

MATHEMATICAL MODEL OF THE
ROAN ANTELOPES, RUMA PARK

BY

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ABSTRACT

Roan antelopes that were once abundant in the country in the 1880s have been reduced to a remnant population of less than fifty individuals in the last estimate, November 2009. Oksendal and Lungu developed population growth model in a crowded environment by introducing randomness in their differential equation via additional noise term. Magin and Kock in their roan antelope recovery plan in the Ruma National Park considered poaching as a major factor affecting population growth of roans which saw a slight population growth before experiencing stagnation since the year 2003 to date. The Kenya Wildlife Service (KWS) has since taken necessary measures to curb poaching. This reduced the risk of poaching as a major factor that accelerated roans' population decay. Lambwe valley is believed to have uranium deposits that could affect fertility. Inbreeding in small populations is known to have substantial effects on population growth rate. We have therefore incorporated in our model genetic defect that was not incorporated by Magin and Kock. This was made possible by making appropriate adjustments to Vortex Version 9.99 which is a computerized program for the simulation of the extinction processes. We noted that there is a high correlation between inbreeding and population growth in small populations. It is hoped that this study will help The Kenya Wildlife Service (KWS) in the management of their complex ecosystem.

Chapter 1

Introduction

The main insitu measure enacted for the conservation of the roan antelope in Kenya was the creation in 1966 of the Lambwe Valley Game Reserve, later gazetted as Ruma National Park in 1983 to provide a legally protected stronghold for the species in Western Kenya [17]. The Park covers an area of 126 km² and situated in Homa Bay County, approximately 30km South-West of Homa Bay Town [1]. Ruma has important surviving population of herbivores such as the Oribi, Jackson's hartebeest, and Rothschild's giraffe.

In the 1880s, the roan antelope occupied large areas including Mount Elgon, Cherangani and Chyulu Hills. By the early 1960s, the distribution had reduced and the species was declining further in most of the scattered localities [43].

In the period, 1985-1993 inadequate levels of protection and high level poaching caused a dramatic decline in roans' population. Consequently, by December 1995, there were only 27 known individuals left [34].

Kenya Wildlife Service recognized the need to protect roan antelopes. In October 1995, IUCN, Antelope Specialist Group funded a three-week recovery plan of another antelope *hirola* (the hunter's antelope). During this session, KWS scientists dedicated a small amount of time to consider the situation of roan antelopes in Kenya. They came up with a detailed document on key aspects of roan antelopes' ecology and biology based on data from Ruma and set a strategy for its conservation. They emphasized need for insitu measures as opposed to translocation to better-protected areas given that roans are sedentary, terrestrial species and selective feeders that are likely to lose out in competition with more generalist grazers [28].

Today, their range in Kenya is highly restricted with a remnant population of less than fifty individuals in the last estimate in November 2009 at Ruma National Park.

Mature female roans usually calve each year, though there is no strictly defined calving season. First conception occurs at around two years old, and with the gestation period of approximately 270 days [50]. Females leave the herd to calve, leaving the calf in hiding as the mother joins the rest. This hiding may last approximately six weeks before joining the herd. The young calves are almost odourless, thus decreasing their chance of detection by predators [17]. The assumption that there is the presence of uranium deposit could be associated with sterility in the valley thus causing genetic drift.

According to Wilson & Hirst [50] adult roans are susceptible to predation by lions, spotted hyenas, wild dogs, while leopards may also prey upon calves. Although not so much of a threat to wild populations, in the current circumstances in Ruma, predation by spotted hyena and leopard, particularly on young animals, may be limiting the roan population's natural recovery rate. Roans are susceptible to the diseases of livestock and their water dependence brings them in regular contact with livestock. Roans have had a stagnated population growth of approximately 48 ± 3 since the year 2003 to date.

Other than drought, diseases, competition for resources, the main cause of decline was believed to be poaching as game meat was highly prized for both wedding and burial ceremonies amongst the locals.

According to Magin & Cock [28] hundred wire snares were located in a mere one kilometre transect of the park by Moi University Survey [28]. The presence of higher ground outside the park allowed the poachers to monitor activities of the Kenya Wildlife Services (KWS) personnel. The high density of agricultural settlements right up to the Park boundaries provided large population of potential poachers.

The wire fence provided abundant supplies of wires for use in snares. Flooding also made roads impassable thus difficulties in control by KWS vehicles. The KWS has since taken necessary measures to curb poaching. In the case of Ruma, two factors that could be potentially targeted for the stagnating population growth are juvenile mortality (mainly predation) and inbreeding. Predator control is not advised due to difficulty in implementation, unpredictable consequences in the ecosystem and conflict with the general KWS management policies of protected areas.

Realistically, the management has to continue to accept juvenile mortality through predation. This therefore leaves management of genetic defect as the only option.

1.1 Background of The Study

Many wildlife populations (like roans) that were once widespread numerous and occupying contiguous habitat have been reduced to small isolated population. The causes of the original decline could be habitat loss, competition for resources and predation. Even if the original causes are removed small isolated population are vulnerable to additional forces, intrinsic to small population and may drive population to extinction [41, 42].

Of particular impact on small population are stochastic processes. With exception of aging, virtually all events in the life of an organism are stochastic. Genetic drift, mating, reproduction and even gene transmission can be described by probability distribution.

Genetic drift is the cumulative and non-adaptive fluctuation in allele frequencies resulting from random sampling of genes in each generation. This can impede or accelerate wildlife population [23]. Inbreeding is not strictly a component of genetic drift but correlated with it. In small populations, inbreeding has been documented to cause loss of fitness in a wide variety of species including virtually all sexually producing animals [9, 51].

Even if the immediate loss of fitness of individual is not large, the loss of genetic variation that results from genetic drift may reduce the ability of the population to adapt to future changes in the environment [11, 40].

The effect of genetic drift and consequent loss of genetic variation in individuals and population have negative impact on demographic rates and increases the susceptibility to environmental perturbations and catastrophes. Reduced population growth and greater fluctuations in numbers in turn accelerate genetic drift [6]. Gilpin [14] described these synergistic destabilizing effects of stochastic process on small wildlife population as extinction vortices.

Most population growth processes are inherently stochastic yet much theoretical analysis involves deterministic models with the assumption that biological systems consist of large collection of individuals in the same ecological interaction. This assumption implies that dynamics of measure (mean) is sufficient description and ignores the influence of variance [49].

Oksendal & Lungu [35] proposed a stochastic logistic model in estimating population growth at any time. We have worked along this line and derived a mathematical model that estimates population growth of roan antelopes by incorporating genetic defect that was not considered by Magin & Kock [28].

1.2 Statement Of the Problem

Robert Brown [36] observed that the path of a given particle is very irregular having a tangent at no point and that the motion of the two distinct particles appear to be independent.

Randomness is an intrinsic property of biological observation which makes deterministic models incomplete.

$$P_t = \lambda P_t \left(1 - \frac{P_t}{M}\right) dt \quad (1.1)$$

where

λ is the intrinsic growth rate,

P_t is the population at any time t and

M is the carrying capacity.

Growth process is therefore subject to many random factors. This has the implication that a complete model should incorporate randomness, to show more realistic results when tested. To complete the deterministic model, inclusion of measurement error (noise term) is necessary. Such models will therefore have both deterministic and stochastic part.

Kock a zoologist [22] estimated the growth of roans in Ruma National Park using Vortex Version 7(a computer simulation of the extinction process),where he considered poaching as a key factor accelerating roans' population decay. Following his recommendations, KWS took necessary measures thus reducing the risk of poaching as a potential factor. However, roans' population has stagnated since the year 2000 to date, with no substantial increase in population.

We have to develop mathematical growth model for the roan antelopes by incorporating genetic defect that was not considered by Kock and Magin in their model.

1.3 Objective Of the Study

The aim of this study is to develop a mathematical model that incorporates genetic defect in estimating the growth of roan antelopes.

1.4 Significance Of the study

It is evident from many ecological texts on species tending towards extinction that their population decline due to a variety of factors including habitat loss, diseases, drought, forest fires, predation and genetic defects. It is hoped that this study will help the Kenya Wildlife Service (KWS) management to determine the urgency of action to take and evaluate appropriate options to improve on its growth rate. Roans' have enormous economic significance in the locality and the nation at large.

1.5 Research Methodology

We have used Vortex Version 9.99, a stochastic simulation of the extinction process software in the management of complex ecological systems [24]. We have also collected related data on roan antelopes from Ruma National Park which is the only park with the remnant population of the roans in the country. This data is used in analysing results on existing population viability.

Chapter 2

Review of Related Literature

Attempts to understand population process dates back to the middle ages with Sir William who by 1300 A.D. composed a table of how people might have doubled over several ages. In his model he started with 8 people one year after the flood of 2700 B.C. and doubling at first every ten years but then at successively longer intervals of time to arrive at 320 million [48]. About the same period Fibonacci proposed a population growth model for rabbits;

$$y_t = \sum_{k=1}^{\infty} y_{k,t},$$

where

$y_{k,t}$ is the number of k old pair of rabbits at a time t ,

y_t is the total number of pairs of rabbits at a time t ,

resulting into the famous Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.....

