B}$. Clearly, a subalgebra is also an algebra. We say that $\mathfrak{A}$ is unital if it possesses an identity.

Given that $\mathbb{K}$ is either a set of complex numbers, $\mathbb{C}$, or real numbers,$\mathbb{R}$, a normed algebra over $\mathbb{K}$ is an algebra $\mathfrak{A}$ over $\mathbb{K}$, which also carries a norm $||.||$, which is sub-multiplicative, in the sense that:

- $||xy|| \leq ||x||.||y||$ for all $x, y \in \mathfrak{A}$

Let $\mathcal{V}$ be a vector space. Clearly, the set of linear maps in $\mathcal{V}$, denoted by $L(\mathcal{V})$, is an algebra. An identity of an algebra, $\mathfrak{A}$, is an element $I \in \mathfrak{A}$ such that $a = Ia = aI$, for all $a \in \mathfrak{A}$. Any algebra has at most one identity. In fact, if I_1, I_2 are identities, then $I_1 = I_1 I_2 = I_2$. We say that $\mathfrak{A}$ is unital if it possesses an identity.

Definition 1.1.3 (Banach Algebra)
We recall that an algebra $\mathfrak{A}$ over $\mathbb{C}$ is called a normed algebra, if it is equipped with a norm $\mathfrak{A} \ni a \mapsto ||a|| \in \mathbb{R}$ such that,

$$||ab|| \leq ||a||||b||$$

It is called a *Banach algebra* if it is complete in the norm, $||.||$

Examples

(1). If X is a normed vector space, denote by $B(X)$ the set of all bounded linear maps from X to itself (the operators on X). Then $B(X)$ is a normed algebra with the point-wise defined operations for addition and scalar multiplication. Multiplication given by $(u, v) \mapsto uov$, and the norm is the operator norm:

$$\begin{aligned} ||u|| &= sup_{x\neq 0}\frac{||u(x)||}{||x||} \\ &= sup_{||x||\leq 1}||u(x)|| \end{aligned}$$

(2). If S is a set, $\ell^{\infty}(S)$, the set of all bounded complex valued functions on S, is a unital Banach algebra where the operations are defined point-wise:

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \\ (fg)(x) &= f(x)g(x) \\ (\lambda f)(x) &= \lambda f(x), \end{aligned}$$

and the norm is the sup-norm;

$$||f||_{\infty} = sup_{x\in S}|f(x)|$$

Definition 1.1.4 (Ideals)

Let $\mathfrak{A}$ be some algebra over a field $\mathbb{K}$.

(a). A subset $\mathfrak{J} \subset \mathfrak{A}$ is called *left ideal* in $\mathfrak{A}$, if

- $\mathfrak{J}$ is a linear subspace of $\mathfrak{A}$;
- $ax \in \mathfrak{J}$, $\forall a \in \mathfrak{A}$, $x \in \mathfrak{J}$.

(b). A subset $\mathfrak{J} \subset \mathfrak{A}$ is called a right ideal in $\mathfrak{A}$, if

- $\mathfrak{J}$ is a linear subspace of $\mathfrak{A}$;
- $xa \in \mathfrak{J}$, $\forall a \in \mathfrak{A}$, $x \in \mathfrak{J}$.

(c). A subset $\mathfrak{J} \subset \mathfrak{A}$ is called a *two sided ideal* in $\mathfrak{A}$, if $\mathfrak{J}$ is both a left and right ideal in $\mathfrak{A}$. Of course, if $\mathfrak{A}$ is commutative, these three notions coincide. In the study of (Banach) algebras, one deals almost exclusively with two sided ideals. The reason is the following;

FACT: If $\mathfrak{J}$ is a two sided ideal in an algebra $\mathfrak{A}$, then the quotient space $\mathfrak{A}/\mathfrak{J}$ carries a unique algebra structure, which makes the quotient map

$$Q : \mathfrak{A} \ni a \quad \mapsto \quad [a] \in \mathfrak{A}/\mathfrak{J}$$

an algebra homomorphism. Moreover, if $\mathfrak{A}$ has a unit 1, and $\mathfrak{J} \subset \mathfrak{A}$, then the algebra $\mathfrak{A}/\mathfrak{J}$ has unit $1 = Q(1)$

Theorem 1.1.5

Let $\mathfrak{A}$ be a Banach algebra, then;

(i). If $\mathfrak{J}$ is a left (or right, two sided) ideal in $\mathfrak{A}$, then so is its closure, $\overline{\mathfrak{J}}$.

(ii). If $\mathfrak{A}$ is a unital Banach algebra, and $\mathfrak{J} \subset \mathfrak{A}$ is a left (or right, or two sided) ideal, then $\overline{\mathfrak{J}} \subset \mathfrak{A}$.

(iii). If $\mathfrak{J}$ is a closed two sided ideal in $\mathfrak{A}$, then when equipped with the quotient norm, the algebra $\mathfrak{A}/\mathfrak{J}$ is a unital Banach algebra. Moreover, if $\mathfrak{A}$ is unital, and $\mathfrak{J} \subset \mathfrak{A}$, then $\mathfrak{A}/\mathfrak{J}$ is a unital Banach algebra.

PROOF. (i). Assume $\mathfrak{J}$ is a left ideal. Clearly, $\mathfrak{J}$ is a linear subspace, so we only need to check the second condition. Fix $a \in \mathfrak{A}$ and $x \in \overline{\mathfrak{J}}$, let us prove that $ax \in \overline{\mathfrak{J}}$. Since $x \in \overline{\mathfrak{J}}$, there exists a sequence $(x_n)_{n=1}^{\infty} \subset \mathfrak{J}$, with $x = lim_{n\to\infty}x_n$. Then the continuity of the left multiplication map $L_a : \mathfrak{A} \to \mathfrak{A}$ gives,

$$\begin{aligned} ax &= L_a x \\ &= lim_{n\to\infty} L_a x_n \\ &= lim_{n\to\infty} a x_n \end{aligned}$$

Since $ax_n \in \mathfrak{J}$, $\forall n \geq 1$, this forces $ax \in \overline{\mathfrak{J}}$. In the case where $\mathfrak{J}$ is a right ideal, the proof is identical. In the case where $\mathfrak{J}$ is two sided, we use the above cases.

(ii). Assume $\mathfrak{J}$ is a left ideal. We argue by contradiction. Suppose $\overline{\mathfrak{J}} = \mathfrak{A}$. in particular, it follows that $\overline{\mathfrak{J}}$ contains 1, the unit in $\mathfrak{A}$. Since the set $GL(A)$ of all invertible elements is open, and contains 1, the fact that 1 belongs to $\overline{\mathfrak{J}}$ gives the fact that the intersection $GL(A) \bigcap \mathfrak{J}$ is non-empty. In particular, this means that $\mathfrak{J}$ contains some invertible elements of x. Of course, if we define $a = x^{-1} \in \mathfrak{A}$, we have $\mathfrak{J} \ni ax = 1$. Since $\mathfrak{J}$

contains 1, it will contain all elements of $\mathfrak{A}$, which contradicts the strict inclusion that $\mathfrak{J} \subset \mathfrak{A}$. In the case when $\mathfrak{J}$ is a right ideal, the proof is identical. In the case when $\mathfrak{J}$ is two sided, we use the above cases.

