ON NORMS OF ELEMENTARY OPERATORS

BY

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Abstract

The study of elementary operators has been of great interest to many mathematicians for the past two decades. Of special interest has been to determine the norms of these operators. The norm problem for elementary operators involves finding a formula which describes the norm of an elementary operator in terms of its coefficients. The upper estimates of these norms are easy to find but approximating these norms from below has proved to be difficult in general. Several mathematicians have produced known results for special cases on the lower estimates, for example, Mathieu found that for prime C*-algebras, the coefficient is $\frac{2}{3}$, Stacho and Zalar obtained $2(\sqrt{2}-1)$ for standard operator algebras on Hilbert spaces, Cabrera and Rodriguez obtained $\frac{1}{20412}$ for JB*-algebras while Timoney came up with a formula involving the tracial geometric mean to calculate the norm of elementary operators. An operator $T: \mathcal{A} \to \mathcal{A}$ is called an elementary operator if T can be expressed in the form $T(x) = \sum_{i=1}^{n} a_i x b_i, \quad \forall\, x \in \mathcal{A}$ where $\mathcal{A}$ is an algebra and a_i, b_i fixed in $\mathcal{A}$. The norm of an operator T is defined by $\|T\| = \sup\{\|Tx\| : x \in H, \ \|x\| = 1\}$ where H is a Hilbert space. The purpose of this study therefore, has been to determine the lower estimate of the norm of the basic elementary operator on a C*-algebra through tensor products. To do this we needed to have a good background knowledge on functional analysis, general topology, operator theory and C*-algebras by understanding the existing theorems and relevant examples especially on tensor product norms. We used the approach of tensor products in solving our particular problem. We hope that the results obtained shall be useful to applied mathematicians and physicists especially in quantum mechanics.

Chapter 1

INTRODUCTION

The study of elementary operators has been a subject of many papers most of which have been on the norms of elementary operators. They first appeared in a series of notes by Sylvester [7] in the 1880's, in which he computed the eigenvalues of the matrix operators on the $n \times n$-matrices. The term elementary operator was coined by Lumer and Rosenblum in the late 1950's [7]. An operator $T : A \rightarrow A$ is called an elementary operator if T can be expressed in the form $T(x) = \sum_{i=1}^{n} a_i x b_i$, where A is an algebra and a_i, b_i $(1 \leq i \leq n)$ fixed in A. For A, a C*-algebra , one may allow a_i and b_i to be in the multiplier algebra $M(A)$ of A [10, 14].

Properties of elementary operators have been investigated in the past two decades and there are many excellent surveys and expositions of certain aspects.

Elementary operators on C*-algebras were extensively examined by Ara and Mathieu [7]. Curto [7] gave an exhaustive overview of spectral properties of elementary operators, Fialkow [7] comprehensively discussed their structural properties (with an emphasis on Hilbert space aspects and methods), and Bhatia and Rosenthal [7] dealt with their applications to operator equations and linear algebra. Mathieu [11] surveyed some recent topics in the computation of the norm of elementary operators and elementary operators on the Calkin algebra. Through all these studies, it has emerged that for general operators, a full description of their properties is rather intricate since these are often intimately interwoven with the structure of the underlying algebras. Therefore, no general formula describing the norm of an arbitrary elementary operator has been found even for simple algebras such as $B(H)$(the algebra of bounded linear operators on a Hilbert space H). For details see [7, 14, 19, 20, 25, 26].

The first chapter is composed of basic results which are used in the subsequent chapters. Here, we also present terminologies and symbols in addition to some definitions regarding elementary operators.

In chapter two, we investigate tensor products and tensor norms. We look closely at tensor products of vector spaces and functionals, Hilbert spaces, operator spaces, normed spaces and C*-algebras. We also give some results on tensor norms, specifically on projective norm, Haagerup norm, spatial norm and maximal C*-norm. Lastly, we establish the relationship between spatial norm and maximal C*-norm.

In chapter three, we investigate elementary operators and give results on the lower estimates of the norm of the basic of elementary operators.

Finally, in the last chapter we give a summary of our work and recommendations.

1.1 Background Information

We first introduce some essential concepts involving definitions and other notions used in the sequel.

1.2 Algebras, Operators and Functionals

Definition 1.2.1. A **Field** is a set $\mathbf{K}$ together with two operations $(+)$ and $(\cdot)$ for which the following conditions hold:

i. *(Closure)* for all $a, b \in \mathbf{K}$, the sum $a + b$ and the product $a.b$ again belong to $\mathbf{K}$;

ii. *(Associativity)* for all $a, b, c \in \mathbf{K}$, $a+(b+c) = (a+b)+c$ and $a\cdot(b\cdot c) = (a \cdot b) \cdot c$;

iii. *(Commutativity)* for all $a, b \in \mathbf{K}$, $a + b = b + a$ and $a \cdot b = b \cdot a$;

iv. *(Distributive laws)* for all $a, b, c \in \mathbf{K}$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$;

v. *(Existence of an additive identity)* $\exists\, 0 \in \mathbf{K}$ for which $a + 0 = a$ and $0 + a = a$ for all $a \in \mathbf{K}$;

vi. *(Existence of a multiplicative identity)* $\exists\, 1 \in \mathbf{K}$ with $1 \neq 0$ for which $a \cdot 1 = a$ and $1 \cdot a = a$ for all $a \in \mathbf{K}$;

vii. *(Existence of additive inverses)* for each $a \in \mathbf{K}\ \exists\, x \in \mathbf{K}$: $a + x = 0$ and $x + a = 0$, $x = -a$ is the additive inverse of $\mathbf{K}$ (the equation $x + a = 0$ and $a + x = 0$ has a solution $x \in \mathbf{K}$ denoted by $-a$);

viii *(Existence of a multiplicative inverses)* for each $a \in \mathbf{K}$, with $a \neq 0$ the equations $a \cdot x = 1$ and $x \cdot a = 1$ have a solution $x \in \mathbf{K}$, called the multiplicative inverse of a, and denoted by a^{-1}.

Definition 1.2.2. A **vector space** over the field $\mathbf{K}$ is a set X on which two operations are defined, called addition and scalar multiplication, and denoted by $(+)$ and $(\cdot)$ respectively. The operations must satisfy the following conditions;

i. *(Closure)* for all $a \in \mathbf{K}$ and all $u, v \in X$, $u + v$ and the scalar product $a \cdot v$ are uniquely defined and belong to X;

ii. *(Associativity)* for all $a, b \in \mathbf{K}$ and all $u, v, w \in X$, $u + (v + w) = (u + v) + w$ and $a \cdot (b \cdot v) = (a \cdot b) \cdot v$;

iii. *(Commutativity of addition)* for all $u, v \in X$, $u + v = v + u$;

iv. *(Distributive laws)* for all $a, b \in \mathbf{K}$ and all $u, v \in X$, $a \cdot (u + v) = (a \cdot u) + (a \cdot v)$ and $(a + b) \cdot v = (a \cdot v) + (b \cdot v)$;

v. *(Existence of an additive identity)* $\exists\, 0 \in X$ for which $v + 0 = v = 0 + v$ for all $v \in X$;

vi. *(Existence of additive inverses)* for each $v \in X$, $\exists\, x \in X$: $v + x = 0 = x + v$, $x = -v$ is the additive inverse of X (the equation $x + v = 0$ and $v + x = 0$ has a solution $x \in X$ denoted by $-v$);

vii *(Unitary law)* for all $v \in X$, $1 \cdot v = v$.