Modern research into population dynamics can be traced back to over 200 years ago with Thomas Malthus [29] whose publication on Principle of population growth by stating that "population when unchecked increases in geometric ratio"

$$P_t = P_0 e^{(b-d)t} = P_0 e^{\lambda t}, \quad (2.1)$$

where

λ is the growth rate,

b is the birth rate,

d is the death rate,

P_0 is the initial population at time zero,

P_t is the population at time t and

t is time

Equation (2.1) is the Malthusian equation in continuous time [8]. This model predicted an exponential population growth.

McArthur and Wilson [48] help found the field of population Biology. They were proponents of natural equilibrium. They felt that mathematical models should be developed to describe the patterns found in nature instead of just providing simple textual description. They started their work with analysis of ants and birds population data. Through the analysis of population of experimental data, they observed that population size remained steady even though the exact species varied. After working with several species, they concluded that nature has great tendency of balancing things out and reaching a very harmonious equilibrium.

"If nature were left alone equilibrium would exist and population would remain close to them"

In 1838, a Belgium mathematician Pierre Francois Verhulst [45] followed suggestions from his mentor Quetelet, that the resistance to growth should be quadratic, with resistance modeled to velocity. He therefore incorporated density dependent effects in Malthus model. He recognized the fact that populations encounter internal competition as they grow within a closed environment.

This competition as he noted, has the tendency to retard the rate of growth. Verhulst modified Malthus equation (2.1) and obtained

$$\frac{dP_t}{dt} = \lambda P_t \left(1 - \frac{P_t}{M} \right) \quad (2.2)$$

where

λ is the intrinsic growth rate,

M is the saturation level (carrying capacity), and

P_t is the population size at the time t .

One of the first models to incorporate interaction between two species in an ecosystem was Lotka - Volterra named after an American biophysist Alfred Lotka and an Italian mathematician Vito Volterra. They assumed that predator prey is the only determinant of population dynamics and came up with the following model.

$$\begin{aligned} \frac{dP_t}{dt} &= \kappa P_t - \alpha P_t N_t \\ \frac{dN_t}{dt} &= -\tau N_t + \beta P_t N_t \end{aligned} \quad (2.3)$$

where

κ is the natural growth rate of prey in the absence of predator,

τ is the natural death rate of predator in the absence of prey,

α is the function of interaction between prey and predator,

β is the function of interaction between predator and prey

N_t is the population of predators at a time t , and

P_t is the population of prey at time t .

According to Griensen [15] analysis of Volterra model is insightful but has no intra specific competition, i.e. natural resource has no diminishing returns. He further asserts that even two species models may not give realistic results like those of single species. Several variations of Verhulst logistic model have been modified to model population growth, which is useful in resource management.

Oksendal and Lungu [35] developed population growth modeled by introducing randomness in their differential equation via additional noise term. Genetic drift has strong impact on the dynamics of small populations.

However, the effects of genetic stochasticity on small wildlife population is still challenging, important and open problem in both Mathematical Biology and Ecology [14].

Chapter 3

Classic Models

3.1 Malthusian Equation

From the Malthusian equation

$$\frac{dP_t}{dt} = \lambda P_t,$$

where

λ is the intrinsic growth rate,

P_t is the population at time t ,

it implies

$$\lambda = \frac{1}{P_t} \left(\frac{dP_t}{dt} \right) \quad (3.1)$$

Integrating equation (3.1) with respect to t , we obtain

$$\int \left(\lambda = \frac{1}{P_t} \frac{dP_t}{dt} \right) dt = \lambda \int dt \quad (3.2)$$

Let

$$d\theta = \frac{dP_t}{P_t},$$

then equation (3.2) becomes

$$\int \theta^{-1} d\theta = \int \lambda dt$$

this gives

$$\ln |\theta| + \eta_1 = \lambda t + \eta_2$$

where η_1 and η_2 are constants of integration. Making $\ln |\theta|$ the subject of the formula, we obtain

$$\ln |\theta| = \lambda t + (\eta_1 + \eta_2) = \lambda t + \omega$$

Since $P_t > 0$, we have $\ln P_t = \lambda t + \omega$

By taking exponentials of both sides, we obtain

$$P_t = e^{\ln P_t} = e^{\lambda t} e^{\omega}$$

Since ω is an arbitrary constant, we write $P_0 = e^\omega$ so that

$$P_t = P_0 e^{\lambda t} \quad (3.3)$$

From equation (3.3) at $t = 0$, we obtain P_0 and trace as under

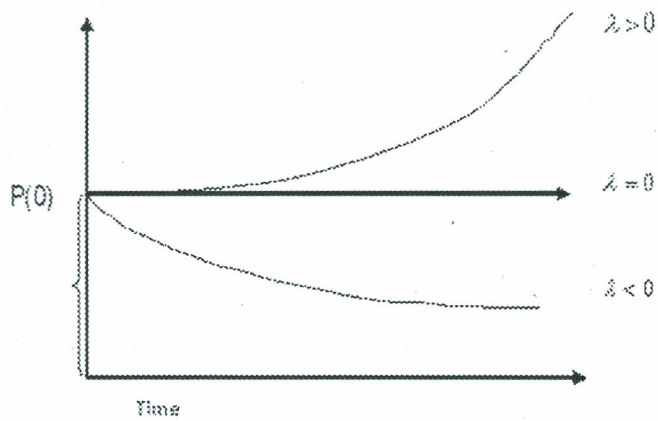


Figure 3.1.1: Exponential Curves

3.2 Verhulst Logistic Model

Consider the population P_t with every member of the population P_t viewing $(P_t - 1)$ as a competitor. Therefore the maximum possible encounter is $(P_t)(P_t - 1)$. Since all interactions are not really for resources, it is only logical to take some small multiples of $(P_t)(P_t - 1)$ say

$$\frac{\nu}{\rho}(P_t)(P_t - 1) \quad (3.4)$$

Combining rate of change equation in Malthus single species and competition in the above expression (3.4) we obtain

$$\frac{dP_t}{dt} = \lambda P_t - \frac{\nu}{\rho}(P_t)(P_t - 1) \quad (3.5)$$

This on expansion gives

$$\begin{aligned} \frac{dP_t}{dt} &= \lambda P_t - \frac{\nu}{\rho} P_t^2 + \frac{\nu}{\rho} P_t \\ &= \left(\lambda + \frac{\nu}{\rho} \right) P_t - \frac{\nu}{\rho} P_t^2 \end{aligned} \quad (3.6)$$

Let $\left(\lambda + \frac{\nu}{\rho} \right) = \mu$ and $\frac{\nu}{\rho} = \alpha$ then

$$\frac{dP_t}{dt} = \mu P_t - \alpha P_t^2 \quad (3.7)$$

Equation (3.7) is the model equation.

Assuming growth rate of roans follows the Verhulst Logistic Model. Then from equation (3.7)

$$\frac{dP_t}{dt} = \mu P_t - \alpha P_t^2$$

If we assume $\mu = \lambda$ and $\alpha = \frac{\lambda}{M}$, then on substitution we obtain

$$\frac{dP_t}{dt} = \lambda P_t - \frac{\lambda}{M} P_t^2 \quad (3.8)$$

where

λ is the intrinsic growth rate coefficient,

P_t is the population size at time t ,

t is time, and

M is the maximum sustainable population limit.

Equation (3.8) may be written as

$$\frac{dP_t}{dt} = \lambda P_t \left(1 - \frac{P_t}{M} \right) \quad (3.9)$$

The solution of equation (3.9) is given by

$$P_t = \frac{MP_0}{(M - P_0)e^{-\lambda t} + P_0} \quad : P(0) = P_0 \quad (3.10)$$

3.2.1 Stability Analysis For Verhulst Logistic Models

The equilibrium is stable if in displacing the system from its equilibrium position by infinitesimal amount and giving each one some small initial velocity [48]. The displacement of different points of the equilibrium remains throughout the course of motion contained within small-prescribed limits. It is important to note that this definition only fits steady state solutions and may not be applicable to chaotic systems as those in turbulent models.

Stability analysis in mathematical modeling implies testing the modeled equation for steadiness (constancy) to disregard those that fail the test and to subject those models that survive for further test.

The major methods for stability analysis are amplification (Neumann) stability analysis, which is based on normal modes and often Fourier superposition. It looks at decay, may be implemented using standard linear algebra, and generally applies to linear systems but apply to nonlinear systems through linearization. The second method is the energy method used in control theory. It looks at variation of functions and normally measures motion amplitude. Consider a general autonomous vector field

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n \quad (3.11)$$

An equilibrium solution of (3.11) is a point $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) = 0$ is a solution that does not change with time, but considering a non-autonomous vector field

$\dot{x} = f(x, t) \quad x \in \mathbb{R}^n$ and in freezing time, t and looking at equilibria of frozen time, vector field (always in fluid mechanics with vector field interpreted as velocity field) the instantaneous fixed points are given by $f(x, t) = 0$

If we find a point, (\bar{x}, \bar{t}) such that $f(x, t) = 0$ and $D_x f(\bar{x}, \bar{t}) \neq 0$ then by implicit function theorem, we find a function $\bar{x}(t)$ with $\bar{x}(\bar{t}) = \bar{x}$ such that $f(\bar{x}(t), t) = 0$ for some small interval \bar{t} . If we let $\bar{x}(t)$ be any solution of equation (3.11) then, $\bar{x}(t)$ is stable if the solution starting close to $\bar{x}(t)$ at a given time remain close to $\bar{x}(t)$ for all times. It is asymptotically stable if nearby solutions not only stay close but also converge to $x(t)$ as $t \rightarrow \infty$.