(iii). Let us prove the first section. We know that the quotient space $\mathfrak{A}/\mathfrak{J}$ is a Banach space, so the only thing we need to prove is the second condition in the definition. Start with two elements $v, w \in \mathfrak{A}/\mathfrak{J}$, and let us prove the inequality;

$$\|vw\|_{\mathfrak{A}/\mathfrak{J}} \leq \|v\|_{\mathfrak{A}/\mathfrak{J}}.\|w\|_{\mathfrak{A}/\mathfrak{J}}$$

For each $\epsilon > 0$, we choose $a_\epsilon \in v$ and $b_\epsilon \in w$, such that;

$$\|a_\epsilon\| \leq \|v\|_{\mathfrak{A}/\mathfrak{J}} + \epsilon$$

and

$$\|b_\epsilon\| \leq \|w\|_{\mathfrak{A}/\mathfrak{J}} + \epsilon$$

Since $a_\epsilon b_\epsilon \in vw$, we immediately get

$$\|vw\|_{\mathfrak{A}/\mathfrak{J}} \leq \|a_\epsilon b_\epsilon\| \leq \|a_\epsilon\|.\|b_\epsilon\| \leq (\|v\|_{\mathfrak{A}/\mathfrak{J}} + \epsilon).(\|v\|_{\mathfrak{A}/\mathfrak{J}} + \epsilon)$$

Since the inequality $\|vw\|_{\mathfrak{A}/\mathfrak{J}} \leq (\|v\|_{\mathfrak{A}/\mathfrak{J}} + \epsilon).(\|v\|_{\mathfrak{A}/\mathfrak{J}} + \epsilon)$ holds for all $\epsilon > 0$ it will clearly imply $\|vw\|_{\mathfrak{A}/\mathfrak{J}} \leq \|v\|_{\mathfrak{A}/\mathfrak{J}}.\|w\|_{\mathfrak{A}/\mathfrak{J}}$

□

Definition 1.1.6 (Involution)

Let $\mathfrak{A}$ be an algebra. Then an **involution** on $\mathfrak{A}$ is a map;

$$\begin{aligned} * : \mathfrak{A} &\to \mathfrak{A} \\ a &\mapsto a^* \end{aligned}$$

satisfying

- $(a^*)^* = a$ for all $a \in \mathfrak{A}$
- $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$ for all $a, b \in \mathfrak{A}$ and all complex numbers α and β.
- $(ab)^* = b^* a^*$ for all $a, b \in \mathfrak{A}$.

If $\mathfrak{A}$ carries an **involution**, we say that $\mathfrak{A}$ is an involutive algebra or a $* - algebra$.

Example

Let S be a subset of a $* - algebra$, $\mathfrak{A}$. We set $S^* = \{a^* : a \in S\}$, and if $S^* = S$, we say S is self-adjoint. A self-adjoint sub-algebra S of $\mathfrak{A}$ is a $* - subalgebra$ of $\mathfrak{A}$ and is a $* - algebra$ when endowed with involution got by restriction.

Definition 1.1.7

A C^* **algebra** is an involutive Banach algebra $\mathfrak{A}$ with a norm satisfying the relations;

- $\|AB\| \leq \|A\|\|B\|$
- $\|A^*\| = \|A\|$
- $\|A^*A\| = \|A\|^2$.

If a C^* algebra has unit element then it is called a unital C^* algebra.

Examples

(1).The following algebras are C^* algebras with involution given by $f \mapsto \overline{f}$:

(a). $\ell^\infty(S)$ where S is a set;

(b). $L^\infty(\Omega, \mu)$, where (Ω, μ) is a measure space;

(c). $C_b(\Omega)$ where Ω is a topological space;

(d). $B_\infty(\Omega)$ where Ω is a measurable space.

(2). The scalar field $\mathbb{C}$ is a unital C^* algebra with involution given by the complex conjugation $\lambda \mapsto \overline{\lambda}$.

(3). $\mathfrak{B}(\mathcal{H})$ is a C^* algebra.

To show this, we argue that $\mathfrak{B}(\mathcal{H})$ is an involutive Banach algebra using the Hilbert space adjoint as our involution. That is, if $\mathcal{H}$ is a Hilbert space, then the map that sends a continous linear operator T to its Hilbert space adjoint T^* is an involution. Thus $\mathfrak{B}(\mathcal{H})$ is an involutive Banach algebra.

We now check that equipped with this involution, $\mathfrak{B}(\mathcal{H})$ satisfies the C^* equation,

$$\|T^*T\| = \|T\|^2$$

for all $T \in \mathfrak{B}(\mathcal{H})$ If $T \in \mathfrak{B}(\mathcal{H})$ then $\|T^*T\| \leq \|T\|^2$. For the reverse inequality, we observe that;

$$\begin{aligned}
\|T\|^2 &= sup_{\|x\|=1}\|Tx\|^2 \\
&= sup_{\|x\|=1}\langle Tx, Tx\rangle \\
&= sup_{\|x\|=1}\langle T^*Tx, x\rangle \\
&= sup_{\|x\|=1}\|T^*T\|\|x\|^2 \\
&= \|T^*T\|
\end{aligned}$$

Thus $\mathfrak{B}(\mathcal{H})$ is a C^* algebra.

Definition 1.1.8

Let $\mathfrak{A}$ be a C^* algebra. A linear functional $\varphi : \mathfrak{A} \mapsto \mathbb{C}$ is called positive if $\varphi(a) \geq 0$ whenever $a \geq 0$ for all $a \in \mathfrak{A}$. More generally, if B is a C^* algebra then a linear map $\varphi : A \to B$ is positive if $\varphi(A^+) \subset B^+$

Definition 1.1.9

A **state** of a C^* algebra is a positive linear functional of norm 1.

Definition 1.1.10

An operator $K \in \mathfrak{B}(\mathcal{H})$ is said to be **compact** if for every bounded sequence (x_n) of vectors of H the sequence (Kx_n) has a convergent subsequence.

Definition 1.1.11

The subspace $\mathcal{K}(\mathcal{H})$ of all compact operator is an ideal of $\mathfrak{B}(\mathcal{H})$.

In order to prove this, we need to show that, if $A, B \in \mathcal{K}(\mathcal{H})$ and $T \in \mathfrak{B}(\mathcal{H})$ then αA. $A+B$, TA and AT are all in $\mathcal{K}(\mathcal{H})$. That is, for any bounded sequence (x_n), we must show that (αAx_n). $([A+B]x_n)$, (TAx_n) and (ATx_n) all have convergent subsequences.

Since A is compact, (Ax_n) has a convergent subsequence (Ax_{n_i}). Then clearly (αAx_{n_i}) is a convergent subsequence of (αAx_n) showing that αA is compact. Also, (x_{n_i}) is a bounded sequence and so, since B is compact, (Bx_{n_i}) has a convergent subsequence $(Bx_{n_{i_j}})$. Then $([A+B]x_{n_{i_j}})$ is a convergent subsequence of $([A+B]x_n)$, showing that $A+B$ is compact.

Again, since $T \in \mathfrak{B}(\mathcal{H})$. T is continous and so (TAx_{n_i}) is a convergent subsequence of (TAx_n) showing that TA is compact. The proof for AT is slightly different. Here, since (x_n) is bounded and $\|Tx_n\| \leq \|T\|.\|x_n\|$ we have that (Tx_n) is bounded and so, since A is compact, (ATx_n) has a convergent subsequence, showing that TA is compact.

Theorem 1.1.12

If $\mathcal{H}$ is a Hilbert space and $\mathfrak{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$, let $\mathcal{K}(\mathcal{H})$ be the subspace of $\mathfrak{B}(\mathcal{H})$ formed by all compact operators, and $[T]$ be the coset of $T \in \mathfrak{B}(\mathcal{H})$ in the Calkin algebra then the quotient space $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is an algebra on the complex plane.

PROOF. Denote the quotient space $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ by:

$$\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) = \{[T] : T \in \mathfrak{B}(\mathcal{H})\},$$

where,

$[T] = \{T + K : K \in \mathcal{K}(\mathcal{H})\}$.

for all $T, N \in \mathfrak{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$,

$[T] + [N] = [T + N] \rightarrow (i)$,

for let $[T]$ and $[N]$ be defined by,

$$\begin{aligned} [T] &= \{[T + K] : K \in \mathcal{K}(\mathcal{H})\} \\ and\ [N] &= \{[N + K] : K \in \mathcal{K}(\mathcal{H})\} \end{aligned}$$

Then,

$$\begin{aligned} [T] + [N] &= \{(T + K) + (N + K) : K \in \mathcal{K}(\mathcal{H})\} \\ &= \{(T + K + N + K : K \in \mathcal{K}(\mathcal{H})\} \\ &= \{T + N + \underbrace{K + K} : K \in \mathcal{K}(\mathcal{H})\} \\ &= \{(T + N) + K : K \in \mathcal{K}(\mathcal{H})\} \\ &= [T + K] \end{aligned}$$

Also, $\lambda[T] = [\lambda T] \rightarrow (ii)$.