Definition 1.2.3. Given a vector space X over a field $\mathbf{K}$, a subset W of X is called a **subspace** if W is a vector space over $\mathbf{K}$ and under the operations already defined on X.

Definition 1.2.4. Let M be a non-void subset of a linear space $(X, \mathbf{K})$. The set of all linear combinations of elements of M is called the space **spanned by** M and is represented by $[M]$. That is,

$$[M] = \{\alpha_1 x_1 + \cdots + \alpha_n x_n\} : n \in \mathbf{N},\ x_i \in M \textit{ and } \alpha_i \in \mathbf{K} \quad (i = 1, \ldots, n).$$

Definition 1.2.5. Let X be a vector space over $\mathbb{C}$. A mapping $\langle ., . \rangle : X \times X \to \mathbb{C}$ is called an **inner product** if $\forall\ x,\ x'$ and $y \in X$ and $\alpha \in \mathbb{C}$, the following conditions are satisfied:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,

(ii) $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$,

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,

(iv) $\overline{\langle x, y \rangle} = \langle y, x \rangle$.

The pair $(X, \langle ., . \rangle)$ is called an **inner product space** over $\mathbb{C}$.

Definition 1.2.6. A real valued function $\|.\| : V \to \mathbb{R}$, where V is a vector space over the field $\mathbb{K}$ is called a norm if it satisfies the following conditions: i.e $\forall\ x,\ y \in V$, and $\alpha \in \mathbb{K}$,

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\alpha x\| = |\alpha| \|x\|$,

(3) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 1.2.7. An **operator** is a mapping of a vector space X onto itself or to another vector space.

Definition 1.2.8. Let X and Y be linear spaces. Then a function $T : X \to Y$ is called a **linear operator** if and only if $\forall\ x,\ y \in X$ and $\alpha, \beta \in \mathbf{K}$, $\quad T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

Definition 1.2.9. Let X, Y be linear spaces. A linear operator $T : X \to Y$ is called **bounded** if and only if there exists a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$.

Definition 1.2.10. Let $B(X, Y)$ be the set of bounded linear operators mapping elements of X to Y. Let $T \in B(X, Y)$ then the **norm of** T is defined as:

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \neq 0\right\}.$$

Definition 1.2.11. A **basis** S for a vector space X is a nonempty set of linearly independent vectors that span X.

Definition 1.2.12. Let $(X, \mathbf{K})$ be an inner product space. Then $\forall\, x,\ y \in X$, x and y are said to be orthonormal if $\langle x, y\rangle = 0$ and $\|x\| = \|y\| = 1$. An orthonormal set of all vectors of the form x and y which form a basis is called an **orthonormal basis.**

Definition 1.2.13. A **Hilbert space** is a complete inner product space i.e a Banach space whose norm is generated by an inner product.

Definition 1.2.14. Let X be a vector space with a scalar field **K**, an **algebra** is a vector space X together with a bilinear map $X \times X \to X$ defined by $(a, b) \to ab \quad \forall\, a, b \in X$ such that $a(bc) = (ab)c \quad \forall\, a, b, c \in X$.

Definition 1.2.15. A **subalgebra** of an algebra A is a vector subspace B such that $\forall\, b, b' \in B$ we have $bb' \in B$.

Definition 1.2.16. A norm $\|.\|$ on an algebra A is said to be sub-multiplicative if it satisfies $\|ab\| \leq \|a\|\|b\|\ \forall\, a,\ b \in A$. An algebra A with the norm $\|.\|$ which is sub-multiplicative, is said to be a **normed algebra.**

Definition 1.2.17. If a normed algebra A admits a unit e such that $ae = ea = a \quad \forall\, a \in A$ and $\|e\| = 1$, then we say that A is a **Unital normed algebra**, otherwise it is non-unital.

Definition 1.2.18. A complete normed algebra A is called a **Banach algebra**.

Definition 1.2.19. An algebra A is called **commutative** (*abelian*) if $ab = ba$, $\forall\, a, b \in A$. It is *non-abelian* if the product is non-commutative.

Definition 1.2.20. Let A be an algebra. A mapping from $A \to A$ defined by $a \mapsto a^* \quad \forall\, a, a^* \in A$ is called an **involution** on A if $\forall\, a, b \in A$ and $\alpha \in \mathbf{K}$, it satisfies the following four conditions:

(i) $(a+b)^* = a^* + b^*$

(ii) $(\alpha a)^* = \bar{\alpha} a^*$

(iii) $(ab)^* = b^* a^*$

(iv) $a^{**} = a$.

Definition 1.2.21. An algebra A with an involution i.e $a \mapsto a^*$ is called a ***-algebra**.

Definition 1.2.22. A Banach algebra A with an involution $a \mapsto a^*$ satisfying the property $\|a\| = \|a^*\|, \quad \forall\, a \in A$ is called a **Banach *-algebra**.

Definition 1.2.23. A Banach *-algebra A with the property $\|a^* a\| = \|a\|^2$, $\forall\, a \in A$ is called a **C*-algebra**.

Example 1.2.24. We consider $B(H)$, the set of all bounded linear operators on a Hilbert space H. We prove that $B(H)$ is a C*-algebra.

Proof. **$B(H)$ is an algebra:**

Let $T \in B(H)$ where $T : H \to H$. Now, multiplication is defined pointwise in $B(H)$. Thus, $ST(x) = S(T(x))$ $\forall$ S, $T \in B(H)$ and $x \in H$.

$B(H)$ is a normed algebra:

$B(H)$ is a normed space, consequently, a normed algebra. For if we let $T \in B(H)$ then $\|T\|$ satisfies the axioms of a norm as below;

(i) Clearly, $\|T\| \geq 0$ and $\|T\| = 0$ if and only if $T = 0$.

(ii)

$$\begin{aligned}
\|\alpha T\| &= \sup\left\{\frac{\|(\alpha T)x\|}{\|x\|} : x \neq 0\right\} \\
&= \sup\left\{\frac{\|\alpha(Tx)\|}{\|x\|} : x \neq 0\right\} \\
&= \sup\left\{\frac{|\alpha|\|Tx\|}{\|x\|} : x \neq 0\right\} \\
&= |\alpha| \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} \\
&= |\alpha|\|T\|.
\end{aligned}$$

(iii)

$$\begin{aligned}
\|T + S\| &= \sup\left\{\frac{\|(T+S)(x)\|}{\|x\|} : x \neq 0\right\} \\
&= \sup\left\{\frac{\|Tx + Sx\|}{\|x\|} : x \neq 0\right\} \\
&\leq \sup\left\{\frac{\|Tx\|}{\|x\|} + \frac{\|Sx\|}{\|x\|} : x \neq 0\right\} \\
&\leq \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} + \sup\left\{\frac{\|Sx\|}{\|x\|} : x \neq 0\right\} \\
&= \|T\| + \|S\|.
\end{aligned}$$

(iv)

$$\begin{aligned}
\|TS\| &= \sup\left\{\frac{\|(TS)(x)\|}{\|x\|} : x \neq 0\right\} \\
&= \sup\left\{\frac{\|T(Sx)\|}{\|x\|} : x \neq 0\right\} \\
&= \sup\left\{\frac{\|T(S(x))\|}{\|S(x)\|}\frac{\|S(x)\|}{\|x\|} : S(x) \neq 0,\ x \neq 0\right\} \\
&\leq \sup\left\{\frac{\|T(Sx)\|}{\|S(x)\|} : S(x) \neq 0\right\} \sup\left\{\frac{\|S(x)\|}{\|x\|} : x \neq 0\right\} \\
&= \|T\|\|S\|.
\end{aligned}$$

$B(H)$ is a *-algebra:

Since $B(H)$ is an algebra and $T \in B(H)$, it has an involution from $B(H)$ to $B(H)$ defined by $T \mapsto T^*$, where T^* is the adjoint of T (see definition 1.2.32) But T is a bounded linear operator so we have,

(i) $(T+S)^* = T^* + S^*$.