$\bar{x}(t)$ is said to be Lyapunov stable if given $\varepsilon > 0$, then there exist a $\delta = \delta(\varepsilon) > 0$ such that for any other solution $y(t)$ of equation (3.11) satisfying $|\bar{x}(t_0) - y(t_0)| < \delta$ where $|\cdot|$ is a norm on \mathbb{R}^n . Then, $|\bar{x}(t) - y(t)| < \varepsilon$ for $t > 0$, $t_0 \in \mathbb{R}$ [48].

We remark that a solution which is not stable is unstable. $\bar{x}(t)$ is said to be asymptotically stable if it is Lyapunov stable and for any other solution $y(t)$ of equation (3.11) there exist a constant $b > 0$ such that if $|\bar{x}(t_0) - y(t_0)| < b$

$$\lim_{t \rightarrow \infty} |\bar{x}(t) - y(t)| = 0$$

And considering a differential equation of the form

$$\frac{dP_t}{dt} = f(t, P_t) \quad (3.12)$$

If we let P_t be any solution of equation (3.12), then P_t is Lyapunov stable if the solution starting close to P_t at any given time remain close to P_t for all later times. It is asymptotically stable if nearby solutions not only stay close but also converge to P_t as $t \rightarrow \infty$. An equilibrium solution of equation(3.12) is a point $P_t \in \mathbb{R}$ such that $f'(P_t) = 0$

Example 1.

$$\frac{dP_t}{dt} = \lambda \left(1 - \frac{P_t}{M} \right) P_t \quad (3.13)$$

We obtain the equilibrium levels by setting

$$\frac{dP_t}{dt} = 0$$

This gives $\lambda \left(1 - \frac{P_t}{M} \right) P_t = 0$ and solving for P_t , we obtain the solution $P_t = 0$ and $P_t = M$. An equilibrium level R for the differential equation $\frac{dP_t}{dt} = fP_t$ is stable if there exist a neighborhood N of R with the property that whenever $P_0 > N$ then the solution P_t with the initial condition $P_t = P_0$

1. Is finite $\forall t > t_0$

2. Has

$$\lim_{t \rightarrow \infty} P_t = \mathbb{R}, \quad N \text{ is the neighborhood of stability.}$$

Figure (3.3.1) below illustrates the notion of equilibrium level R and (a, b) is the neighborhood of \mathbb{R} . If the population graph P_t finds itself between the lines a or b i.e. $a < P_t < b$ then $P_t \rightarrow \mathbb{R}$ as $t \rightarrow \infty$

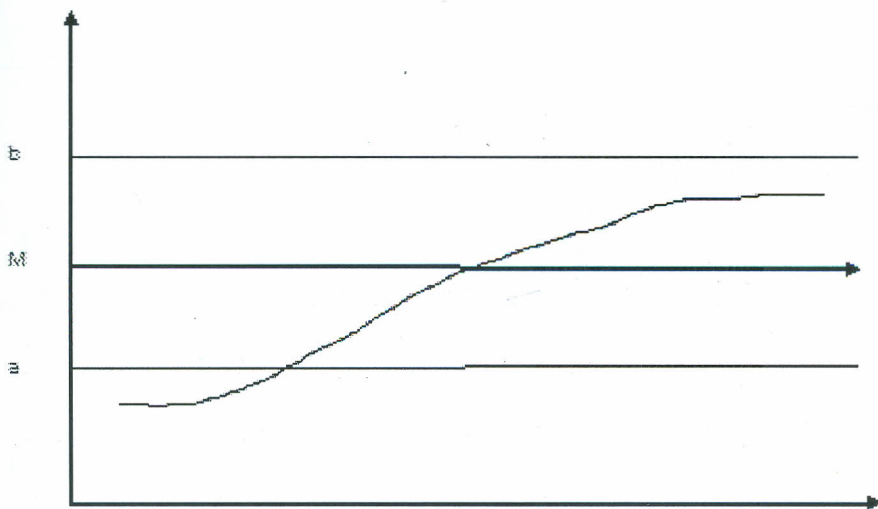


Figure 3.2.1: Population curve attracted to equilibrium R

We now carry out the stability analysis for the Verhulst logistic differential equation

$$\frac{dP_t}{dt} = fP_t = \lambda P_t \left(1 - \frac{P_t}{M} \right)$$

We observe that 0 and M are the equilibrium levels of the equation and letting P_0 be the initial value corresponding to t_0 , the solution to the differential equation becomes

$$P_t = \frac{MP_0e^{-\lambda t}}{(M - P_0) + P_0e^{\lambda(t-t_0)}} \quad : P_0 \neq M \quad (3.14)$$

$$\Rightarrow (M - P_0) + P_0e^{\lambda(t_*-t_0)} = 0$$

whenever t_* is a point of jump discontinuity with $t_* > t_0$

$$e^{\lambda(t_*-t_0)} = \left(\frac{P_0 - M}{P_0} \right)$$

Since $P_0 > M$ and $P_0 \neq 0$ the RHS > 0 therefore a positive logarithm (raising to the log).

$$\lambda(t_* - t_0) = \ln \left(\frac{P_0 - M}{P_0} \right) > 0$$

$$t_* - t_0 = \frac{1}{\lambda} \ln \left(\frac{P_0 - M}{P_0} \right) > 0 \quad (3.15)$$

When $t_0 = 0$ then

$$t_* = \frac{1}{\lambda} \ln \left(\frac{P_0 - M}{P_0} \right) > 0$$

Now solving for t we obtain a value say $t_* > t_0$ and if we let $P_0 > 0, P_0 \neq M$ then equation (3.14) is the form of the solution. We look for singularity of the denominator by attempting to solve this algebraically and this leads to equation (3.15). However if $P_0 > 0, P_0 > M$ then $\left(\frac{P_0 - M}{P_0} \right) < 0$ has no algorithm and therefore equation (3.15) has no solution for real t . However since $\left(\frac{P_0 - M}{P_0} \right) < 1$, $\ln \left(\frac{P_0 - M}{P_0} \right) < 0$ and since $\lambda > 0$ the solution of t_* in equation (3.15) will be such that $t_* < t_0$.

Given our main concern is time values after initial condition and finding singularities fail. Thus Lyapunov stability holds for any solution corresponding to the initial condition P_0 where $P_0 > 0, P_0 > M$ since asymptotic stability holds in relation to the equilibrium M . The set of all positive numbers forms neighborhood of stability of M and the point M is the stable equilibrium level and the neighborhood for stability is $(0, \infty)$ [48]

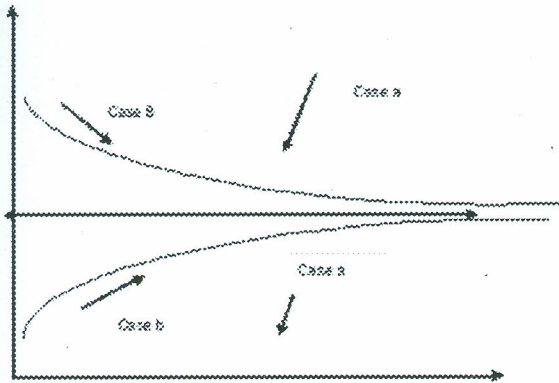


Figure 3.2.2: Two types of solutions to the Logistic differential equation.

Case (i) Equilibrium Solution $P_t = 0, P_t = M$

Case ii) Equilibrium solution with +ve initial value $P_0 > 0$

The logistic growth equation has proved useful in population ecology despite simplified assumptions in its derivative assumption. Verhulst logistic equation has been used to describe the population growth model for Peruvian anchovies. Morgan [31] also used the same equation to describe the herding behavior of African elephants.

Chapter 4

Formulation of Model Equation

4.1 Stochastic Processes

A dynamical system is a mathematical structure used to model the deterministic evolution of some physical phenomena (system) in time. Ordinary differential equations (ODE) normally interpreted as describing evolution in time and hence deterministic dynamical system. Dynamical systems are deterministic since their future is (in principle) completely predictable from knowledge of present state. However, for Pivato if some intrinsic randomness in the system which makes perfect prediction of the future impossible but strong trends or correlation exists, the mathematical structure used to model

this phenomenon is stochastic process [37]. Stochastic process consist of space, time and probability measure

Definition 4.1. If Ω is a given set, then a σ algebra F on Ω is a family F of subset of Ω with the following properties.