Let $[T]$ be defined as above, then,

$$\lambda[T] = \{\lambda(T + K) : K \in \mathcal{K}(\mathcal{H})\}$$

$$
\begin{aligned}
&= \{\lambda T + \lambda K : K \in \mathcal{K}(\mathcal{H})\} \\
&= \{\lambda T + K : K \in \mathcal{K}(\mathcal{H})\} \\
&= [\lambda T]
\end{aligned}
$$

And lastly, $[T][N] = [TN] \to (iii)$.

Let $[T]$ and $[N]$ be defined as in (i) above then,

$$
\begin{aligned}
[T][N] &= \{(T + K)(N + K) : K \in \mathcal{K}(\mathcal{H})\} \\
&= \{(TN + \underbrace{TK + KN + K^2}: K \in \mathcal{K}(\mathcal{H})\} \\
&= \{TN + K : K \in \mathcal{K}(\mathcal{H})\} \\
&= [TN]
\end{aligned}
$$

The quotient space $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ provided with equations (i) and (ii) is a vector space on the complex plane and the map defined on $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ into $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ by $([T],[N]) \mapsto [TN]$ is associative bilinear with unit element $[I]$. Hence $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is an algebra on complex numbers with the unit element $[I]$. □

Definition 1.1.13

For a bounded linear operator T on a separable Hilbert Space, $\mathcal{H}$, the numerical range of $T \in \mathfrak{B}(\mathcal{H})$ is by definition the set,

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

The following properties of the numerical range are immediate:

$$\begin{aligned} W(\alpha I + \beta T) &= \alpha + \beta W(T) \\ W(T^*) &= \{\bar{\lambda}, \lambda \in W(T)\} \\ W(U^*TU) &= W(T), \end{aligned}$$

(see, for example [10])

for $\alpha, \beta \in \mathbb{C}$ and for any unitary operator U. An element $U \in \mathfrak{B}(\mathcal{H})$ is called *unitary* if U is invertible and its inverse given by U^*.

Definition 1.1.14

The *numerical radius* $w(T)$ of an operator T on $\mathcal{H}$ is given by,

$w(T) = sup\{|\lambda|, \lambda \in W(T)\}$ (see, [10])

Notice that, for any vector $x \in \mathcal{H}$, we have,

$|\langle Tx, x\rangle| \leq w(T)\|x\|^2$

Definition 1.1.15

The *spectrum* of an operator $T \in \mathfrak{B}(\mathcal{H})$, denoted by $\sigma(T)$ is defined by,

$\sigma(T) = \{\lambda : T - \lambda I \text{ is not invertible}\}$

Proposition 1.1.16

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathfrak{B}(\mathcal{H})$. Then:

$$\sigma(T) = \sigma(T^*)^* \quad := \quad \{\bar{\lambda} : \lambda \in \sigma(T)\}$$

PROOF. If λ is not contained in $\sigma(T)$, let $R = (\lambda - T)^{-1}$. For all $x, y \in \mathcal{H}$,

$$\langle x, y\rangle \quad = \quad \langle R(\lambda - T)x, y\rangle$$

$$
\begin{aligned}
&= \langle(\lambda - T)x, R^*y\rangle \\
&= \langle x, (\lambda - T)^* R^* y\rangle
\end{aligned}
$$

Thus $(\lambda - T)^* R^* = I$, and similarly, $R^*(\lambda - T)^* = I$. But $(\lambda - T)^* = \overline{\lambda} - T^*$, so that;

$$
\begin{aligned}
R^* &= (\overline{\lambda} - T^*)^{-1} \\
&= [(\lambda - T)^*]^{-1}
\end{aligned}
$$

Thus $\rho(T)^* \subseteq \rho(T^*)$. Moreover,

$$
\begin{aligned}
\rho(T^*)^* &\subseteq \rho(T^{**}) \\
&= \rho(T)
\end{aligned}
$$

In other words, $\rho(T^*) \subseteq \rho(T)^*$. We conclude that $\sigma(T) = \sigma(T^*)^*$.

□

Definition 1.1.17

The point spectrum of an operator $T \in \mathfrak{B}(\mathcal{H})$, $\sigma_p(T)$ is defined as;

$\sigma_p(T) = \{\lambda \in \sigma(T), Tf = \lambda f \ for \ some \ f \in \mathcal{H}\}$

Those λ not in the spectrum $\sigma(T)$ are called the *resolvent set*, $\rho(T)$ of T and thereupon ,the operator $(T - \lambda I)^{-1}$ is called the *resolvent operator* for T

Definition 1.1.18

The *spectral radius* $r(T)$ of an operator T on $\mathcal{H}$ is given by,

$r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\}$ (see, [10])

Definition 1.1.19

If $\Pi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, defined by, $T \mapsto T + K$, $K \in \mathcal{K}(\mathcal{H})$

is the map from $\mathfrak{B}(\mathcal{H})$ onto the Calkin-algebra, $\mathfrak{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, then the *essential numerical range* of T denoted by $W_e(T)$ is the numerical range of the coset containing T in the Calkin algebra.

Stampfli, Williams and Fillmore [5] gave the definition of $W_e(T)$ as follows;

$W_e(T) = \bigcap \overline{W(T+K)}$ where the intersection runs over the compact operators $K \in \mathcal{K}(\mathcal{H})$.

Definition 1.1.20

The essential spectrum, $\sigma_e(T)$ is the spectrum of the coset $[T]$ in the Calkin algebra. More precisely,

$\sigma_e(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})}\{\ \sigma(T+K)\}$, where $T \in \mathfrak{B}(\mathcal{H})$. (see,[11])

Definition 1.1.21

The essential numerical radius $w_e(T)$ of an operator T on $\mathcal{H}$ is given by,

$w_e(T) = sup\{|\lambda|, \lambda \in W_e(T)\}$

Proposition 1.1.22

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathfrak{B}(\mathcal{H})$. Then $\sigma(T) = \sigma_e(T) \bigcup \sigma_p(T) \bigcup \sigma_p(T^*)^*$, where $\sigma_p(T^*)^* = \{\overline{\lambda} : \lambda \in \sigma_p(T^*)\}$.

PROOF. Suppose λ is not an element of $\bigcup \sigma_p(T) \bigcup \sigma_p(T^*)^*$. Then $nul(T - \lambda) = nul(T-\lambda)^* = 0$. Thus $(T-\lambda)$ is injective and has dense range. If λ is not an element of $\sigma_e(T)$, then $(T-\lambda)$ is Fredholm and thus $ran(T-\lambda)$ is closed. But then $(T-\lambda)$ is bijective and hence λ is not an element of $\sigma(T)$. Thus $\sigma(T) \subseteq \sigma_e(T) \bigcup \sigma_p(T) \bigcup \sigma_p(T^*)^*$. The other inclusion is obvious.

□

Definition 1.1.23

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathfrak{B}(\mathcal{H})$. Then the **semi-Fredholm domain** $\rho_{sF}(T)$ of T is the set of all complex numbers λ such that $\lambda 1 - \Pi(T)$ is either left or right invertible in the Calkin algebra. If $\mu \in \mathbb{C}$, then μ is called a (T) - **singular point** if the function $\lambda \mapsto P_{ker(T-\lambda)}$ is discontinous at μ. Otherwise, μ is said to be (T) - **regular**.

If $\mu \in \rho_{sF}(T)$ and μ is singular (respectively, μ is regular), then we write $\mu \in \rho^s_{sF}(T)$, (respectively, $\rho^r_{sF}(T)$)

Theorem 1.1.24

Let $T \in \mathfrak{B}(\mathcal{H})$ and suppose that $\lambda \in \delta\sigma(T)$. Then either λ is an isolated point or $\lambda \in \sigma_e(T)$.