But, $\langle (T+S)x, y\rangle = \langle x, (T+S)^*y\rangle$, $\forall\, x,\ y \in H$.

Also,

$$\begin{aligned}
\langle (T+S)x, y\rangle &= \langle Tx + Sx, y\rangle \\
&= \langle Tx, y\rangle + \langle Sx, y\rangle \\
&= \langle x, T^*y\rangle + \langle x, S^*y\rangle.
\end{aligned}$$

Thus, $\langle x, (T+S)^*y\rangle = \langle x, T^*y + S^*y\rangle$, $\forall\, x,\ y \in H$.

(ii) $(\alpha T)^* = \overline{\alpha} T^*$.

Now,

$$\langle (\alpha T)x, y\rangle = \langle x, (\alpha T)^*y\rangle. \tag{1.2.1}$$

Also,

$$\langle(\alpha T)x, y\rangle = \alpha\langle T(x), y\rangle = \langle x, (\alpha T)^* y\rangle\rangle = \langle x, \overline{\alpha} T^*(y)\rangle. \qquad (1.2.2)$$

Equations (1.2.1) and (1.2.2) shows that $\langle x, (\alpha T)^* y\rangle = \langle x, \overline{\alpha} T^*(y)\rangle$.

(iii) $(TS)^* = S^*T^*$.

Since

$$\begin{aligned}
(TS)x &= T(S(x)), \\
\langle(TS)(x), y\rangle &= \langle T(S(x)), y\rangle \\
&= \langle Sx, T^*y\rangle \\
&= \langle x, S^*(T^*(y))\rangle \\
&= \langle x, (S^*T^*)(y)\rangle.
\end{aligned}$$

On the other hand, $\langle(TS)(x), y\rangle = \langle x, (TS)^*(y)\rangle$

i.e $(TS)^* = S^*T^*$

(iv) $T^{**} = T$.

Now $\langle Tx, y\rangle = \langle x, T^*y\rangle = \langle (T^*)^*x, y\rangle$.

So $\langle (T - T^{**})x, y\rangle = 0 \; \forall \; x, y \in H$.

Therefore, $T - T^{**} = 0$ and hence $T^{**} = T$.

$B(H)$ is a Banach *-algebra.

For all $T \in B(H)$, $\|T\| = \|T^*\|$.

Now, $\forall\, x \in H$,

$$\begin{aligned}\|T^*(x)\|^2 &= \langle T^*x, T^*x\rangle \\ &= \langle T(T^*(x)), x\rangle \\ &\leq \|T(T^*(x))\|\|x\| \\ &\leq \|T\|\|(T^*(x))\|\|x\| \\ \|T^*(x)\| &\leq \|T\|\|x\|\end{aligned}$$

i.e

$$\|T^*\| \leq \|T\|. \tag{1.2.3}$$

Also, $\|T^{**}\| \leq \|T^*\|$, but $T^{**} = T$. Therefore,

$$\|T\| \leq \|T^*\| \tag{1.2.4}$$

and hence by (1.2.3) and (1.2.4), $\|T\| = \|T^*\|$.

$B(H)$ is a C*-algebra.

We need to show that it satisfies the property $\|T^*T\| = \|T\|^2$, $\forall\, T \in B(H)$.

Now, $\|T^*T(x)\| \leq \|T^*\|\|x\|\|T\| = \|T\|^2\|x\|$

$$\Rightarrow \|T^*T\| \leq \|T\|^2. \tag{1.2.5}$$

Also, $\forall\, x \in H$,

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle T^*Tx, x\rangle \\ &\le \|T^*Tx\|\|x\| \\ &\le \|T^*T\|\|x\|^2 \end{aligned}$$

i.e

$$\|T\|^2 \le \|T^*T\|. \tag{1.2.6}$$

By (1.2.5) and (1.2.6), $\|T^*T\| = \|T\|^2$, so $B(H)$ is a C*-algebra. □

Definition 1.2.25. Let X be a vector space over $\mathbf{K}(\mathbb{C}\ or\ \mathbb{R})$. A mapping $f : X \to \mathbf{K}$ is called a **functional.**

Definition 1.2.26. A functional f on a vector space X over $\mathbf{K}$ is called a **linear functional** if $f : X \to \mathbf{K}$ is a complex-valued linear operator.

Definition 1.2.27. A linear functional f is said to be **bounded** if and only if there exists a constant $C > 0$ such that $|f(x)| \le C\|x\| \quad \forall\, x \in X$.

Definition 1.2.28. Let f be a bounded linear functional on X. Then the **norm of** f is defined as $\|f\| = \sup\left\{\frac{|f(x)|}{\|x\|} : x \neq 0\right\}$.

Definition 1.2.29. Let X be a vector space and X^* the set of all linear functionals on X then X^* is called the **dual space** of X.

Definition 1.2.30. A **positive linear functional** is a linear functional on a Banach algebra A with an involution that satisfies the condition

$$f(aa^*) \ge 0, \ \ \forall\, a \in A.$$

Definition 1.2.31. Let A an algebra with involution. Then the linear functional f is called a **state** on A if f is positive and $\|f\| = f(e) = 1$ where e is a unit element in A.

Definition 1.2.32. If $T \in B(H, K)$, where H and K are Hilbert spaces, then the linear operator $T^* \in B(K, H)$ satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$ $\forall\, x \in H$ and $\forall\, y \in K$ is called the **(Hilbert space) Adjoint** of T.

Definition 1.2.33. A bounded operator $T \in B(H)$ is said to be **self-adjoint** if $T^* = T$. Thus, T is **Hermitian** and $\mathfrak{D}(T) = H$ if and only if T is self-adjoint.

Definition 1.2.34. A bounded linear operator T on a Hilbert space H is said to be **normal** if it commutes with its adjoint i.e $TT^* = T^*T$.

Definition 1.2.35. A **unitary operator** is a bounded linear operator U on a Hilbert space satisfying: $U^*U = UU^* = I$, where U^* is the adjoint operator.

This property is equivalent to the following:

(i) U preserves inner product on the Hilbert space, so that for all vectors x and y in the Hilbert space H, $\langle Ux, Uy\rangle = \langle x, y\rangle$.

(ii) U is a surjective isometry (distance preserving map) i.e

$$\|U(x - y)\| = \|x - y\|.$$

Definition 1.2.36. If H is a Hilbert space, then an operator $T \in B(H)$ is a **finite rank** operator if the dimension of the range of T is finite, and a **compact** operator if for every bounded sequence $\{x_n\}$ in H, the sequence $\{Tx_n\}$ contains a convergent subsequence.

Definition 1.2.37. Let $D = (\lambda_{jk})$ $(j,\ k = 1, ..., n)$ be an n-rowed square matrix. Then the sum of its eigenvalues equals to the **trace** of D, that is, the sum of the elements of the principal diagonal: $trace\, D = \lambda_{11} + \cdots + \lambda_{nn}$.

Definition 1.2.38. A bounded linear operator $P : H \to H$ on a Hilbert space H is a **projection** if and only if P is self-adjoint $(P^* = P)$ and idempotent $(P^2 = P)$.

Definition 1.2.39. Let H be a Hilbert space and $B(H)$ the algebra of bounded linear operators on H. Then $T : B(H) \to B(H)$ is called an **elementary operator** if T has a representation $T(x) = \sum_{i=1}^{n} a_i x b_i$ where a_i and b_i are fixed in $B(H)$.