(i) $\Phi \in F$

(ii) $f \in F \Rightarrow f^c \in F$

where $f^c = \Omega/F$ is the compliment of F in Ω

(iii)

$$A_1, A_2, \dots \in F \Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$$

The pair (Ω, F) is called a measurable space.

A probability measure P on a measurable space (Ω, F) is a function $P : F \rightarrow [0, 1]$ such that

(a) $P(\Phi) = 0, P(\Omega) = 1$

(b) If $A_1, A_2, \dots \in F$ and

$$\left(A_i \right)_{i=1}^{\infty} \text{ is a disjoint i.e. } \left(A_i \cap A_j = \Phi \quad : i \neq j \right)$$

then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The triple (Ω, F, P) is called a probability space. It is called a complete probability space if F contains all the subsets of G of Ω with P outer measure zero.

$$P^*(G) = \inf\{P(F) : f \in F, G \subset F\} = 0$$

Given any family μ of subsets of Ω there is a smallest σ algebra H_μ containing μ namely

$$H_\mu = \bigcap \{H : H \text{ } \sigma \text{ algebra of } \Omega, \mu \subset H\}$$

Let (Ω, F, P) denote a complete given probability space, then a random variable X is F measurable function $X : \Omega \rightarrow \mathbb{R}^n$. Every random variable induces probability measure μ_x on \mathbb{R}^n defined by $\mu_x(B) = P(x^{-1}(B))$, μ_x is the distribution of X .

If

$$\int_{\Omega} |X(\omega)| dP(\omega) < \infty$$

then

$$E[X] = \int_{\Omega} X(\omega) dP(\omega) = \int_{R^n} x d\mu_x$$

where x is called the expectation of $X(w.r.t.P)$

Definition 4.2. A stochastic process is a parameterized collection of random variables $\{X_t\}_{t \in T}$ and defined on probability space (Ω, F, P) and assuming values in \mathbb{R}^n [50].

The parameter space T is usually half line $[0, \infty)$ but may belong to $[a, b]$ the non-negative integers and even subsets of R^n for $n \geq 1$ such that for each $t \in T$ fixed we have a random variable $W \rightarrow X_t(\omega) : \omega \in \Omega$ and on fixing $\omega \in \Omega, t \in T$ which is called the path of X_t

For clarity $X_t \equiv X(t)$.

A stochastic process $X = \{X(t), t \in T\}$ is a collection of random variables. For each T in the index set $T, X(t)$ is a random variable with t interpreted as time and X_t the state of the process at a time t [38]. If we let X be some set, time for some other set and we let W be some σ -algebra on X the W measurable stochastic process on the state space X over

time T is a probability measure W [37]

Stochastic processes are sequences of events governed by probabilistic laws. These systems occupy one state at a given time and could make transition probabilities from one state to another. The set X of possible status may be finite or infinite depending on application. X consist of discrete elements X_i for $i = 0, 1, 2, \dots$ with element X_i being possible states of the systems at any time t .

The probability $P_{i,j}(t)$ of the system making transition from the state i to j in the interval time t is the conditional probability defined as

$$P_{i,j}(t) = \Pr\{X_{t_0+1}/X_{t_0} = X_i\} \quad (4.1)$$

where X_{t_0} is the state of the state of the system at the time t_0 . The index set T is a countable set and X discrete time stochastic process or continuous time stochastic if it forms a continuum.

Definition 4.3. A discrete time stochastic process is the probability measure on $(X^+, \otimes_n \in T)$, [31]. Discrete time stochastic processes are ranked in increasing order of complexity.

This hierarchy follows either Bernoulli or Markov processes. Discrete time processes can be demonstrated by random walks with probability p of a particle moving to the right and probability $[(p - 1) = q]$ of particle moving to the left.

Let $P_{i,j}$ be the transition probability then

$$P_{i,j+1} = P = 1 - P_{i,j-1} : i = \pm 1, \pm 2, \pm 3, \dots$$

suppose for arbitrary time i, x in a random variable X_i takes $p = 1, q = -1$ and X_i are independent and identically distributed (iid) with identity function.

$$\rho\delta(x - 1) - (1 - q)d(x + 1)$$

$$E[X] = 2p - 1$$

and

$$Var[X] = 4p(1 - p)$$

If the n^{th} partial sum of the random variable

$$Y_n = X_1 + X_2 + \dots + X_n = \sum_{i=1}^n X_i$$

Then the sequence for the random variable

$$\{Y_1, Y_2, \dots, Y_n\}$$

is the random walk with the probability distribution

$$E[Y_n] = n(2p - 1)$$

and

$$Var[Y_n] = 4npq$$

at stage n . If we let μ and σ^2 be the mean and variance respectively then

$$E[Y_n] = n\mu$$

and

$$Var[Y_n] = n\sigma^2$$

Definition 4.4. Let X be some set and time t be some open set and closed interval in \mathbb{R} representing an interval of time and \mathcal{W} be some σ algebra, then W be some measurable continuous time stochastic process on state space X over time interval T is the probability measure W Continuous time stochastic process $\{X(t), t \geq 0\}$ has independent increments if

$\forall t_0 < t_1 < t_n$ the random variables

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$$

are independent.

They may make stationary increments if $X(t+s) - X(t)$ has distribution values $\forall t$ i.e. the distribution only depends on s .

This implies that for n time points the random variables set

$$\{X(t_1), X(t_2), \dots, X(t_n)\}$$

and

$$\{X(t_1 + s), X(t_2 + s), \dots, X(t_n + s)\}$$

has the same joint probability distribution thus

$$E[E(t)] = E[X(t+s)]$$

Markov process is a continuous time $X = \{X(t), t \geq 0\}$ with the

$$\Pr\{X(t) \leq x | X(\mu), \mu \in [0, s] = \Pr\{X(t) \leq x | X(s)\}$$

Markov processes are stochastic processes for which all its future knowledge is summarized in current value. Examples of these processes are Brownian motion, stable processes, Poisson processes and even Levy processes. We can therefore ascertain that stochastic processes are variable with both the expected variable term (drift term) and random term (diffusion term).

The drift-coefficient term, models dominant actions while diffusion-coefficients represents randomness along the dominant curve. Roan antelope population growth varies in random number and represents stochastic process.

4.2 Brownian Motion and Stochastic Differential Equations

An irregular movement of pollen grains suspended in water as was observed by a botanist Robert Brown in 1828 has a wide range of application. Nobert Weiner came up with a concise and rigorous mathematical definition of Brownian motion, sometimes called Weiner Process.

Definition 4.5. A Brownian motion or Wiener process is a stochastic process $\xi(t) \geq 0$ satisfying

(i) $\xi(0) = 0$

(ii) For any $0 \leq t_0 < t_1 \dots < t_n$ the random variables $\xi(t_k), -x(t_k) (1 \leq k \leq n)$ are independent

(iii) If $0 \leq s \leq t, x(t) - x(s)$ is normally distributed with

$$E(P(t) - P(s)) = (t - s)\mu E(\xi(t) - \xi(s)^2) = (t - s)\sigma^2,$$

where μ and σ are constants, $\sigma \neq 0$

If $\xi(t)$ is a Brownian motion, then μ is the drift and σ^2 is the variance. Brownian motion can be a Wiener process $dW = \varepsilon\sqrt{dt}$: ε is a random drawn from standard normalized if $\mu = 0$ and $\sigma^2 = 1$ any continuous time process with stationary independent increments and can be proved to be Brownian motion. Brownian motions are used in models that resemble random movements of particles. A (μ, σ) Brownian motion $\xi = \{\xi(t), t \geq 0\}$ can be expressed as a Wiener process i.e

$$\xi(t) = \mu t + \sigma W_t$$

and a normal variable with mean of zero and a variance of one. The values of dW for any two intervals are independent such that small infinite change can be written as $\Delta W_t = \xi\sqrt{\Delta t}$ adding up each of those intervals, we obtain

$$W_t - W(0) = \lim_{t \rightarrow 0} \left\{ \sum_{i=1}^n \varepsilon_i \sqrt{\Delta t} \right\}$$

One dimensional Weiner process has $\xi(t)$ determined by the stochastic differential equation(SDE) of the form of

$$d\xi(t) = \mu dt + \sigma dW_t \quad : P(0) = P_t, \quad (4.2)$$

where μ (drift rate) and σ standard deviation.