PROOF. Suppose λ is not contained in $\sigma_e(T)$. Then by Proposition 1.1.22, we may assume that $\lambda \in \sigma_p(T)$ (otherwise consider $\overline{\lambda}$ and T^*). Since $\lambda \in \delta\sigma(T)$, we can find a sequence $\{\lambda\}_n \subseteq \rho(T)$ such that $\lambda = lim_{n\to\infty}\lambda_n$.

Since $ker(T-\lambda_n) = \{0\}$ for all $n \geq 1$ while $ker(T-\lambda) \neq \{0\}$, we conclude that $\lambda \in \rho^s_{sF}(T)$. Since $\rho^s_{sF}(T)$ has no accumulation points in $\rho_{sF}(T)$, and since λ is not contained in $\sigma_e(T)$, we conclude that λ is isolated in $\sigma(T)$.

□

Definition 1.1.25

Let $A \in \mathfrak{B}(\mathcal{H})$, $A \geq 0$. The *trace* of A is:

$$Tr(A) = \sum_i \langle Av_i, v_i \rangle \in [0, \infty]$$

If $Tr(A) < \infty$ then A is called *trace class*.

Definition 1.1.26 (Eigen values and Eigen vectors)

If v is a non-zero vector such that Tv is a scalar multiple of v, then the line through 0 and v is an invariant set under T and v and is called a *characteristic vector or eigen vector*. The scalar λ such that $Tv = \lambda v$ is called a *characteristic value or eigen value*.

Definition 1.1.27 (Dual of a normed space)

Let $\mathfrak{X}$ be a normed vector space. The dual space to $\mathfrak{X}$, denoted by $\mathfrak{X}^*$ is the space of bounded linear functionals. In other words, $\mathfrak{X} := \mathfrak{B}(\mathfrak{X}, \mathbb{K})$. clearly,

$$||v|| := sup_{||x|| \leq 1} |\langle v, x \rangle| < \infty$$

Definition 1.1.28

An operator T is called *normal* if $T^*T = TT^*$. Normal operators may be regarded as a generalization of self-adjoint operators T in which T^* need not be exactly T but commutes with T. (see, [10])

1.2 Statement of the problem

First, we review the properties of the numerical range available in literature. The relationship between the spectrum of the operator $T \in \mathfrak{B}(\mathcal{H})$

and the numerical range, $W(T)$, of the operator is then reviewed. Secondly, the properties of the essential numerical range, $W_e(T)$, defined for the Calkin algebra are studied. Then we find the relationships between the essential spectrum, $\sigma_e(T)$, and the essential numerical range of the operator T. Thirdly, the properties shared by both the numerical range and the essential numerical range are determined. Finally, we investigate the role of the essential numerical range to the field of operator theory.

1.3 Objectives of the study

- To investigate the properties of the essential numerical range
- To establish the relationships between the numerical range and the essential numerical range
- To find out the role of the essential numerical range to the field of operator theory.

1.4 Significance of the study

The findings of this study are aimed at striking the relationships between the usual numerical range and the essential numerical range, reveal the properties of the essential numerical range and the roles of the essential numerical range in Operator theory. Thus the findings of this study will contribute immensely to the field of Operator theory.

1.5 Research Methodology

This involved reading various texts and articles on the numerical range and the essential numerical range, solving problems, discussions with supervisors, browsing the internet for journals on the topic and visiting university libraries for research materials.

1.6 Notations and organization of the study

Let $\mathcal{H}$ denote Hilbert space. Unless otherwise stated we will assume that the underlying field is the complex field $\mathbb{C}$[1] and the norm on $\mathcal{H}$ is $\|.\|$. $\mathfrak{B}(\mathcal{H})$ will denote the space of all bounded linear operators from the linear space $\mathcal{H}$ to linear space $\mathcal{H}$. We write $\mathfrak{B}(\mathcal{H}) := \mathfrak{B}(\mathcal{H}, \mathcal{H})$.

This work is divided into four broad Chapters. In *Chapter One*, we attempt to give simple proofs and definitions that we anticipate to use in *Chapter Two*, *Chapter Three* and *Chapter Four*. In *Chapter Two*, we give a detailed review of the numerical range. We mostly refer to facts that are already available in literature. We prove various properties of the numerical range and also prove the spectral inclusion theorem. In *Chapter Three*, we proceed to build our study by defining the essential numerical range and give its properties. Here, we also prove most of the properties of the essential numerical range. In *Chapter Four* and *Chapter Five*, we continue outlining more results of our study. This Chapter is devoted to giving the relationships between the numerical range and the essential

[1] $\mathbb{C}$ $\mathbb{R}$ and $\mathbb{Q}$ denote complex, real, and rational number fields respectively, while $\mathbb{Z}$ and $\mathbb{N}$ denote integers and positive integers respectively

numerical range based on their properties. We also give the relationships between the numerical range and the essential numerical range based on the theorems by J. Christophe and the J. S. Lancaster. Finally, we discuss the roles of the essential numerical range in operator theory. We identify some of the classes of operators that can be identified by the property that 0 is in the essential numerical range.

Chapter 2

Properties of the Numerical Range

2.1 Introduction

In this Chapter, we review the properties of the numerical range of an operator T acting on a fixed complex separable infinite dimensional Hilbert space. We give proofs of a number of properties and show the relationship between the spectrum of an operator and the numerical range of the operator.

Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $\langle , \rangle$, and let $\mathfrak{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. We recall that the numerical range (also known as the field of values) $W(T)$ of $T \in \mathfrak{B}(\mathcal{H})$ is the collection of all complex numbers of the form $\langle Tx, x \rangle$ where x is a unit vector in $\mathcal{H}$. i.e.

$$W(T) \;=\; \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

For an operator T on a Hilbert space H, the following are known results on the numerical rage:

(a). $W(T)$ is invariant under unitary similarity.

(b). $W(T)$ lies in the closed disc of radius $||T||$ centred at the origin.

(c). $W(T)$ contains all eigenvalues of T

(d). $W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$

(e). $W(I) = \{1\}$

More generally, if α and β are complex numbers and T is a bounded operator on $\mathcal{H}$, then,

$W(\alpha I + \beta I) = \alpha W(T) + \beta$

(f). If $\mathcal{H}$ is finite dimensional then $W(T)$ is compact.

The last fact follows from the compactness of the unit sphere of $\mathcal{H}$ and continuity of the quadratic form associated with T. If $\mathcal{H}$ is infinite dimensional, then it supports bounded operators with non-closed numerical range.

If $\mathfrak{A}$ is a Banach Algebra with unit e, then the Algebraic Numerical Range of an arbitrary element $a \in \mathfrak{A}$ is defined by,

$$V(a) = \{f(a) : f \in \mathfrak{A}', ||f|| = f(1) = 1\}$$

Here, $\mathfrak{A}'$ denotes the space of all continuous linear functionals on $\mathfrak{A}$. If $a = T$ and $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$ then $V(T) = \overline{W(T)}$ and $V(T)$ is a non-empty, compact and convex set. (see, [8])

2.2 Elliptic range theorem

We start the proof of the elliptic range theorem as a pre-requisite to proving the *Toeplitz-Hausdorff theorem*

Theorem 2.2.1 (Elliptic Range Theorem)

Let T be an operator on a two-dimensional space. Then $W(T)$ is an ellipse whose foci are the eigenvalues of T (See for example, [4])

PROOF. Without loss of generality, we can choose T as an upper triangular matrix i.e. the Schur decomposition theorem guarantees that any square matrix T may be transformed by unitary similarity transformation to upper triangular form. with its eigenvalues on the diagonal. Also, since $W(T)$ is invariant under similarity transformation, it suffices to consider only upper triangular matrices in this proof.

Thus let

$$T = \begin{pmatrix} \lambda_1 & a \\ 0 & \lambda_1 \end{pmatrix}$$

where λ_1 and λ_2 are the eigenvalues of T.

If $\lambda_1 = \lambda_2 = \lambda$, we have,

$$T - \lambda = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

and,

$W(T-\lambda) = \{z : |z| \leq \frac{|a|}{2}\}$, and $W(T)$ is a circle with centre at λ and radius $\frac{|a|}{2}$.