Definition 1.2.40. Let H be a Hilbert space and $T : H \to H$ be a linear operator then:

(i) A number $\lambda \in \mathbb{C}$ is called the **eigenvalue** of T if there is a non-zero $x \in H$ such that $Tx = \lambda x$; the vector x is then called an **eigenvector** for T corresponding to the eigenvalue λ.

(ii) The set $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$ is called a **numerical range** if $T \in B(H)$.

Definition 1.2.41. Let X be a linear space. A subset M of the linear space X is **convex** if for all $x, y \in M$ and for any positive real number t satisfying $0 < t < 1$ we have $tx + (1-t)y \in M$.

Definition 1.2.42. If M is a subset of a linear space X, then a **convex hull** M, represented by $conv(M)$ is the smallest convex subset of X containing M and it is the intersection of all the convex subsets of X that contain M.

Definition 1.2.43. For a tuple $(c_1, ..., c_n)$ of operators $c_i \in B(H)$, we denote $W_m(c_1, ..., c_n)$ the **matrix numerical range** by:

$$W_m(c_1, ..., c_n) = \{(\langle c_j^* c_i \xi, \xi \rangle)_{i,j=1}^n : \xi \in H, \|\xi\| = 1\} \subset M_n.$$

The closure of W_m is called the **extremal numerical range** defined by:

$$W_{m,e}(c_1, ..., c_n) = \{\alpha \in \overline{W_m(c_1, ..., c_n)} : trace(\alpha) = \|\sum_{i=1}^n c_i^* c_i\|\}.$$

Definition 1.2.44. The **rank** of a matrix D is defined as the order of the largest square array in D with a nonzero determinant.

Definition 1.2.45. Let X be a non-empty set and $\mathbf{K}$ be the field of real or complex numbers. Let $\mathbf{K}_X$ be the set of all finite linear combinations of elements of X such that $\mathbf{K}_X = \{\sum_{i=1}^n \alpha_i x_i : \ x_i \in X, \alpha_i \in \mathbf{K}\}$ where the operations are as $\alpha x_i + \beta x_i = (\alpha + \beta)x_i$ and $\alpha(\beta x_i) = (\alpha\beta)x_i$. Then the vector space K_X over $\mathbf{K}$ is called the **free vector space.**

Remark 1.2.46. The term *free* is used to connote the fact that there is no relationship between the elements of X.

Definition 1.2.47. Let X and Y be two vector spaces over $\mathbf{K}$, and let T be the subspace of the free vector space $\mathbf{K}_{X\times Y}$ generated by all the vectors of the form $\alpha(x, y) + \beta(x', y) - (\alpha x + \beta x', y)$ and $\alpha(x, y) + \beta(x, y') - (x, \alpha y + \beta y')$ $\forall$ $\alpha, \beta \in \mathbf{K}$ and $x, x' \in X, \ y, y' \in Y$. Then the quotient space $\mathbf{K}_{X\times Y}/T$ is called the **tensor product** of X and Y and is denoted by $X \otimes Y$.

An element of $X \otimes Y$ has the form $\sum \alpha_i(x_i, y_i) + T$. The coset $(x, y) + T$ is denoted by $x \otimes y$ and therefore any element μ of $X \otimes Y$ has the form $\mu = \Sigma_i x_i \otimes y_i$.

Definition 1.2.48. If x and y are elements of a Hilbert space H we define the operator $x \otimes y$ on H by $(x \otimes y)(z) = \langle z, y \rangle x$.

Lemma 1.2.49. *If x and y are elements of a Hilbert space H then for the operator $x \otimes y$ on H, $\|x \otimes y\| = \|x\|\|y\|$.*

Proof. From the above definition, $(x \otimes y)(z) = \langle z, y \rangle x$. Since y is fixed, let us denote $(x \otimes y)(z)$ by $T_y z$.
Now, by the definition of an operator norm,

$$\begin{aligned} \|T_y\| &= \sup\{\|T_y z\| : z \in H, \|z\| = 1\} \\ &= \sup\{\|(x \otimes y)(z)\| : z \in H, \|z\| = 1\} \\ &= \sup\{\|\langle z, y \rangle x\| : z \in H, \|z\| = 1\} \\ &= \sup\{|\langle z, y \rangle| \|x\| : z \in H, \|z\| = 1\} \\ &= \|x\| \sup\{|\langle z, y \rangle| : z \in H, \|z\| = 1\}. \end{aligned}$$

But $|\langle z, y \rangle|$ is maximum when $z = \frac{y}{\|y\|}$ with $y \neq 0$.
Hence,

$$\begin{aligned} \|T_y\| &= \|x\| |\langle \frac{y}{\|y\|}, y \rangle| \\ &= \|x\| \frac{1}{\|y\|} \langle y, y \rangle \\ &= \|x\| \frac{\|y\|^2}{\|y\|} \\ &= \|x\|\|y\|. \end{aligned}$$

Therefore, $\|x \otimes y\| = \|x\|\|y\|$. □

Definition 1.2.50. Suppose A is a complex algebra and f is a linear functional on A which is not identically zero. Then if $f(a,b) = f(a)f(b)$ $\forall\, a,\, b \in A$ then f is called a **complex homomorphism** on A.

Definition 1.2.51. Suppose A and B are C*-algebras. A mapping $\phi : A \to B$ is said to be a **C*-homomorphism** if for any $\alpha,\, \beta \in \mathbb{C}$ and $a,\, b \in A$ the following conditions are satisfied:
(i) $\phi(\alpha a + \beta b) = \alpha\phi(a) + \beta\phi(b)$
(ii) $\phi(ab) = \phi(a)\phi(b)$
(iii) $\phi(a^*) = (\phi(a))^*$
(iv) ϕ maps a unit in A to a unit in B.
Further, if ϕ is $1-1$ we say that the mapping ϕ is a C*-isomorphism. i.e. for all $a,\, b \in A$ and $a \neq b$ we have $\phi(a) \neq \phi(b)$ and so A and B are isomorphic.

Definition 1.2.52. A **representation** of a C*-algebra A is defined as the pair (H, ϕ), where H is a complex Hilbert space and ϕ is a *-morphism of A into $B(H)$. The representation (H, ϕ) is said to be faithful if and only if ϕ is a *-isomorphism between A and $\phi(A)$.
The space H is called the representation space, the operators $\phi(a)$ are called the representatives of A and by implicit identification of ϕ and the set of representatives, we say that ϕ is a representation of A on H.

1.3 Completion of normed spaces

Definition 1.3.1. Let $\{x_n\}$, $\{y_n\}$ be Cauchy sequences in $(X,\, d)$ then $\{x_n\}$ is said to be **equivalent** to $\{y_n\}$ denoted by $\{x_n\} \sim \{y_n\}$ if and only if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

The collection of all **equivalence classes** in this case is denoted by X^*. See details of equivalence relations and classes in [8].

Definition 1.3.2. A mapping $A : X \to Y$ where X, Y are normed linear spaces is said to be a **congruence** if it is simultaneously an isometry and an isomorphism.

Let (X, d) be an arbitrary metric space. Then the complete metric space (X^*, d^*) is said to be a **completion** of (X, d) if:

1. (X, d) is isometric to a subspace (X_0, d^*) of (X^*, d^*).
2. The closure of X_0, $\overline{X_0}$ is all of X^* i.e $\overline{X_0} = X^*$.