Thus $d\xi(t)$ is the sum of the deterministic term dt and the stochastic term (dW_t) and in the short term interval $[t_{i-1}, t]$ and the increase may be given by

$$\xi_i(t) - \xi_{i-1}(t) = \mu \int_{t_{i-1}}^t dt + \sigma \int_{t_{i-1}}^t dW_t \quad (4.3)$$

With a general solution of the form

$$\xi(t) = \xi_{i-1}(t_{i-1}) + \mu(t_i - t_{i-1}) + \sigma(W(t_i) - W(t_{i-1})) \quad (4.4)$$

and in particular if the interval is $[0,1]$ the equation (4.3)

becomes

$$\xi(t) = \xi_0 + \mu \int_0^t dt + \sigma \int_0^t dW_t \quad (4.5)$$

whose solution is

$$\xi_t = \xi_0 + \mu t + \sigma W_t \quad (4.6)$$

with $\xi(0) = 0$ and $\lambda W(0) = 0$

A generalized Weiner process with non-constant coefficient

$$d\xi = \mu(\xi, t)dt + \sigma(\xi, t)dW_t \quad (4.7)$$

where $\mu(\xi, t)$ and $\sigma(\xi, t)$ are functions of variable ξ and time t is called Ito's process if it solves the equation

$$\xi(t) = \xi_0 + \int_0^t \mu(\xi(t), t)dt + \int_0^t \sigma(\xi(t), t)dW_t : t \geq 0 \quad (4.8)$$

where ξ_0 is the initial value, $\mu(\xi(t), t)$ is the drift term and $\sigma(\xi(t), t)$ is the diffusion term. A special type of Ito's with linear coefficient is the geometric Brownian motion (gBm) and has the stochastic differential equation of the form

$$d\xi(t) = \mu\xi(t)dt + \sigma\xi(t)dW_t \quad : \mu > 0, \sigma > 0, \quad (4.9)$$

where μ is the mean growth rate and σ is the rate of diffusion. Equation (4.9) can be expressed as a growth function

$$\frac{d\xi(t)}{\xi(t)} = \mu dt + \sigma dW_t, \quad \xi(0) = \xi_t \quad (4.10)$$

over infinitely short time interval $(t, t + \Delta t)$.

Solutions to equation (4.10) can not be obtained from standard Reinman Calculus formula for total derivative. If we let $f(x, t)$ be a continuous function with $(x, t) \in \mathbb{R} \times [0, \infty)$ together with its derivatives f_t, f_x, f_{xx} then the process $f(\xi(t), t)$ has the SDE (4.11). Ito achieved a rigorous treatment for integrating such Weiner like differential equation, thus Ito calculus, [13].

The solution to equation (4.10) is the stochastic differential equation

$$df(\xi(t), t) = [f_t(\xi(t), t) + f_x(\xi(t), t)\mu(t) + \frac{1}{2}(\xi(t), t)b^2(t)]dt + f_x(\xi(t), t)\sigma(t)dW_t \quad (4.11)$$

This is called Ito's formula. It is noticeable that if $W(t)$ were continuously differentiable in t then by Reinman calculus the term $\frac{1}{2}f_{xx}b^2dt$ would not appear.

Proof. This will be divided into several steps.

Step 1

For any $m \geq 2$

$$d(W(t))^m = m(w(t))^{m-1} + \frac{1}{2}m(m-1)(w(t))^{m-2}dt \quad (4.12)$$

Theorem 4.6. *If*

$$d\xi(t) = \mu_i(t)dt + \sigma_i(t)dW_t \quad i = 1, 2$$

then

$$d(\xi_1(t)\xi_2(t)) = \xi_1(t)d\xi_2(t) + \xi_2(t)d\xi_1(t) + \sigma_1(t)\sigma_2(t) \quad (4.13)$$

The integrated form of equation(4.13) for any $0 \leq t_1 < t_2 \leq T$

is

$$\begin{aligned}
\xi_1(t_2)\xi_2(t_2) - \xi_1(t_1)\xi_2(t_1) &= \int_{t_1}^{t_2} \xi_1(t)\mu_2(t)dt + \\
&\int_{t_1}^{t_2} \xi_1(t)\sigma_2(t)dW_t + \\
&\int_{t_1}^{t_2} \xi_2(t)\mu_1(t)dt + \\
&\int_{t_1}^{t_2} \xi_2(t)\sigma_1(t)dW_t + \\
&\int_{t_1}^{t_2} \sigma_1(t)\sigma_2(t)dt \quad (4.14)
\end{aligned}$$

and by linearity of the stochastic differentials we obtain

$$dQ(w(t)) = Q'(w(t))dW(t) + \frac{1}{2}Q''(w(t))dt \quad (4.15)$$

for any polynomial Q

Step 2

Let $G(x, t) = Q(x)g(t)$ where $Q(x)$ a polynomial and $g(t)$ is continuously differentiable for $t \geq 0$ By Theorem (4.6) and

equation (4.15), we have

$$\begin{aligned}dG(w(t)) &= f(w(t))dg(t) + g(t)df(w(t)) \\ &= \left[f(w(t))g'(t) + \frac{1}{2}g(t)f''(w(t)) \right] dt + \\ &\quad g(t)f'(w(t))dW(t) \\ &0 \leq t_1 < t_2 \leq T\end{aligned}$$

and

$$\begin{aligned}G(w(t_2), t_2) - G(w(t_1), t_1) &= \int_{t_1}^{t_2} \left[G_t(w(t), t) + \right. \\ &\quad \left. \frac{1}{2}G_{xx}(w(t), t) \right] dt + \\ &\quad \int_{t_1}^{t_2} G_x(w(t), t)dW_t\end{aligned}\tag{4.16}$$

Step 3

Formula (4.16) remains valid if

$$G(x, t) = \sum_{i=1}^m f_i(x)g_i(t)$$

where $f_i(x)$ are polynomials and $g_i(t)$ are continuously differentiable.

Now letting $G_n(x, t)$ be a polynomial in x and t such that

$$\begin{aligned} G_n(x, t) &\rightarrow f(x, t) & \frac{\partial}{\partial t} G_n(x, t) &\rightarrow f_t(x, t) \\ \frac{\partial}{\partial x} G_n(x, t) &\rightarrow f_x(x, t) & \frac{\partial^2}{\partial x^2} G_n(x, t) &\rightarrow f_{xx}(x, t) \end{aligned}$$

Uniformly on compact subset of $(x, t) \in \mathbb{R} \times [0, \infty)$ We have,

$$\begin{aligned} G_n(w(t_2), t_2) - G_n(w(t_1), t_1) &= \int_{t_1}^{t_2} \left[\frac{\partial}{\partial t} G_n(w(t), t) + \right. \\ &\quad \left. \frac{1}{2} \frac{\partial^2}{\partial x^2} G_n(w(t), t) \right] dt + \\ &\quad \int_{t_1}^{t_2} \frac{\partial}{\partial x} G_n(w(t), t) dW_t \end{aligned} \tag{4.17}$$

It is clear that

$$\int_{t_1}^{t_2} \left[\frac{\partial}{\partial t} G_n(w(t), t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} G_n(w(t), t) \right] dt \rightarrow$$

$$\int_{t_1}^{t_2} \left[f_t(w(t), t) - \frac{1}{2} f_{xx}(w(t), t) \right] dt$$

a.s

$$\int_{t_1}^{t_2} \left| \frac{\partial}{\partial x} G_n(w(t), t) - f_x(w(t), t) \right|^2 dt \rightarrow 0$$

a.s, hence taking $n \rightarrow \infty$ in equation (4.17) we get the relation

$$f(w(t_2), t_2) - f(w(t_1), t_1) = \int_{t_1}^{t_2} \left[f_t(w(t), t) + \frac{1}{2} f_{xx}(w(t), t) \right] dt +$$

$$\int_{t_1}^{t_2} f_x(w(t), t) dW_t$$

(4.18)

Step 4

Formula (4.18) extends to the process

$$\Phi(w(t), t) = f(\xi_1 + \mu_1 t + \sigma_1(w(t), t)),$$

where ξ_1, μ_1, σ_1 are random variables measurable with respect

to f_{t_1} , we have

$$\Phi(w(t_2), t_2) - \Phi(w(t_1), t_1) = \int_{t_1}^{t_2} \left[f_t(\bar{\xi}(t), t) + f_x(\bar{\xi}(t), t)\mu + \frac{1}{2}f_{xx}(\bar{\xi}(t), t)\sigma_1^2 \right] dt + \int_{t_1}^{t_2} f_x(\bar{\xi}(t), t)\sigma dW_t, \quad (4.19)$$

where

$$\bar{\xi}(t) = \xi_1 + \mu_1 t + \sigma_1 w(t)$$

The proof of equation (4.19) is equivalent to that of equation (4.0.18) with changes resulting from the formula (3.9), we have

$$\partial(\bar{\xi}(t))^m = m(\bar{\xi}(t))^{m-1}[\mu_1 dt + \sigma_1 dw(t)] + \frac{1}{2}m(m-1)(\bar{\xi}(t))^{m-2}\sigma_1^2 dt$$

This replaces equation (4.12)

Step 5

If μ_1 and $\sigma(t)$ are step functions, then

$$\begin{aligned} f(\xi(t_2), t_2) - f(\xi(t_1), t_1) &= \int_{t_1}^{t_2} \left[f_t(\xi(t), t) + f_x(\xi(t), t)\mu(t) \right. \\ &\quad \left. + \frac{1}{2}f_{xx}(\xi(t), t)\sigma^2(t) \right] dt \\ &\quad + \int_{t_1}^{t_2} f_x(\xi(t), t)\sigma(t)dW_t \end{aligned} \tag{4.20}$$