If $\lambda_1 = \lambda_2$ and $a = 0$, we have,

$$T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

If $x = (f, g)$,

$$\begin{aligned} \langle Tx, x\rangle &= \lambda_1|f|^2 + \lambda_2|g|^2 \\ &= t\lambda_1 + (1-t)\lambda_2, \end{aligned}$$

where $t = |f|^2$ and $|f|^2 + |g|^2 = 1$.

So $W(T)$ is the set of convex combinations of λ_1 and λ_2 and is the segment joining them.

If If $\lambda_1 \neq \lambda_2$ and $a \neq 0$, we have,

$$T - \frac{\lambda_1+\lambda_2}{2} = \begin{pmatrix} \frac{\lambda_1-\lambda_2}{2} & a \\ 0 & \frac{\lambda_2-\lambda_1}{2} \end{pmatrix},$$

$$\begin{aligned} e^{-i\Theta}[T - \frac{\lambda_1+\lambda_2}{2}] &= \begin{pmatrix} r & ae^{-i\Theta} \\ 0 & -r \end{pmatrix} \\ &= B \end{aligned}$$

,

where $\frac{\lambda_1-\lambda_2}{2} = re^{i\Theta}$.

$W(B)$ is an ellipse with centre at $(0,0)$, and minor axis $|a|$, and foci at $(r,0)$ and $(-r,0)$.

Thus $W(T)$ is an ellipse with foci at λ_1, λ_2 and the major axis has an inclination of Θ with the real axis. □

2.3 Convexity of the numerical range

Theorem 2.3.1 (Toeplitz-Hausdorf Theorem)

The numerical range of an operator $T \in B(H)$ is convex. (see, [10])

PROOF. Let $\alpha, \beta \in W(T)$ such that $\alpha = \langle Tf, f\rangle$, $\beta = \langle Tg, g\rangle$ and $\|f\| = \|g\| = 1$.

We need to show that the segment containing α and β is also contained in $W(T)$. Let V be the subspace spanned by f and g and E be the orthogonal projection of H on V, so that $Ef = f$ and $Eg = g$. We also have for the operator ETE on V,

$$\begin{aligned} \langle ETEf, f\rangle &= \langle Tf, f\rangle; \\ \langle ETEg, g\rangle &= \langle Tg, g\rangle \end{aligned}$$

By the *elliptic range theorem*, $W(ETE)$ is an ellipse. Hence $W(ETE)$ contains the segment joining α and β. It is easy to see that $W(ETE) \subset W(T)$ and that $W(T)$ contains the segment joining α and β.

□

2.4 Non-similarity invariance of the numerical range

Theorem 2.4.1 (Non-similarity invariance of the numerical range) *(see, [10])*

PROOF. Let T_λ be the operator associated with the matrix

$$\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix},$$

and $T_\lambda = \lambda T_1$, where.

$$T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Thus $W(T_\lambda) = \lambda W(T_1)$, the closed disc of radius $|\lambda|$ centred at the origin, so all the different operators T_λ have different numerical ranges. But for $\lambda \neq 0$ all these operators are similar.

Indeed, $S_\lambda := \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$

is non-singular and $S_\lambda T_1 S_\lambda^{-1} = T_\lambda$ (See for example, [1])

□

2.5 The Projection property

The ***projection property*** holds for the numerical range. According to the projection property;

$ReW(T) = W(ReT)$, where $T \in \mathfrak{B}(\mathcal{H})$ and Re stands for the real part. This comes about as a result of the fact that every operator $T \in \mathfrak{B}(\mathcal{H})$ can be written as; $T = ReT + iIMT$. (see, [19])

2.6 Extreme points of the closure numerical range

Theorem 2.6.1

The extreme points of the closure of the numerical range $W(T)$ of a normal operator T are eigenvalues of T if and only if $W(T)$ is closed. (see, [10])

PROOF. Let $W(T)$ be closed. We can assume that the extreme point is $z = 0$ and that $W(T) \subseteq \{\lambda : Im\lambda \geq 0\}$ and $\langle Tx, x\rangle = 0 \in W(T)$; hence $\langle (T - T^*)x, x\rangle = 0$. Since the operator $\frac{1}{i}(T - T^*) \geq 0$, it follows that $(T - T^*)x = 0$. Consequently, x is an element of the closed subspace $\{f : Tf = T^*f\} = N$. Since T is normal, we have;

$$\begin{aligned} T^*Tx &= TT^*x \\ &= TTx \end{aligned}$$

and hence the subspace N is invariant for T and $T|_N$ is self-adjoint. Obviously, $W(T|_N) \subset W(T)$ and $W(T|_N) \subset \mathcal{R}$. Hence $W(T|_N) \subset W(T)\bigcap\mathcal{R}$, and thus $T|_N = 0$ and $Tx = 0$, i.e. $0 \in W_e(T)$. The converse is true for any operator T. The compact convex set $\overline{W(T)}$ is the convex hull of its extreme points. When the latter are eigenvalues of T, as assumed in the theorem, we have;

$$\begin{aligned}\overline{W(T)} &\subset co(\sigma_p(T)) \\ &\subset co(W(T)) \\ &= W(T)\end{aligned}$$

□

2.7 Spectral Inclusion

The spectrum of an operator T consists of those complex numbers λ such that $T - \lambda I$ is not invertible. For the purpose of showing that the spectrum of an operator is contained in the closure of the numerical range, it is enough to look at the boundary of the spectrum. The boundary of the spectrum is contained in the approximate point spectrum, $\sigma_{app}(T)$, which consists of complex numbers λ for which there exists a sequence of unit vectors $\{x_n\}$ with $\|(T - \lambda I)x_n\| \to 0$. Since $W(T)$ is convex, it suffices to show that $\sigma_{app}(T) \subset W(T)$.

Theorem 2.7.1 (Spectral Inclusion theorem)
The spectrum of an operator is contained in the closure of its numerical range. (see, [10])

PROOF. Consider any $\lambda \in \sigma_{app}(T)$ and a sequence $\{x_n\}$ of unit vectors with $\|(T - \lambda I)x_n\| \to 0$

By the Schwarz inequality,

$$|\langle (T - \lambda I)x_n, x_n\rangle| \leq \|(T - \lambda I)x_n\| \to 0$$

Thus $\langle Tx_n, x_n\rangle \to \lambda$.

So, $\lambda \in \overline{W(T)}$

□

Chapter 3

The Essential Numerical Range

3.1 Introduction

In this chapter, we investigate the properties of the essential numerical range, for instance, convexity, unitary invariance and projection property. We also determine the essential numerical range of an essentially normal operator and find how the essential numerical range is connected to the diagonal set among other properties.

We recall that Stampfli, Williams and Fillmore [5] gave the definition of $W_e(T)$ as follows:

$W_e(T) = \bigcap \overline{W(T+K)}$ where the intersection runs over the compact operators $K \in \mathcal{K}(\mathcal{H})$.

3.2 Properties of the essential numerical range

A. G. Chacon and R. G. Chacon [4] gave the the properties of the essential numerical range as follows:

Let $T \in \mathfrak{B}(\mathcal{H})$ then:

(1). $W_e(T)$ is a non-void compact and convex set.

(2). $W_e(T) = \{0\}$ if and only if T is compact.

(3). If T is an essentially normal operator, then $W_e(T) = co(\sigma_e(T))$ and the essential numerical radius, $w_e(T) = \|T\|_e$.

(4). If M is a closed linear subspace of $\mathcal{H}$ such that $M^{\perp}$ has finite dimension. Then $W_e(T) = W_e(P_M T|_M)$, where P_M denotes the orthogonal projection onto M.

According to A. G. Chacon and R. G. Chacon [4], $W_e(T)$ is a closed subset of $\overline{W(T)}$ and the essential spectrum, $\sigma_e(T)$ is always a compact subset contained in $\sigma(T)$.

Theorem 3.2.1

$W_e(T) = \overline{W(T)}$ if and only if $Ext(W(T)) \subset W_e(T)$. Therefore, $W_e(T) = \overline{W(T)}$ if $W(T)$ has no extreme points. Here, $Ext(W(T))$ denotes the set of extreme points of $W(T)$.