Statement (2) is equivalent to saying that X_0 is dense in X^*, that is, every point of X^* is either a point or limit point of X_0 (i.e for any point $x \in X^*$, $\exists \{x_n\} \in X_0$ that converges to x) [9].
The two properties above are proved in the theorem below.

Theorem 1.3.3. *Every metric space (X, d) has a completion (X^*, d^*) and furthermore, if (X^{**}, d^{**}) is also a completion of (X, d) then (X^*, d^*) is isometric to (X^{**}, d^{**}), i.e the completion of a space is unique to within an isometry. See proof in [2].*

Equipped with Theorem (1.3.3) we can now look at the completion of a normed linear space.

Theorem 1.3.4. *For every normed linear space X there's a complete X^* such that X is congruent to a dense subset X_0 of X^* and the norm on X^* extends the norm on X.*

Proof. (From [2]) Let X be a normed linear space and consider the distance function d defined by taking

$$d(x, y) = \|x - y\|, \ \forall\, x, y \in X.$$

We call d as a norm derived metric.

From Theorem (1.3.3), we have (X, d) as a metric space and it's completion (X^*, d^*) also a complete metric space.

We identify $x \in X$ with its isometric image in X^*. Our prime aim is to show that, after defining vector addition and scalar multiplication, X^* will be a complete normed linear space with the property that not only is X isometric to a dense subset of X^* but is also isomorphic to this dense subset.

Further, we show that the norm on X^* will extend the norm on X, the extension made by the above identification in mind.

Thus we exhibit $X_0 \subset X^*$ such that

i. $\overline{X}_0 = X^*$,

ii. X is isomorphic and isometric to X_0 i.e X and X_0 are congruent.

Now let $x^*, y^* \in X^*$ (i.e equivalence classes of Cauchy sequences of X). Let

$$\{x_n\} \in x^* \quad and \quad \{y_n\} \in y^* \tag{1.3.1}$$

We define $x^* + y^*$ to be the equivalence class containing $\{x_n + y_n\}$ and we call it z^*.

Now we show that $\{x_n + y_n\}$ is a Cauchy sequences.

To show this, we note that

$$\|x_n + y_n - (x_m + y_m)\| \leq \|x_n - x_m\| + \|y_n - y_m\|.$$

We now show that the operation is well defined.

Suppose $\hat{x}_n \sim x_n$ and $\hat{y}_n \sim y_n$ then we recall from definition (1.3.1) what is meant for two sequences to be equivalent and we can show that

$$\{x_n + y_n\} \sim \{\hat{x}_n + \hat{y}_n\}$$

by noting that

$$\|x_n + y_n - (\hat{x}_n + \hat{y}_n)\| \leq \|x_n - \hat{x}_n\| + \|y_n - \hat{y}_n\|.$$

Let $\alpha \in \mathbf{K}$. With $\{x_n\} \in x^*$ as in equation (1.3.1), we define αx^* to be the class containing $\{\alpha x_n\}$.

Therefore, $\{\alpha x_n\}$ is a Cauchy sequence and that the operations of scalar multiplication is well defined. Hence X^* with these two operations is, indeed a linear space.

Now we introduce a norm on X^*.

With $\{x_n\} \in x^*$ as in equation (1.3.1), we define

$$\|x^*\| = \lim_{n\to\infty} \|x_n\| \tag{1.3.2}$$

We show that the limit in equation (1.3.2) exists.

Since

$$| \, \|x_m\| - \|x_n\| \, | \;\; \leq \;\; \|x_m - x_n\|,$$

it is easy to see that the sequence of real numbers $\{\|x_n\|\}$ is Cauchy and hence the limit exists.

We suppose that $x_n \sim \hat{x}_n$. Since

$$| \, \|x_n\| - \|\hat{x}_n\| \, | \leq \|x_n - \hat{x}_n\|$$

and the term on the right goes to zero, the norm in equation (1.3.2) is well defined.

Now we show that the mapping in equation (1.3.2) is truly a norm.

As in equation (1.3.1),

(i). The mapping is non-negative and equals to zero if and only if $x^* = 0^*$.

Suppose $\|x^*\| = 0$, $\Rightarrow \lim_{n\to\infty} \|x_n\| = 0$. This implies $x_n \to 0$.

Thus $\{x_n\} \sim (0, 0, 0, ...)$ or $\{x_n\} \in 0^*$ and $x^* = 0^*$.

(ii). $\|\alpha x^*\| = \lim_{n\to\infty} \|\alpha x_n\| = |\alpha| \lim_{n\to\infty} \|x_n\| = |\alpha| \|x^*\|$.

(iii). $\|x^* + y^*\| = \lim_{n\to\infty} \|x_n + y_n\| \leq \lim_{n\to\infty} \|x_n\| + \lim_{n\to\infty} \|y_n\|$

or

$$\|x^* + y^*\| \leq \|x^*\| + \|y^*\|.$$

Hence equation (1.3.2) determines a norm on X^*.

Next we show that X^* is complete with respect to the distance function determined by this norm, denoted by d_N. i.e we need to show that d_N and d^* agree (d^* as in Theorem (1.3.3)).

Now

$$\begin{aligned} d_N(x^*, y^*) &= \|x^* - y^*\| \\ &= \lim_{n\to\infty} \|x_n - y_n\| \\ &= \lim_{n\to\infty} d(x_n, y_n) \\ &= d^*(x^*, y^*) \end{aligned}$$

hence we conclude that X^* is a complete normed linear space.
As in Theorem (1.3.3), we have an isometry A between X and X_0 of X^* :
The set of all equivalence classes of X^* containing all elements of the form $(x, x, x, ...) \in x'$, $x \in X$.
We show that A establishes an isomorphism between X and X_0. Its already known that A is onto X_0, and since its an isometry, it is one-to-one. So we only need to show that it preserves linear combination.
Suppose x, $y \in X$ and let $Ax = x'$ and $Ay = y'$.
Now consider

$$A(x + y) = (x + y)'.$$

Since

$$(x + y,\ x + y, ...) \in (x + y)'$$

and

$$(x + y,\ x + y, ...) = (x, x, ...) + (y, y, ...)$$

we can say by our rule for addition, that

$$(x + y)' = x' + y'$$

and A preserves vector addition.

For scalar multiplication, let $\alpha \in \mathbf{K}$ and $x \in X$.

Now, let $Ax = x'$. So for $\alpha \in \mathbf{K}$,

$$A(\alpha x) = \alpha(Ax) = \alpha x'$$

and

$$(\alpha x, \alpha x, ...) \in (\alpha x)'$$

so we can say by our rule for scalar multiplication, that

$$(\alpha x)' = \alpha x'$$

and A preserves scalar multiplication. □

1.4 Literature review

The term elementary operator came as a result of basic elementary operators [4, 5]. If A is an algebra, then given $a, b \in A$ we define a basic elementary operator $M_{a,b} : A \to A$ by $M_{a,b}(x) = axb$. Therefore, an elementary operator is the sum $T = \sum_{i=1}^{n} M_{a_i,b_i}$ of the basic ones, see definition (1.2.39). On detailed study of the norm of elementary operators, a number of results have been shown. Trivially, for the basic elementary operator, $\|M_{a,b}\| \leq 2\|a\|\|b\|$. For the Jordan elementary operator $\mathcal{U} = \|M_{a,b} + M_{b,a}\|$, $\|M_{a,b} + M_{b,a}\| \leq 2\|a\|\|b\|$ for the upper estimate. Considering the lower estimates, Mathieu proved that for prime C*- algebras, $\|M_{a,b} + M_{b,a}\| \geq \frac{2}{3}\|a\|\|b\|$, Cabrera and Rodriguez proved that for JB* algebras, $\|M_{a,b} + M_{b,a}\| \geq \frac{1}{20412}\|a\|\|b\|$ while Stacho and Zalar [21] proved that for standard operator algebras on Hilbert spaces $\|M_{a,b} + M_{b,a}\| \geq 2(\sqrt{2} - 1)\|a\|\|b\|$. Recently, Timoney [24, 25] showed that $\|M_{a,b} + M_{b,a}\| \geq \|a\|\|b\|$ and further came up with a formula for calculating the norm of a general elementary operator involving matrix numerical range using the notion of tracial geometric mean [27].