Indeed, by denoting I_1, \dots, I_n the successive interval in $[t_1, t_2]$ in which μ, σ are constants. If we apply equation (4.19) with t_1, t_2 replaced by the end points of I_i and sum over i the formula (4.20) follows

Step 6

Let μ_1, σ_1 be a non-anticipative step function such that

$$\int_0^T |\mu_i(t) - \mu(t)| dt \rightarrow 0 \tag{4.21}$$

a.s

$$\int_0^t |\sigma_i(t) - \sigma(t)|^2 dt \rightarrow 0 \tag{4.22}$$

and letting

$$\xi_i(t) = \xi(0) + \int_0^t \mu_i(s) ds + \int_0^t \sigma_i(s) dW_s$$

Then $\sup |\xi_i(t) - \xi(t)| \rightarrow 0$ hence the subsequence $\{i'\}$

$$\sup_{0 < t < T} |\xi_{i'}(t) - \xi(t)| \rightarrow 0 \quad a.s \quad \text{if } i = i' \rightarrow \infty \quad (4.23)$$

This and equation (4.22) imply that

$$\int_0^T |f_x(\xi_i(t), t, \sigma_i(t) - f_x(\xi(t), t)\sigma(t)|^2 dt \rightarrow 0$$

If $i = i' \rightarrow \infty$ it follows that

$$\int_{t_1}^{t_2} f_x(\xi_i(t), t)\sigma_i dW_t \rightarrow \int_{t_1}^{t_2} f_x(\xi(t), t)\sigma(t) dW_t$$

If $i = i' \rightarrow \infty$ then from equations (4.21 - 4.23)

$$\int_{t_1}^{t_2} \left[f_t(\xi_i(t), t) + f_x(\xi(t), t)\mu_i(t) + \frac{1}{2}f_{xx}(\xi_i(t), t)(\sigma_i(t))^2 \right] dt \rightarrow \int_{t_1}^{t_2} \left[f_t(\xi(t), t) + f_x(\xi(t), t)\mu(t) + \frac{1}{2}f_{xx}(\xi(t), t)\sigma^2(t) \right] dt$$

If $i = i' \rightarrow \infty$ If writing equation (4.0.20) and taking $\mu = \mu_i, \sigma = \sigma_i, \xi = \xi_i, i = i' \rightarrow \infty$ then formula (4.20) follows the general μ, σ and this completes the proof for the theorem. \square

Theorem 4.7. Let $d\xi_i(x) = \mu_i(t)dt + \sigma_i(t)d\xi \quad : 1 \leq i \leq m$ and let $f(x_1, \dots, x_m, t)$ be a continuous function in (x, t) where $x = (x_1, \dots, x_m) \in R^m, t \geq 0$ together with its first t derivative and second x derivative then $f(\xi_1(t), \dots, \xi_m, t)$ stochastic differential given by [13]

$$df(X(t), t) = \left[f_t(X(t), t) + \sum_{i=1}^m f_{x_i}(X(t), t)\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^m f_{x_i x_j}(X(t), t)\sigma_i(t)\sigma_j(t) \right] dt + \sum_{i=1}^m f_{x_i}(X(t), t)\sigma_i(t)dW_t \tag{4.24}$$

where $X(t) = (\xi_1(t), \dots, \xi_m(t))$

Equation (4.24) is the Ito's formula. From theorem (4.7), equation (4.24) the geometric Brownian motion (gBm) is given by

$$d\xi(t) = \mu\xi(t)dt + \sigma\xi(t)dW_t, \tag{4.25}$$

where $\mu\xi(t)dt$ is the drift and $\sigma\xi(t)dW_t$ is the diffusion term $dW_t = \varepsilon\sqrt{dt}$

Dividing both sides of equation (4.25) by $\xi(t)$, we obtain

$$\frac{d\xi(t)}{\xi(t)} = \mu dt + \sigma dW_t \quad (4.26)$$

and in order to get the strong solution of equation (4.26) we let $f(\xi(t), t)$ be a function of ξ and t twice differentiable in ξ and once in t such that

$$f(\xi(t), t) = \ln \xi(t)$$

Note

$$\frac{d\xi(t)}{\xi(t)} = \mu dt + \sigma dW_t$$

suggests the nature of $f(\xi(t), t)$ differentiating $f(\xi(t), t)$ twice with respect to ξ and once with respect to t gives

$$\frac{d(\xi(t), t)}{d\xi} = \frac{1}{\xi} \quad \frac{\partial^2(\xi(t), t)}{d\xi^2} = \frac{-1}{\xi^2} \quad \frac{\partial(\xi(t), t)}{\partial t} = 0$$

and by equation (4.11) we have integral in the form

$$df(\xi(t), t) = d(\ln \xi(t)) = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \varepsilon \sqrt{t} \quad (4.27)$$

Equation (4.27) follows a generalized Weiner process with

the drift rate $\left(\mu - \frac{\sigma^2}{2}\right)$ and diffusion coefficient σ , which are constants. The distribution of this process is given by

$$df(\xi(t), t) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)dt, \sigma\sqrt{dt}\right)$$

or

$$\partial\left(\ln \xi(t) \sim N\left(\mu - \frac{\sigma^2}{2}\right)dt, \sigma\sqrt{dt}\right)$$

whose solution over the interval (t_{i-1}, t_i) is given by

$$\ln \xi(t) = \ln \xi(t_{i-1}) + \left(\mu - \frac{\sigma^2}{2}\right)(t_{i-1}, t_i) + \sigma\xi_i(\sqrt{t_{i-1}, t_i}) \quad (4.28)$$

Moreover, on putting like terms together, we obtain

$$\ln \left(\frac{\xi(t_i)}{\xi(t_{i-1})}\right) = \left(\mu - \frac{\sigma^2}{2}\right)(t_{i-1}, t_i) + \sigma\varepsilon(\sqrt{t_{i-1}, t_i}) \quad (4.29)$$

And in considering the interval $(0, 1)$ then equation (4.27) becomes

$$\ln \xi(t) = \ln \xi_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\varepsilon\sqrt{t} \quad \xi(0) = \xi_0 > 0 \quad (4.30)$$

Thus $\ln \xi(t)$ is normally distributed for any time t with the mean given by $\ln \xi_0 + \left(\mu - \frac{\sigma^2}{2}\right)t$ and variance by $\sigma^2 t$ and the change in logarithm of the population size in the interval $0, 1$

results in

$$\ln \xi(t) - \ln \xi_0 = \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma \varepsilon \sqrt{t} \quad (4.31)$$

with the corresponding distribution given by

$$\ln \xi(t) - \ln \xi_0 \sim N \left(\left(\mu - \frac{\sigma^2}{2} \right) t, \sigma \sqrt{t} \right)$$

From equation (4.31) the strong solution becomes

$$\xi(t) = \xi_0 \exp \left[\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma \varepsilon \sqrt{t} \right] \quad (4.32)$$

which has the log-normal distribution given by

$$\xi(t) \sim \text{log-normal} \left(\xi_0 \exp \mu t, \xi_0 \sqrt{\exp(2\mu t), \exp(\sigma^2 t)^{-1}} \right)$$

such that if $\sigma = 0$ then equation (4.31) becomes

$$\xi(t) = \xi_0 \exp(\mu t)$$

Thus $\xi(t)$ has the exponential growth with the expectation $\xi_0 \exp(\mu t)$ and variance zero.

4.3 Model Equation

In addition to competition for resources and predation. We consider the genetic defect on the population growth rate for the roan antelopes. From the Verhulst logistic growth rate equation (1.1) and adding genetic growth component to the logistic growth model we have

$$dP_t = \lambda P_t \left(1 - \frac{P_t}{M}\right) dt - \Psi(P_t), \quad (4.33)$$

where

λ is the growth ratio

P_t is the population at time t

M is the carrying capacity

$\Psi(P_t)$ is the function of P_t representing genetic defect

Letting $\Psi(P_t) = \gamma$ a constant then equation (4.33) becomes

$$\frac{dP_t}{dt} = \lambda P_t \left(1 - \frac{P_t}{M}\right) - \gamma \quad (4.34)$$

And equating equation (4.34) to zero we obtain

$$\lambda P_t^2 - \lambda M P_t + \gamma M = 0 \quad (4.35)$$

whose solution is given by

$$P_t = \frac{\lambda M \pm \sqrt{(\lambda^2 M^2 - 4\lambda\gamma M)}}{2\lambda} : P(0) = P_0 \quad (4.36)$$

The nature of solution of equation (4.36) depends on the genetic defect γ such that

$\gamma > \frac{\lambda M}{4}$ there is no real valued function implying genetic defect rate leads to extinction,

$\gamma = \frac{\lambda M}{4}$ has unique solution thus absolute growth rate in the absence of genetic defect and

$\gamma < \frac{\lambda M}{4}$ has positive growth rate with genetic defect.