Proof. If $Ext(W(T)) \subset W_e(T)$, then $Ext(\overline{W(T)}) \subset W_e(T) \subset \overline{W(T)}$. Taking convex hulls, we obtain $\overline{W(T)} = W_e(T)$. The reverse implication is obvious. (see, [21])

□

Theorem 3.2.2
If $\overline{W(T)}$ has a corner at the point λ, then λ is in $W_e(T)$ or λ is a reducing eigenvalue of finite multiplicity for T which is an isolated point of $\sigma(T)$. (see, [21])

PROOF. Recall that λ is called a corner of a convex set C if $\lambda \in C$ and C is contained in a sector of vertex λ and opening less than π. The proof proceeds as follows: If λ is not in $W_e(T)$, then λ must be a corner of $W(T)$. Thus λ is an eigenvalue of T. Thus the eigenspace corresponding to λ reduces T. The rest of the proof follows from elementary Fredholm theory. $T - \lambda$ is Fredholm (λ is not in $\sigma_e(T)$) of index 0 (λ reduces T); in particular, λ has finite multiplicity. If λ is not isolated, then there exist $\lambda_n \in \sigma(T)$ such that $\lambda_n \neq \lambda$, $\lambda_n \to \lambda$, and $T - \lambda_n$ is Fredholm of index 0 (this set is open). Hence, there exists $x_n \in \mathcal{H}$ such that $\|x_n\| = 1$, $x_n \in rg(T - \lambda)$ and $Tx_n = \lambda_n x_n$. Thus the weak limits of x_n lie in the $rg(T-\lambda) \bigcap ker(T-\lambda) = 0$. This yields a contradiction since $x_n \to 0$ weakly and $\langle Tx_n, x_n\rangle \to \lambda$ implies $\lambda \in W_e(T)$.

□

3.2.1 Unitary invariance

The essential numerical range is *unitarily invariant.* For, let T be an operator on $\mathfrak{B}(\mathcal{H})$ then $W_e(T)$ is unitarily invariant. That is;

$W_e(U^*TU) = W_e(T)$ for any unitary operator $T \in \mathfrak{B}(\mathcal{H})$. (see, [11])

3.2.2 Essential numerical range of identity

The essential numerical range behave in a nice and predictable way under affine transformations of T. That is;

$W_e(\alpha T + \beta I) = \alpha W_e(T) + \beta$, for all $\alpha, \beta \in \mathbb{C}$. Thus it is easy to see that $W_e(I) = \{1\}$ (see, [11])

3.2.3 The projection property

The essential numerical range obeys the projection property. For instance; for an operator $T \in \mathfrak{B}(\mathcal{H})$ we have,

$ReW_e(T) = W_e(ReT)$, where $T \in \mathfrak{B}(\mathcal{H})$ and Re stands for the real part. This comes about as a result of the fact that every operator $T \in \mathfrak{B}(\mathcal{H})$ can be written as: $T = ReT + iIMT$. (see, [19])

3.2.4 Convexity

Theorem 3.2.3

The essential numerical range of an operator $T \in \mathfrak{B}(\mathcal{H})$ is convex.

PROOF. The essential numerical range is defined as $W_e(T) = \bigcap_{K\in\mathcal{K}(\mathcal{H})} \overline{W(T+K)}$. Therefore, since each $W(T+K)$ is convex by the Toeplitz-Hausdorff theorem, $\overline{W(T+K)}$ is convex as well. Consequently, $\bigcap \overline{W(T+K)}$ is convex as well. Therefore, $W_e(T)$ is convex.

□

3.2.5 Spectral inclusion

The essential spectrum, $\sigma_e(T)$, is contained in the essential numerical range, $W_e(T)$. (see, [21])

3.2.6 Essential numerical range of a scalar

For any scalar λ, we have;

$$W_e(T+\lambda) = W_e(T)+\lambda$$

for all $\lambda \in \mathbb{C}$. (see, [12])

3.2.7 Essential numerical range of an essentially normal operator

Let X be an essentially normal operator i.e. $X^*X - XX^*$ is compact. Then $W_e(X) = co\sigma_e(X)$. Indeed for such an operator, the essential norm, $\|X\|_e$ equals the essential spectral radius, $\rho_e(X)$ i.e.

$$\|X\|_e = \rho_e(X)$$

Note that $e^{i\theta}X + \mu I = Y$ is also an essentially normal operator for any $\theta \in \mathbb{R}$ and $\mu \in \mathbb{C}$. Let z be an extremal point of of $W_e(X)$. With suitable θ and μ, we have;

$$e^{i\theta}z + \mu = W_e(Y)$$

$$= max\{|y| : y \in W_e(Y)\},$$

the maximum being attained at the single point $e^{i\theta}z+\mu$. Since $co\sigma_e(T) \subset W_e(Y)$ and $\rho_e(Y) = W_e(Y)$, this implies that $e^{i\theta}z + \mu \in \sigma_e(Y)$. Hence $z \in \sigma_e(Y)$, so that $W_e(X) = co\sigma_e(X)$. (see, [19])

3.2.8 Essential numerical range of an operator in a C^* subalgebra with no finite projections

Let X be an operator lying in a C^* subalgebra of $\mathfrak{B}(\mathcal{H})$ with no finite dimensional projections. Then for any real θ, we have

$$\overline{W(Re\ e^{i\theta}\ X)} = W_e(Re\ e^{i\theta}X)$$

Thus from the projection property for $W(.)$ and $W_e(.)$, we infer that $W_e(X) = \overline{W(X)}$ (see, [19])

3.2.9 The essential numerical range and the diagonal set

For an operator $A \in \mathfrak{B}(\mathcal{H})$ the diagonal set, $\Delta(A)$, is defined as;

$$\Delta(A) = \{\lambda : there\ is\ a\ basis \{e_n\}_{n=1}^{\infty}\ with\ \langle e_n, Ae_n\rangle = \lambda\}.$$

This definition was given by J. Christophe [19]. He also gave an alternative definition of the essential numerical range of an operator $A \in \mathfrak{B}(\mathcal{H})$

as;

$$W_e(A) = \{\lambda : there\ is\ an\ orthonormal\ system \{e_n\}_{n=1}^{\infty}$$
$$with\ lim_{n=\infty}\langle e_n, Ae_n\rangle = \lambda\}.$$

The equivalence of $\Delta(A)$ and $W_e(A)$ was checked by J. Christophe [19] as follows; Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal system such that $lim_{n=\infty}\langle x_n, Ax_n\rangle = \lambda$. If span $\{x_n\}_{n=1}^{\infty}$ is of finite co-dimension p, we immediately get a basis,

$$\begin{aligned} e_1, \ldots\ldots, e_p; e_{p+1} &= x_1, \ldots\ldots; e_{p+n} \\ &= x_n, \ldots\ldots\ldots\ldots\ldots \end{aligned}$$

such that $lim_{n=\infty}\langle e_n, Ae_n\rangle = \lambda$. If span $\{x_n\}_{n=1}^{\infty}$ is of infinite co-dimension, we may complete this system with $\{y_n\}_{n=1}^{\infty}$ in order to obtain a basis. Let P_j be the subspace spanned by y_j and $\{x_n : 2^{j-1} \leq n < 2^j\}$. By Parker's theorem, there is a basis of P_j, say $\{e_l^j\}_{l\in\Lambda_j}$, with;

$$\langle e_l^j, Ae_l^j\rangle = \frac{1}{dim\ P_j} Tr\ AP_j.$$

Since $\frac{1}{dim\ P_j} Tr\ AP_j \to \lambda$ as $j \to \infty$, it is possible to index $\{e_l^j\}_{j\in\mathbb{N};l\in\Lambda_j}$ in order to obtain a basis $\{f_n\}_{n=1}^{\infty}$ such that;

$$lim_{n=\infty}\langle f_n, Af_n\rangle = \lambda$$

Thus from the foregoing proof of equivalence, J. Christophe [19] in Proposition 1.2 gives the relationship as;

$$int\ W_e(A) \subset \Delta(A) \subset W_e(A)$$

where *int* $W_e(A)$ denotes the interior of $W_e(A)$.