The tracial geometric mean of the positive (semi-definite) $n \times n$-matrices D, E is $tgm(D, E) = trace\sqrt{\sqrt{D}E\sqrt{D}}$ where $\sqrt{.}$ denotes the positive square root.

Theorem 1.4.1. *For $a = [a_1, ..., a_n] \in B(H)^n$ (a row matrix of operators $a_i \in B(H)$), $b = [b_1, ..., b_n]^t \in B(H)^n$ (a column matrix of operators $b_i \in B(H)$) and $Tx = \sum_{i=1}^{n} a_i x b_i$ an elementary operator, we have $\|T\| = \sup\{tgm(Q(a^*, \xi), Q(b, \eta)) : \xi, \eta \in H, \|\xi\| = 1, \|\eta\| = 1\}$. For proof, see [27].*

Through the idea of tensor products [1, 6], the norm of elementary operators can be determined. The Haagerup norm, for example, of an element $w \in B(H) \otimes B(H)$ (of the algebraic tensor product) is defined by

$$\|w\|_h^2 = \inf \left\{ \left\| \sum_{i=1}^{n} a_i a_i^* \right\| \left\| \sum_{i=1}^{n} b_i^* b_i \right\| \right\}$$

where the infimum is taken over all possible representations

$$w = \sum_{i=1}^{n} a_i \otimes b_i.$$

A well known estimate of an operator T due to Haagerup states that if $T = \sum_{i=1}^{n} a_i \otimes b_i$ then $\|T\| \leq \|T\|_{cb} \leq \{\| \sum_{i=1}^{n} a_i a_i^* \| \| \sum_{i=1}^{n} b_i^* b_i \|\}^{\frac{1}{2}}$ where

$\|T\|_{cb}$ is the completely bounded norm of T. We have shown that the Haagerup norm is actually a norm in chapter two.

Theorem 1.4.2. *Let $A = B(H)$ (where A is an algebra) and let $T \in \varepsilon\ell(B(H))$ be as above then*

$$\|T\| \leq \|T\|_{cb} \leq \frac{1}{2}\left(\left\{\left\|\sum_{i=1}^{n} a_i a_i^*\right\|\right\} + \left\{\left\|\sum_{i=1}^{n} b_i^* b_i\right\|\right\}\right)$$

if and only if $W_{m,e}(a_1^*, ..., a_n^*) \cap W_{m,e}(b_1^*, ..., b_n^*)$ is nonempty. See proof in [26].

Agure and Nyamwala [14] also used the spectral resolution theorem to calculate the norm of an elementary operator induced by normal operators in a finite dimensional Hilbert space.

Lemma 1.4.3. *Let T be a normal operator such that $T : H \rightarrow H$ where H is a finite dimensional Hilbert space then*

$$\|T\| = \left(\sum_{j=1}^{m} \mid \lambda_j \mid^2\right)^{\frac{1}{2}}$$

where λ_j are distinct eigenvalues of T for corresponding eigenspaces (M_j, $j = 1, ..., m$). See [14] for proof.

Theorem 1.4.4. *Let $T_{a,b} : B(H) \rightarrow B(H)$ be an elementary operator defined by $T_{a,b}(x) = \sum_{i=1}^{n} a_i x b_i$ where a_i and b_i are normal operators and H a finite dimensional Hilbert space then*

$$\|T\| = \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m} \mid \alpha_{i,j} \mid^2 \mid \beta_{i,j} \mid^2\right)\right)^{\frac{1}{2}}$$

where $\alpha_{i,j}$ and $\beta_{i,j}$ are distinct eigenvalues of a_i and b_i respectively. See [14] for proof.

Our main interest therefore, has been to further investigate the norm of elementary operators where we precisely aimed at determining the norm of the basic elementary operator$M_{a,b} : B(H) \longrightarrow B(H)$ defined by $M_{a,b}(x) = axb \;\; \forall \, x \in B(H),\; a,\; b$ fixed in $B(H)$, the algebra of all bounded linear operators on a Hilbert space H (see example 1.2.24).

1.5 Statement of the problem

Let H be a complex Hilbert space, $T : H \longrightarrow H$ be a bounded linear operator and $B(H)$ the set of bounded linear operators on H. $B(H)$ is an algebra, in fact a C*-algebra. The norm of T is defined as:

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\}.$$

In our study we include the basic elementary operator $M_{a,b} : B(H) \longrightarrow B(H)$ defined by $M_{a,b} = axb$, $\forall \, x \in B(H)$ and a, b fixed in $B(H)$. The upper estimate of the norm of a basic elementary operator are easy to find. Therefore, we determine $\|M\|$, specifically, we concentrate on determining the lower estimate of this norm through tensor products.

1.6 Objective of the study

The purpose of this study is to determine the lower estimate of the norm of the basic elementary operator through tensor products.

1.7 Significance of the study

The results obtained are a contribution to the field of elementary operators and a motivation for further research to aspiring mathematicians in this particular field of study. Further, we hope that the results obtained shall be useful to applied mathematicians and physicists especially in quantum mechanics.

1.8 Research methodology

For a successful completion of this research, we developed a good background knowledge of the theory of operators, especially C*-algebras, General Topology and Functional Analysis . We have restated some known results which we found useful to our work however, for most parts of this work we omitted the proofs. Instead, we indicated where the proofs may be found. In some cases we provided alternative proofs to the known results by taking advantage of the operator theory results constructed here. Lastly, we used the technical approach of tensor products in solving the stated problem. We initially examined the algebraic properties of tensor products, their norm properties and applicability in our case before applying it in finding a solution to our problem.

Chapter 2

TENSOR PRODUCTS AND TENSOR NORMS

2.1 Introduction

In this chapter we study tensor products and tensor norms. We look closely at tensor products of vector spaces and functionals, Hilbert spaces, operator spaces, normed spaces and C*-algebras. We also give some results on tensor norms, especially on projective norm, Haagerup norm, spatial norm and maximal C*-norm. Lastly, we establish the relationship between spatial norm and maximal C*-norm.

2.1.1 Bilinear maps and tensor products.

Let X and Y be vector spaces over $\mathbf{K}$. A function $f : X \times Y \to \mathbf{K}$ is **bilinear** if it is linear in both variables separately, that is,

$$f(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 f(x_1, y) + \alpha_2 f(x_2, y)$$

and

$$f(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 f(x, y_1) + \beta_2 f(x, y_2)$$

for all $x,\ x_1,\ x_2 \in X$ and $y,\ y_1,\ y_2 \in Y$.

We write $B(X, Y; \mathbf{K})$ to denote the set of all bilinear functions from $X \times Y$ to $\mathbf{K}$. A bilinear function $f : X \times Y \to \mathbf{K}$ with values in the base field is called a **bilinear form** on $X \times Y$. See [6, 13] for more details on bilinear forms.