Suppose we have a genetic defect at the rate proportional to P_t and if we let $\Psi = \gamma P_t dt$ then equation (4.33) becomes

$$\frac{dP_t}{dt} = \lambda P_t \left(1 - \frac{P_t}{M}\right) - \gamma P_t \quad (4.37)$$

Integrating equation (4.37) and solving for P_t we obtain the solution

$$P_t = \frac{(\lambda M - \gamma)P_0}{[\lambda(M - P_0) - \gamma]e^{-(\lambda M - \gamma)t} + \lambda P_0} : P(0) = P_0$$

As $t \rightarrow \infty, P_t \rightarrow P_0$ and $t \rightarrow \infty, P_t \rightarrow \frac{(\lambda - \gamma)M}{\lambda}$ with the following steady states

$$P_t = 0, \quad P_t = \frac{(\lambda - \gamma)M}{\lambda}$$

Stochastic models are probabilistic in structure. This helps in solving the effects of uncertainty in ecological models. Hence, analysis of systems with white noise gives better results. If we consider population growth process

$$\frac{1}{P_t} \frac{dP(t)}{dt} = \lambda(M - P_t)$$

adding noise to the continuous growth process above, we obtain

$$\frac{1}{P_t(M - P_t)} \frac{dP_t}{dt} = \lambda dt + \text{noise} \quad (4.38)$$

If noise = $\sigma dW_t = \sigma \varepsilon \sqrt{dt}, \varepsilon \sim N(0, 1)$, equation (4.38) can be written as

$$\frac{1}{P_t(M - P_t)} \frac{dP_t}{dt} = \lambda dt + \sigma dW_t, \quad M \neq P_t \quad (4.39)$$

On making dP_t the subject of the formula, we obtain the logistic stochastic differential equation

$$dP_t = \lambda P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t \quad (4.40)$$

with the distribution

$$\left[dP_t \sim N(\lambda P_t)dt, \sigma P_t(M - P_t)\sqrt{dt} \right]$$

On using the variable

$$Y(t) = \log \left(\frac{P(t)}{|M - P(t)|} \right) \quad M \neq P_t$$

and simplifying equation (4.39) we obtain

$$dY = \lambda Mdt + \sigma M dW_t \quad (4.41)$$

Equation (4.41) is the generalised Weiner process with λMdt as drift and σMdt as variance. Equation (4.41) has the explicit solution

$$Y(t) = Y(0) + \lambda M(t - t_0) + \sigma M W_t, \quad W_0 = 0 \quad (4.42)$$

If we let

$$Y(t) = \log \left(\frac{P_t}{M - P(t)} \right) \text{ and } Y(0) = \left(\frac{P(0)}{M - P(0)} \right)$$

Equation (4.41) becomes

$$\log \left(\frac{P(t)}{M - P(t)} \right) = \log \left(\frac{P(0)}{M - P(0)} \right) + \lambda M(t - t_0) + \sigma MW_t \quad (4.43)$$

and making P_t the subject of the formula we have the Verhulst Logistic Brownian motion

$$P_t = \frac{MP_0}{(M - P_0)e^{-\{\lambda M(t-t_0) + \sigma MW_t\}} + P_0} : P(0) = P_0 \quad (4.44)$$

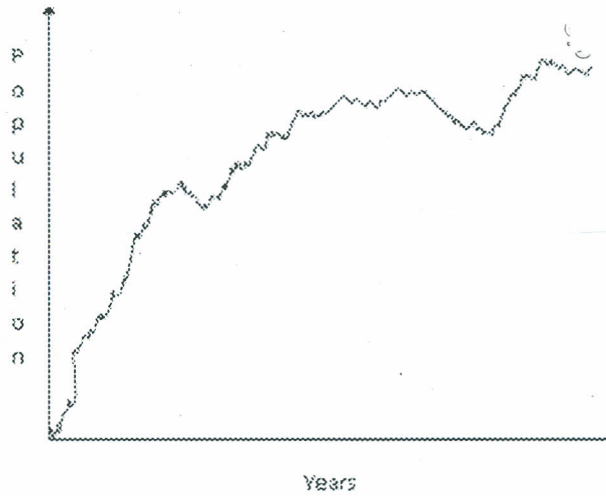


Figure 4.3.1: Trajectory of logistic Brownian motion.

Considering roans resources whose population P_t varies randomly due to natural factors (e.g predation, diseases) according to autonomous diffusion process

$$dP_t = \lambda P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t \quad (4.45)$$

Equation (4.45) is an Ito process called logistic geometric Brownian motion, and can be solved by use of Ito's lemma. Let $F(P_t, t)$ be function of P_t and t be twice differentiable in P_t and once in t , we have

$$dF(P_t, t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial P_t}dP_t + \frac{1}{2} \frac{\partial^2 F}{\partial P_t^2} dP_t^2$$

But

$$dP_t = \lambda P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t$$

Hence $dP_t^2 = \sigma^2 P_t^2(M - P_t)^2 dt$ and by Ito's calculus we obtain

$$\begin{aligned} dF(P_t, t) &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial P_t} \lambda P_t(M - P_t) dt + \\ &\quad \frac{\partial F}{\partial P_t} \sigma P_t(M - P_t) dW_t + \\ &\quad \frac{1}{2} \frac{\partial F}{\partial P_t^2} \sigma^2 P_t^2(M - P_t)^2 dt \end{aligned} \quad (4.46)$$

We can rewrite equation (4.46) in the form

$$\begin{aligned} dF(P_t, t) &= \left\{ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial P_t} \lambda P_t(M - P_t) + \right. \\ &\quad \left. \frac{1}{2} \frac{\partial F}{\partial P_t^2} \sigma^2 P_t^2(M - P_t)^2 \right\} dt + \\ &\quad \frac{\partial F}{\partial P_t} \sigma P_t(M - P_t) dW_t \end{aligned} \quad (4.47)$$

If we use the variable $F = \ln \left(\frac{P_t}{M - P_t} \right)$ then

$$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial P_t} = \frac{M}{P_t(M - P_t)}, \quad \frac{\partial F}{\partial P_t^2} = \frac{2M(P_t - M)}{P_t^2(M - P_t)^2}$$

Substituting this in equation (4.46), we obtain

$$dF(P_t, t) = \left\{ \lambda M - \frac{1}{2} \sigma^2 (M^2 - 2MP_t) \right\} dt + \sigma M dW_t \quad (4.48)$$

Equation (4.48) is similar to to the Brownian motion in equation (4.40). Its solution is got by integration. Thus

$$dF(P_t, t) \sim \lambda M - \frac{1}{2}\sigma^2(M - 2MP_t)dt, \sigma M dW_t$$

It can be solved by Ito calculus. When $\sigma = 0$ then equation (4.48) is a deterministic differential equation given by

$$dF(P_t, t) = \lambda M dt = \left(\frac{M}{P_t(M - P_t)} \right) dP_t$$

and making dP_t the subject of the subject of the formula, we obtain

$$dP_t = \lambda P_t(M - P_t)dt$$

If we let

$$F(P_t, t) = \ln \left(\frac{P_t}{(M - P_t)} \right) \text{ then } dF(P_t, t) = \left(\frac{M}{P_t(M - P_t)} \right) dP_t$$

and rewriting equation (4.45)

$$dP_t = \lambda P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t$$

we obtain

$$\frac{dP_t}{P_t(M - P_t)} = \lambda dt + \sigma dW_t \quad (4.49)$$

But

$$dF(P_t, t)P_t(M - P_t) = MdP_t$$

hence

$$dP_t = \frac{dF(P_t, t)P_t(M - P_t)}{M}$$

and when substituted in equation (4.0.49) we obtain

$$\frac{dF(P_t, t)P_t(M - P_t)}{MP_t(M - P_t)} = \lambda dt + \sigma M dW_t$$

$$dF(P_t, t) = \lambda M dt + \sigma dW_t \quad (4.50)$$

This is a generalised Weiner process with $\lambda M dt$ as the drift and $\sigma M dt$ as the variance. It has the explicit solution

$$F(P_t, t) = F(P_0, 0) + \lambda M_t + \sigma M dW_t$$

which is equivalent to

$$\ln \left(\frac{P_t}{M - P_t} \right) = \ln \left(\frac{P_0}{M - P_0} \right) + \lambda M_t + \sigma M W_t$$

Solving for P_t we obtain

$$P_t = \frac{MP_0}{(M - P_0)e^{-\lambda M_t - \sigma M W_t} + P_0} : P(0) = P_0 \quad (4.51)$$

When $\sigma = 0$ in equation (4.51) we obtain the deterministic logistic differential equation given by

$$P_t = \frac{MP_0}{(M - P_0)e^{-\lambda M_t} + P_0} \quad \text{as } t \rightarrow \infty, e^{-\lambda M_t - \sigma MW_t} \rightarrow 0$$

To take care of fluctuations in the roan antelopes population growth rate due to genetic defect at the rate proportional to $P_t(M - P_t)$ so as to ensure positive population growth rate, we add genetic defect in equation (4.45) to obtain

$$dP_t = (\lambda - \gamma)P_t(M - P_t)dt + \sigma P_t(M - P_t)dW_t \quad (4.52)$$

where

P_t roan antelopes population at time t ,

λ roan antelope growth ratio,

γ genetic defect,

M carrying capacity,

σ diffusion rate and

W_t random variable.