Chapter 4

Results and Discussion

4.1 Introduction

In this chapter we discuss the relationships between the numerical range and the essential numerical range. We prove two theorems in this chapter that show how the numerical range and the essential numerical range are related. Finally, we discuss the roles of the essential numerical range in operator theory and give conclusions and recommendations for further research.

4.2 John Lancaster theorem

The relationship between the numerical range and the essential numerical range is given in the result by John Lancaster.

Theorem 4.2.1 (John Lancaster theorem)

For $T \in \mathfrak{B}(\mathcal{H})$ *we have* $\overline{W(T)} = conv\{W(T) \bigcup W_e(T)\}$ *- (see, [21])* PROOF. Clearly, $W(\alpha T+\beta)=\alpha W(T)+\beta$ for all $\alpha,\beta\in\mathbb{C}$. Therefore by rotation and translation, we can assume that $\overline{W(T)}$ is contained in the closed right half plane and $0\in Ext(\overline{W(T)})-W(T)$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors of $\mathcal{H}$ such that $\langle Tx_n,x_n\rangle\to 0$. By weak sequential compactness of the unit ball of $\mathcal{H}$, we can assume that $\{x_n\}_{n=1}^{\infty}$ converges weakly to $x\in\mathcal{H}$ with $\|x\|\leq 1$. We prove that x is the 0 vector, and hence $0\in W_e(T)$.

If $\|x\|=1$, then $x_n\to x$ strongly. But;

$$\begin{aligned}|\langle Tx,x\rangle| &\leq |\langle T(x-x_n),x\rangle|+|\langle Tx_n,x-x_n\rangle|+|\langle Tx_n,x_n\rangle|\\ &\leq |\langle x-x_n,T^*x\rangle|+\|T\|\|x-x_n\|+|\langle Tx_n,x_n\rangle|\to 0\end{aligned}$$

Hence $\langle Tx,x\rangle=0$ and $0\in W(T)$. So assume $0<\|x\|<1$. Clearly the operator ReT is positive since $W(T)$ is contained in the closed right half plane. Then;

$$\begin{aligned}\|(ReT)^{\frac{1}{2}}x_n\|^2 &= \langle (ReT)x_n,x_n\rangle\\ &= Re\langle Tx_n,x_n\rangle\to 0,\end{aligned}$$

so $\|(ReT)x_n\|\to 0$. This clearly yields $Re\langle Tx,x\rangle=0$ so $\langle Tx,x\rangle$ is purely imaginary. On the other hand;

$$\langle T(x-x_n),x-x_n\rangle = \langle Tx,x-x_n\rangle-\langle Tx_n,x\rangle+\langle Tx_n,x_n\rangle\to -\langle Tx,x\rangle$$

and

$$\|x - x_n\|^2 = 1 - 2Re\langle x - x_n, x\rangle - \|x\|^2,$$

so $\langle Ty_n, y_n\rangle \to -\langle Tx, x\rangle/(1 - \|x\|^2)$ where $y_n = \frac{(x-x_n)}{\|x-x_n\|}$. Thus we have produced a non-zero purely imaginary points in $\overline{W(T)}$ which lie in the upper and lower half planes. However this implies that 0 is a non-extreme point of $\overline{W(T)}$, thus completing the proof of the inclusion. The equality follows from the inclusion by the Krein-Milman theorem.

□

The theorem below by J. Christophe [19] also reinforces the John Lancaster theorem

Theorem 4.2.2

Let T be an operator, then:

(i). If $W_e(T) \subset W(T)$ then $W(T)$ is closed.

(ii). There exist normal finite rank operators R of arbitrarily small norm such that $W(T + R)$ is closed.

Proof. Assertion (i) is due to Theorem 4.2.1. We prove the second assertion and implicitly prove Lancaster's result.

We may find an orthonormal system $\{x_n\}$ such that the closure of the sequence $\{\langle Tx_n, x_n\rangle\}$ contains the boundary of the essential numerical range, $\delta W_e(T)$.

Fix $\epsilon > 0$. It is possible to find an integer p and scalars z_j, $1 < j < p$, with $|z| < \epsilon$ such that;

$co\{\langle x_j, Tx_j\rangle + z_j : 1 < j < p\} \supset \delta W_e(T)$.

Thus, the finite rank operators,

$R = \sum_{1<j<p} z_j x_j \bigotimes x_j$ has the property that $W(T+R)$ contains $W_e(T)$. We need this operator R. Indeed, setting $X = T + R$, we also have $W(X) \supset W_e(X)$. We then claim that $W(X)$ is closed (this claim implies assertion (i)). By the contrary, there would exist,

$z \in \delta\overline{W(X)} \backslash W_e(X)$.

Furthermore, since $\overline{W(X)}$ is the convex hull of its extreme points, we could assume that such a z is an extreme point of $\overline{W(X)}$. By suitable rotation and translation, we could assume that $z = 0$ and that the imaginary axis is a line of support of $\overline{W(X)}$. The projection property for $W(.)$ would imply that $W(ReX) = (x, 0]$ for a certain negative number x, so that $0 \in W_e(ReX)$.

Thus we would deduce from the projection property for $W_e(.)$ that $0 \in W_e(X)$; a contradiction. (see, [19])

□

4.3 Role of the essential numerical range

4.3.1 Operators with the *Small entry property*

This is a matricial property. An operator $T \in \mathfrak{B}(\mathcal{H})$ has the *small entry property* if for every $\epsilon > 0$, there is a basis $\{e_n\}$ such that $|\langle Te_n, e_m\rangle| < \epsilon$ for all n and m. The condition $0 \in W_e(T)$ is equivalent to the fact that the operator T has the *small entry property.* That is; if the operator T has the small entry property, then for any $\epsilon > 0$, there is a basis so that all entries of the matrix of T have absolute value less than ϵ. In particular, the diagonal entries of the matrix must have an accumulation point λ with $|\lambda| < \epsilon$ and since $W_e(T) = \{\lambda : \textit{there is an orthonormal sequence } \{x_n\}_{n=1}^{\infty} \textit{ with } lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda\}$, its evident that $\lambda \in W_e(T)$. And since $W_e(T)$ is closed, $0 \in W_e(T)$. We conclude that $0 \in W_e(T)$ is equivalent to the property that the operator T has the *small entry property.* Thus we infer that the essential numerical range serves to identify the class of operators that satisfy the *small entry property.* (see,[9])

We give the theorem by Q. F. Stout [12] that reinforces that the condition $0 \in W_e(T)$ is equivalent to the fact that the operator T has the *small entry property.*

Theorem 4.3.1

For any $T \in \mathfrak{B}(\mathcal{H})$, the following conditions are equivalent:

(a). $0 \in W_e(T)$

(b). *There is a basis ξ such that $T \in kernel(hull(\kappa_\xi))$.*

(c). *T has the small entry property.*

(d). *There exists a sequence of bases $\xi_{(n)}$ such that $T_{\xi(n)} \to 0$ uniformly in $\mathfrak{B}(\mathfrak{B}(\mathcal{H}))$. (see, [12])*

PROOF. The proof can be found in Q.F. Stout [12], Theorem 2.3

□

4.3.2 Zero diagonal operators

An operator $T \in \mathfrak{B}(\mathcal{H})$ is called *zero diagonal* if there exists an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ such that $\langle Te_n, e_n\rangle = 0$ for all n. We state the theorem below by D. Bakic [9] without proof.

Theorem 4.3.2

Let $T \in \mathfrak{B}(\mathcal{H})$ be a bounded operator on a separable Hilbert space, $\mathcal{H}$. Then there exists an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ such that $lim\langle Te_n, e_n\rangle = 0$ if and only if 0 is in the essential numerical range of T.

Thus from this theorem, we infer that 0 is in the essential numerical range. We conclude that the notion that an operator T is zero diagonal is equivalent to the fact that $0 \in W_e(T)$. Thus the essential numerical range also serves to identify zero diagonal operators. (see, [9])

The essential numerical range plays an important role in solving problems from the operator theory. The list below of mutually equivalent conditions indicates the importance of the essential numerical range.