Lemma 2.1.1. *Let f be a mapping from a cartesian product space to the tensor product space i.e $f : X \times Y \to X \otimes Y$. Then f is a bilinear map.*

Proof. Let $x,\ x_1,\ x_2 \in X$ and $y,\ y_1,\ y_2 \in Y$. Also let $\alpha,\ \beta \in \mathbf{K}$. To show that f is bilinear, it suffices to show that it is linear in each vector space X and Y separately. To show linearity in X, let $f(x, y) = x \otimes y$. Then,

$$\begin{aligned} f(\alpha x_1 + \beta x_2, y) &= (\alpha x_1 + \beta x_2) \otimes y \\ &= (\alpha x_1 \otimes y) + (\beta x_2 \otimes y) \\ &= \alpha(x_1 \otimes y) + \beta(x_2 \otimes y) \\ &= \alpha f(x_1 \otimes y) + \beta f(x_2 \otimes y). \end{aligned}$$

Hence f is linear in X.

To show linearity in Y,

$$\begin{aligned} f(x, \alpha y_1 + \beta y_2) &= x \otimes (\alpha y_1 + \beta y_2) \\ &= (x \otimes \alpha y_1) + (x \otimes \beta y_2) \\ &= \alpha(x \otimes y_1) + \beta(x \otimes y_2) \\ &= \alpha f(x \otimes y_1) + \beta f(x \otimes y_2). \end{aligned}$$

Hence f is linear in Y and therefore, f is a bilinear map. □

2.1.2 Universal property of tensor products

The space of all bilinear maps from $X \times Y$ to another vector space Z is naturally isomorphic to the space of all linear maps from $X \otimes Y$ to Z. This is built into the construction; $X \otimes Y$ has all the relations that are necessary to ensure that a homomorphism from $X \otimes Y$ to Z will be linear.

Theorem 2.1.2. *Let X and Y be vector spaces over the same field K. There exists $X \otimes Y$ called tensor product of X and Y with a canonical bilinear homomorphism $f : X \times Y \to X \otimes Y$ distinguished up to isomorphism by the following universal property; Every bilinear homomorphism $\phi : X \times Y \to Z$ lifts to a unique homomorphism $\tilde{\phi} : X \otimes Y \to Z$ such that $\phi(x, y) = \tilde{\phi}(x \otimes y)$ for all $x \in X$ and $y \in Y$. See [23] for proof.*

2.2 Tensor products of vector spaces

The tensor product, $X \otimes Y$, of the vector spaces X and Y can be constructed as a space of linear functionals on $B(X \times Y)$ in the following way; for $x \in X$, $y \in Y$ we denote by $x \otimes y$ the functional given by evaluation at the point (x, y). In other words,

$$(x \otimes y)(f) = \langle f, x \otimes y \rangle = f(x, y)$$

for the bilinear form f on $X \times Y$. The tensor product $X \otimes Y$ is the subspace of the dual $B(X \times Y)^*$ spanned by these elements. Thus, a typical tensor in $X \otimes Y$ has the form $u = \sum_{i=1}^{n} \alpha_i x_i \otimes y_i$ where n is a natural number, $\alpha_i \in \mathbf{K}$, $x_i \in X$ and $y_i \in Y$.

We note a few elementary facts about tensors. First, if $u = \sum_{i=1}^{n} \alpha_i x_i \otimes y_i$ is a tensor and f a bilinear form, then the action of u on f is given by:

$$u(f) = \left\langle f, \sum_{i=1}^{n} \alpha_i x_i \otimes y_i \right\rangle = \sum_{i=1}^{n} \alpha_i f(x_i, y_i).$$

We note that mapping $(x, y) \mapsto x \otimes y$ is multiplicational on $X \times Y$ with values in the vector space $X \otimes Y$. This product is itself bilinear, so we have, for example,

(i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$,

(ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$,

(iii) $\alpha x \otimes y = (\alpha x) \otimes y = x \otimes (\alpha y)$,

(iv) $0 \otimes y = x \otimes 0 = 0$.

We note that $u = \sum_{i=1}^{n} \alpha_i x_i \otimes y_i$ can rewritten as $u = \sum_{i=1}^{n} x_i \otimes y_i$.

Theorem 2.2.1. *Let X and Y be vector spaces.*

(a) Let E_1 and E_2 be linearly independent subsets of X and Y respectively, then $\{x \otimes y : x \in E_1, y \in E_2\}$ is a linearly independent subset of $X \otimes Y$.

(b) If $E_1 = \{e_i : i \in I\}$ and $E_2 = \{e'_j : j \in J\}$ are bases for X and Y respectively then $E_1 \otimes E_2 = \{e_i \otimes e'_j : e_i \in E_1, e'_j \in E_2\}$ is a basis for $X \otimes Y$. (original proof in [15]).

Proof. (a) Suppose $\sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) = 0$ where $x_i \in E_1$ and $y_i \in E_2$. Let ϕ and φ be linear functionals on X and Y respectively.

Consider the bilinear form defined by $f(x, y) = \phi(x)\varphi(y)$. We have

$$u(f) = 0$$

and so

$$\sum_{i=1}^{n} \alpha_i \phi(x_i) \varphi(y_i) = \varphi\left(\sum_{i=1}^{n} \alpha_i \phi(x_i) y_i\right) = 0.$$

Since this holds for every $\varphi \in Y^*$, we can conclude that

$$\sum_{i=1}^{n} \alpha_i \phi(x_i) y_i = 0,$$

and so by the linear independence of E_2 we have $\alpha_i \phi(x_i) y_i = 0$ for every $\phi \in X^*$. But, by the linear independence of E_1, each x_i is nonzero and it follows that $\alpha_i = 0, \ \forall\, i$.

(b) From (a) we only need to show that $E_1 \otimes E_2$ spans $X \otimes Y$.

Let $x \otimes y \in X \otimes Y$ such that $x = \sum_{i=1}^{n} \alpha_i e_i$ and $y = \sum_{j=1}^{m} \beta_j e'_j$.

We therefore have

$$\begin{aligned} x \otimes y &= \sum_{i=1}^{n} \alpha_i e_i \otimes \sum_{j=1}^{m} \beta_j e'_j \\ &= \sum_{j=1}^{m} \beta_j (\sum_{i=1}^{n} \alpha_i e_i \otimes e'_j) \\ &= \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n} \alpha_i (e_i \otimes e'_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j (e_i \otimes e'_j). \end{aligned}$$

Since $x \otimes y$ was picked arbitrarily in $X \otimes Y$, any vector in $X \otimes Y$ can be expressed as a linear combination of the vectors $e_i \otimes e'_j$. We deduce that $E_1 \otimes E_2$ spans $X \otimes Y$. Therefore, $E_1 \otimes E_2$ is a basis of $X \otimes Y$. □

Theorem 2.2.2. *The following are equivalent for* $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$.