Suppose $F(P_t, t) = F$ is twice differentiable function in P_t and once in t , then by Ito's lemma

$$dF(P_t, t) = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial P_t} dP_t + \frac{1}{2} \frac{\partial^2 F}{\partial P_t^2} dP_t^2$$

which is equivalent to

$$\begin{aligned} dF(P_t, t) = & \left\{ \frac{\partial F}{\partial t} dt + (\lambda - \gamma) P_t (M - P_t) \frac{\partial F}{\partial P_t} + \right. \\ & \left. \frac{1}{2} \sigma^2 P_t^2 (M - P_t)^2 \frac{\partial^2 F}{\partial P_t^2} \right\} dt + \\ & \sigma P_t (M - P_t) \frac{\partial F}{\partial P_t} dW_t \end{aligned} \quad (4.53)$$

Using the variable

$$F(P_t, t) = \ln \left(\frac{P_t}{M - P_t} \right) \quad (4.54)$$

where,

$$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial P_t} = \frac{M}{P_t(M - P_t)}, \quad \frac{\partial^2 F}{\partial P_t^2} = \frac{2M(P_t - M)}{P_t^2(M - P_t)^2}$$

We substitute the above results in equation (4.53) to obtain

$$\begin{aligned} dP_t = & \frac{M}{P_t(M - P_t)} \left[(\lambda - \gamma) P_t (M - P_t) dt + \sigma P_t (M - P_t) dW_t \right] + \\ & \frac{1}{2} \left(\frac{2P_t - M^2}{P_t^2(M - P_t)^2} \right) \sigma^2 P_t^2 (M - P_t)^2 dt \end{aligned}$$

$$\Rightarrow dP_t = M \left\{ (\lambda - \gamma) + \frac{1}{2} \sigma^2 (2P_t - M) \right\} dt + \sigma M dW_t \quad (4.55)$$

with

$$dP_t \sim N \left\{ M(\lambda - \gamma) + \frac{1}{2} \sigma^2 (2P_t - M) dt, \sigma M \sqrt{dt} \right\}$$

On rewriting equation (4.52) as

$$\frac{dP_t}{P_t(M - P_t)} = (\lambda - \gamma) dt + \sigma dW_t \quad (4.56)$$

and using the variable in equation (4.53) we can rewrite equation (4.55) as

$$\partial F(P_t, t) = (\lambda - \gamma) M dt + \sigma dW_t \quad (4.57)$$

Integrating equation (4.57) with respect to t , we obtain

$$F(P_t, t) = F(P_0, 0) + (\lambda - \gamma) M t + \sigma M W_t, \quad (4.58)$$

which on substitution with the variable in equation (4.54), yields

$$\ln \left(\frac{P_t}{M - P_t} \right) = \ln \left(\frac{P_0}{M - P_0} \right) + (\lambda - \gamma) M t + \sigma M W_t \quad (4.59)$$

Solving equation (4.59), we obtain

$$P_t = \frac{MP_0}{(M - P_0)e^{-(\lambda-\gamma)Mt - \sigma MW_t} + P_0} : P(0) = P_0 \quad (4.60)$$

From equation (4.60) when $\lambda = \gamma$, we have

$$P_t = \frac{MP_0}{(M - P_0)e^{-\sigma MW_t} + P_0} : P(0) = P_0 \quad (4.61)$$

Equation (4.61) is a function of random variable W_t only.

This implies that the population may approach extinction.

Chapter 5

Population Viability Analysis

Population viability analysis is the estimation of extinction probabilities and other measures of population performance by analysis that incorporate identifiable threats to population survival into models of the extinction process [14].

For most sexually reproducing species, the ultimate biological extinction is assured whenever the population has declined to a point that it no longer has individuals of both sexes. Extinction is simply defined as the absence of at least one sex. The second option of defining extinction involves assessing the probability of population dropping below user defined minimum viable population-termed as quasi extinction.

Minimum Viable Population (MVP) is the size at which a population has 99 percent probability of persistence for 1000 years,[42]. Extinction could simply be defined as decline in population size. Given the small and stagnated population growth of the roans in the last estimate despite the management policies adopted by the KWS, it is clear that the conservation measures targeting species population increase may be inadequate. Wilson & Hirst ascertains that even small populations of roans can rapidly project positive population growth given favorable conditions,[50]. It is at this point that we want apply the PVA tools to evaluate the probability of extinction and the likely management interventions. Simulations of different scenarios of the Ruma roans' populations were run and their implications discussed.

5.1 Simulations

We have used Vortex version 9.99, which is a stochastic simulation of the extinction process [24]. Vortex is the most appropriate PVA tool to use for this thesis since roans have low fecundity, long lifespan, have change in genetic variation with polygynous breeding system and a local population of approximately less than 50 individuals [24].

We entered biological population parameters of the Ruma roans' in Vortex model. Female roans (heifers) attain maturity at approximately 2 years with 270-320 days gestation period. We therefore set the age of first birth at 3 years. Male roans (bachelors) age of first reproduction was set at 5 years even though they mature slightly earlier approximately 3years but they take time to establish their territories and secure breeding access to females [52].

The maximum reproductive age was set at 12 years for females while that of males was set 10 years. According to Wilson & Hirst, roans have established social structures with a polygynous breeding system [49]. Given that we had six calves with eight adult bulls we estimated, a 75% successful

sire. Roans calve once a year, we set the litter size at 1. Since there were 25 adult cows with only 6 calves, we estimated the proportion of adult females calving annually at 25%. Juvenile mortality is usually as high as (30 – 60%) during the hiding period with a standard deviation of 10 percent due to environmental variation. The birth sex ratio was set at 1:1. Bachelor roans' mortality rates were set 10% slightly higher than their female counter parts (heifers) since they are usually driven out of the herds by adult bulls. Heifers, adult bulls, and adult cows mortality rate was set at 5% which is the default value for large ungulates with a deviation of 3%, [24]

The effect of genetic drift due to inbreeding was included in the simulation model at three different levels. The first simulation represented natural genetic drift due to inbreeding with default lethal equivalent of 3.14, the median value from the study of 40 mammalian species [24]. Secondly we made a more optimistic assumption of 0.09 lethal equivalents (lower quartile). Similarly in the third level we opted to be pessimistic in our assumption 5.62 lethal equivalent (upper quartile).

Neither drought nor diseases related catastrophes were set since none had been recorded in the last two decades. KWS had constructed water points that supplemented water supply during dry seasons. KWS management also set up a veterinary unit that was responsible for preventing and curing wildlife diseases in the park.

Each simulation was run 200 times in a span of 50 years. The carrying capacity was set at 250 individuals. The default carrying capacity is 20 individuals per 1000ha [52]. The initial population size and structure were as at June 2010 from Ruma National Park.

Chapter 6

Results and Conclusions

6.1 Results

It was not possible to model every aspect of potential variation in vortex parameters but instead concentrated on the inbreeding and supplementation. This is to guide the management in their conservation strategy. We set the parameters as in Appendix A with an inbreeding coefficient of 3.14 default lethal equivalents. Environment Variation (EV) in reproduction is left to be concordant. We run the simulations for Lambwe 1 (scenario 1) 50 years with 200 iterations each of 365 days without supplementation.

We noted that inbreeding has high-level impact on population survival. The population started dropping drastically

after the 20th year before going extinct in the 43rd year.

We created scenario two (Lambwe 1 supplemented) with the parameters in Appendix A but incorporated juvenile and adult supplementation. We started our initial supplementation from year 5 and Year 40 as our last year of supplementation.

We noted a decrease in the mean inbreeding coefficients in the supplemented scenario (Lambwe 1 supplemented). We further detected a steady population in almost 50 years as compared to the non-supplemented scenario (Lambwe 1) which showed a drop in the mean survival after only 20 years. Also noted is the fact that over supplementation of both the juvenile and adult roans increased inbreeding in the later years, thus slow rate of population growth.

However, supplementation ensured population survival for a long time. In all the above simulations, we maintained adult death rate at 5% default value for large ungulates.

6.2 Conclusion and Recommendation

Long-term probability of extinction and ensuring high population growth rate are usually the major reasons for intervention by Conservationist groups in supporting the small-threatened species populations. In Ruma, the major factor for consideration is genetic drift caused by inbreeding in controlling juvenile mortality.

Predator control is not advised due to difficulty in implementation, unpredictable consequences to the ecosystem and is in conflict with the general KWS management policies in protected areas. The management is left with the options for controlling juvenile mortality through supplementation.

Supplementation involves importing juveniles from parks like Akagera in Uganda or any other park at intervals of 5 years to curb inbreeding, which is a major threat to population growth in small populations. Severe inbreeding after a decade or so may have deleterious effects.

We therefore recommend that genetic studies be carried out to ascertain the extent to which inbreeding affects population growth. This should be done without causing disruption of the social groups.

We further recommend that juvenile supplementation of two-year-old heifers and three-year-old bachelors. This seems to be a reasonable age for supplementation compared to one year old who are usually vulnerable to predation and other factors relating to increase in their mortality rates.

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