Theorem 4.3.3

For an operator $A \in \mathfrak{B}(\mathcal{H})$ the following conditions are mutually equivalent.

(a). *There exists an orthonormal basis $\{e_n\}$ for H such that $lim_n\langle Ae_n, e_n\rangle = 0$*

(b). $0 \in W_e(A)$

(c). *There exists an orthonormal sequence $\{a_n\}$ in H such that $lim_n\langle Aa_n, a_n\rangle = 0$.*

(d). *There exists a sequence of unit vectors* (x_n) *in* $\mathcal{H}$ *weakly converging to* 0 *such that* $lim_n T x_n = 0$.

(e). *There exists an orthogonal projection* $P \in \mathfrak{B}(\mathcal{H})$ *with an infinite dimensional range such that* PAP *is a compact operator.*

(f). *For each* $\epsilon > 0$ *there exists an orthonormal basis* $\{e_n\}$ *for* $\mathcal{H}$ *such that* $|\langle Ae_n, e_m\rangle| < \epsilon$, *for all* n *and* m.

(g). *For each* $\epsilon > 0$ *and* $p > 1$ *there exists an orthonormal basis* $\{e_n\}$ *for* $\mathcal{H}$ *such that* $\sum_{n=1}^{\infty} |\langle Ae_n, e_n\rangle|^p < \epsilon$

(h). *There exists a sequence of zero diagonal operators* A_n *in* $\mathcal{H}$ *such that* $A = (norm) lim_n A_n$

(i). *There exists a zero diagonal operator* $T \in \mathfrak{B}(\mathcal{H})$ *and a compact operator* $K \in \mathcal{K}(\mathcal{H})$ *such that* $A = T + K$

(j). *There exists an operator* $B \in \mathfrak{B}(\mathcal{H})$ *such that* $A = B^*B - BB^*$. *In this case* A *is self-adjoint necessarily.*

(k). *The spectrum of* A *has at least one non-negative limit point and at least one non-positive limit point. (see. [9])*

PROOF. $(a) \Leftrightarrow (b)$, This is due to the assertion of *Theorem 4.3.2* above.

$(b), (c), (d)$ and (e) are equivalent, (see, [5]).

$(e) \Leftrightarrow (f) \Leftrightarrow (g)$

$(h) \Leftrightarrow (b)$

$(i) \Leftrightarrow (a)$

$(a) \Rightarrow (i)$: Let us take the orthonormal basis from (a) and define $K \in \mathfrak{B}(\mathcal{H})$ by $Ke_n = \langle Ae_n, e_n \rangle e_n$ for all n. Since $\langle Ae_n, e_n \to 0$, K is compact. Obviously, $T = A - K$ is zero diagonal.

$(j) \Leftrightarrow (k) \Leftrightarrow (b)$

□

4.4 Conclusions and Recommendations

In line with the objectives of our study, we have studied the properties of the essential numerical range. We have also looked at the relationships between numerical range and the essential numerical range. Finally, we have discussed the role of the essential numerical range in operator theory. We have found out that, just like the familiar numerical range, the essential numerical range is convex. Both the essential numerical range and the familiar numerical range satisfy the properties, for instance, unitary invariance and the projection property. Our study has also shown that the essential spectrum is contained in the essential numerical range. This is contrary to the well known fact that the spectrum is contained in the closure of the numerical range. We have revealed that the essential numerical range is non-void and compact set. We have also given the essential numerical range of an essentially normal operator as well as the connection between the essential numerical range and the diagonal set. We have shown that the essential numerical range is a subset of the closure of the numerical range. This is as a result of the John Lancaster theorem and the J. Christophe theorem. This study has also revealed some of the roles of the essential numerical range. We have found out that the essential numerical range can be used to identify zero diagonal operators as well as the class of operators that satisfy the 'small entry property'. We have also given a list of mutually equivalent conditions that show the importance of the essential numerical range.

Although the numerical range has had a great role in various aspects of operator theory e.g. similarity of operators e.t.c.,we wish to recommend that there is a need to investigate in which of such areas can the numerical range be replaced with the essential numerical range in such a way that the results still carry through. Consequently, the equality of spectra of a given pair of operators has been considered by several authors. But the equality of essential spectra for the same operator is still wanting in the extant literature of operator theory. We wish to recommend this area for research.

References

[1] A. Brown, P. R. Halmos, C. Pearcy, *Commutators of Operators on Hilbert spaces*, Canadian J. Math. vol 17 (1965), pg (695 - 708)

[2] A. Brown, C. Pearcy, *Structure of commutators of operators*, Annals Of Math.(2) vol 82(1965), pg. 112 - 127

[3] A. P. Fillmore, *On Similarity and the diagonal of a matrix*, American Moth. Monthly. vol 76(1969). pg 167 - 169

[4] A. G. Chacon, R. G. Chacon, *Some properties of Composition Operators on the Dirichlet space*, Acta Math. Univ. Comenianae,vol 1 (2000) pg. 7

[5] A. P. Fillmore, J. G. Stampfli, J. P. Williams, *On the essential numerical range, the essential spectrum and a problem of Halmos*, Acta Sci. Math. (Szeged) vol 33 (1972), pg. 179 - 192

[6] A. H. Joel, J. G. Stampfli, *Commutators and Compressions*, Israel J. Math. vol 10 (1971), 433 - 441

[7] A. P. Fillmore, J. P. Williams, *On operator ranges*, Advances in Math.vol 7(1971) pg. 254 - 281

[8] B. Magajna, *On the essential numerical range of a generalized derivation*, Proceedings of Amer. Soc. vol.99, No. 1, January 1987.

[9] D. Bakic, *Compact operators, essential spectrum and the essential numerical range*, Mathematical communications vol 3(1998), 103 - 108

[10] D. Rao, K. Gustafson, *Numerical Range*, Springer Verlag, 1997.

[11] F. Bonsall, J. Duncan, *Numerical Ranges II*, Cambridge University Press, 1973, London Math. Soc. Series 10.

[12] F. Q. Stout, *Schur products of operators and the essential numerical range*, Trans. Amer. Math. Soc vol 264 (1981), pg. 39 - 47

[13] G. J. Stampfli, *The norm of a derivation*, Pacific Journal of Mathematics, vol. 33, No. 3,1970

[14] G. J. Stampfli, J. P. Williams, *Growth conditions and the numerical range in a Banach algebra*, Tohoku Math. J.vol 20(1968), 417 - 424

[15] H. Radjavi, *Structure of $A^*A - AA^*$*, J. Math. Mech. vol 16(1966), pg. 19 - 26.

[16] I. D. Berg, *An extentension of the Weyl-von Neumann theorem to normal operators*, Trans. Amer. Math. Soc vol. 160(1971), pg. 365 - 371

[17] I. D. Berg, B. Sims, *Denseness of operators which attain their numerical ranges*, J. Austral. Math. Soc. vol 160(1984) pg. 130 - 133

[18] I. J. Fujii, S. Zumino, M. Nakamura, *On Hummel's theorem*, Mathematica Japonica vol 37(1992), pg. 985 - 986

[19] J. Christophe, *Compressions and Pinchings*, J. operator theory, vol 50(2003) pg. 211 - 220

[20] J. Conway, *A course in functional analysis*, Springer Verlag, 1985.

[21] J. S. Lancaster, *The boundary of the numerical range*, Proceedings of Amer. Math. Soc. vol. 49, No. 2, June 1975.

[22] K. Muroi, K. Tamaki, *On Ringrose's characterization of compact operators*, Mathematica Japonica vol 19(1974), pg. 259 - 261

[23] P. Bourdon, J. Shapiro, *The numerical range of automorphic composition operators*. J. Math. Anal. Appl. vol 251 (2000) 839 - 854.

[24] P. Bourdon, D. Levi, S. Narayan, J. Shapiro, *Which linear-fractional composition operators are essentially normal?* J. Math. Anal. Appl. vol 280 (2003) 30 - 53.

[25] P. Fan, *On the diagonal of an operator*, Trans. Amer. Math. Soc. vol 283 (1984), 239 - 251.