(i) $u = 0$

(ii) $\sum_{i=1}^{n} \phi(x_i)\varphi(y_i) = 0, \ \ \forall\, \phi \in X^*, \ \varphi \in Y^*.$

(iii) $\sum_{i=1}^{n} \phi(x_i)y_i = 0, \ \ \forall\, \phi \in X^*.$

(iv) $\sum_{i=1}^{n} \varphi(y_i)x_i = 0, \ \ \forall\, \varphi \in Y^*.$ *(original proof in [15])*

Proof. *(i)* $\Rightarrow$ *(ii)*

Since $u = \sum_{i=1}^{n} x_i \otimes y_i$, we note that

$$\begin{aligned} 0 &= u(f) \\ &= \left\langle f, \sum_{i=1}^{n} x_i \otimes y_i \right\rangle \\ &= \sum_{i=1}^{n} f(x_i, y_i) \\ &= \sum_{i=1}^{n} \phi(x_i)\varphi(y_i), \ \ \forall\, \phi \in X^*, \ \varphi \in Y^*. \end{aligned}$$

(ii) $\Rightarrow$ *(iii)*

Now,

$$\sum_{i=1}^{n} \phi(x_i)\varphi(y_i) = 0, \ \ \forall\, \phi \in X^*, \ \varphi \in Y^*.$$

$$\Rightarrow \varphi\left(\sum_{i=1}^{n} \phi(x_i)y_i\right) = 0, \ \ \forall\, \varphi \in Y^*.$$

$$\Rightarrow \sum_{i=1}^{n} \phi(x_i)y_i = 0, \ \ \forall\, \phi \in X^*.$$

(iii) $\Rightarrow$ *(iv)*

From

$$\sum_{i=1}^{n} \phi(x_i)y_i = 0, \ \ \forall\, \phi \in X^*$$

we have

$$\varphi\left(\sum_{i=1}^{n}\phi(x_i)y_i\right)=0,\quad \forall\,\varphi\in Y^*.$$

So

$$\sum_{i=1}^{n}\phi(x_i)\varphi(y_i)=0,\quad \forall\,\phi\in X^*,\ \varphi\in Y^*.$$

But,

$$\phi\left(\sum_{i=1}^{n}\varphi(y_i)x_i\right)=0,\quad \forall\,\phi\in X^*.$$

$$\Rightarrow\sum_{i=1}^{n}\varphi(y_i)x_i=0,\quad \forall\,\varphi\in Y^*.$$

$(iv)\Rightarrow(i)$

Suppose $\sum_{i=1}^{n}\varphi(y_i)x_i=0,\ \ \forall\,\varphi\in Y^*$. Let $f\in B(X\times Y)$. Further, let $E,\ F$ be the subspaces of $X,\ Y$ respectively spanned by $\{x_1,...,x_n\}$, $\{y_1,...,y_n\}$ respectively and let B denote the restriction of f to $E\times F$.

Choosing bases for the finite dimensional space $E,\ F$ and expanding the bilinear form B relative to these bases yields a representation for B of the form $B(x,\ y)=\sum_{j=1}^{m}\pi_j(x)\tau_j(y)$ where $\pi_j\in E^*$ and $\tau_j\in F^*$. See [7].

Now we may extend the domain of π_j,τ_j to all of $X,\ Y$ respectively in the following manner: choose algebraic complements , P, Q for E, F respectively, so that $X=E\oplus P$ and $Y=F\oplus Q$. Then, if $x=x_1+x_2\in X$ with $x_1\in E,\ x_2\in P$, let $\pi_j(x)=\pi_j(x_1)$. The functionals τ_j are defined in Y in a similar way.

We now consider B as a bilinear form on $X\times Y$ by using the representation of B given above. Now f and B may be different bilinear forms on

$X \times Y$, but they agree on $E \times F$. Thus we have

$$\begin{aligned} u(f) &= \sum_{i=1}^{n} f(x_i, y_i) \\ &= \sum_{i=1}^{n} B(x_i, y_i) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \pi_j(x_i)\tau_j(y_i) \\ &= \sum_{j=1}^{m} \pi_j \left(\sum_{i=1}^{n} \tau_j(y_i)x_i \right) = 0 \qquad (\text{by } (iv)). \end{aligned}$$

Thus $u(f) = 0, \ \forall\, f \in B(X \times Y)$. □

Theorem 2.2.3. *Let X and Y be finite dimensional vector spaces. Then $X^* \otimes Y^* \approx (X \otimes Y)^*$ via the isomorphism $\tau : X^* \otimes Y^* \to (X \otimes Y)^*$ defined by $\tau(\phi \otimes \varphi)(x \otimes y) = \phi(x)\varphi(y)$.*

Proof. (From [23]) We need to show that τ is an isomorphism. Let us fix $\phi \in X^*$ and $\varphi \in Y^*$, and consider the map $\sigma_{\phi,\varphi} : X \times Y \to \mathbf{K}$ defined by

$$\sigma_{\phi,\varphi}(x, y) = \phi(x)\varphi(y).$$

This map is bilinear, and so by the universal property of tensor products implies that there exists a unique linear map $\hat{\sigma}_{\phi,\varphi} : X \otimes Y \to \mathbf{K}$ for which

$$\hat{\sigma}_{\phi,\varphi}(x \otimes y) = \sigma_{\phi,\varphi}(x, y) = \phi(x)\varphi(y).$$

Thus $\hat{\sigma}_{\phi,\varphi} \in (X \otimes Y)^*$. Now we define a map $\sigma : X^* \times Y^* \to (X \otimes Y)^*$ by

$$\sigma(\phi, \varphi) = \hat{\sigma}_{\phi,\varphi}.$$

This map is also bilinear. For instance,

$$\begin{aligned}\sigma(\alpha\phi+\beta\varphi,\,\psi)(x\otimes y) &= (\alpha\phi+\beta\varphi)(x)\psi(y)\\ &= \alpha\phi(x)\psi(y)+\beta\varphi(x)\psi(y)\\ &= \alpha\sigma(\phi,\,\psi)(x,\,y)+\beta\sigma(\varphi,\,\psi)(x,\,y)\\ &= [\alpha\sigma(\phi,\psi)+\beta\sigma(\varphi,\psi)](x,\,y)\end{aligned}$$

and so

$$\sigma(\alpha\phi+\beta\varphi,\,\psi)=\alpha\sigma(\phi,\,\psi)+\beta\sigma(\varphi,\,\psi)$$

which shows that σ is linear in its first coordinate. Similarly, it's linear in it's second coordinate and hence bilinear. Therefore, the universal property implies that there exists a unique linear map $\tau: X^*\otimes Y^*\to (X\otimes Y)^*$ for which

$$\tau(\phi\otimes\varphi)=\sigma(\phi,\,\varphi)$$

that is,

$$\tau(\phi\otimes\varphi)(x\otimes y)=\sigma(\phi\otimes\varphi)(x\otimes y)=\hat{\sigma}_{\phi\otimes\varphi}(x\otimes y)=\phi(x)\varphi(y).$$

To show that τ is an isomorphism, let $\mathfrak{B}=\{\mathbf{b}_i\}$ be a basis for X, with the dual basis $\mathfrak{B}'=\{\varphi_i\}$, and let $\mathfrak{C}=\{\mathbf{c}_i\}$ be a basis for Y, with the dual basis $\mathfrak{C}'=\{\psi_i\}$. Then

$$\tau(\varphi_i\otimes\psi_j)(b_x\otimes c_y)=\varphi_i(b_x)\psi_j(c_y)=\delta_{i,x}\delta_{j,y}=\delta_{(i,j)(x,y)}$$

and so $\tau(\varphi_i\otimes\psi_j)\in(X\otimes Y)^*$ is a dual basis vector to the basis $\{b_x\otimes c_y\}$ for $X\otimes Y$. Thus, τ takes the basis $\{\varphi_i\otimes\psi_j\}$ for $X^*\otimes Y^*$ to the basis

$\{\tau(\varphi_i \otimes \psi_j)\}$. Hence τ is an isomorphism. □

2.3 Tensor products of Hilbert spaces

Definition 2.3.1. Let H, K be Hilbert spaces. The pair $(H \otimes K, \vartheta)$, where $\vartheta : H \times K \rightarrow H \otimes K$ is a bilinear operator acting by the rule $(x, y) \mapsto x \otimes y$, is called the ***Hilbert tensor product***.

Theorem 2.3.2. *Let H and K be Hilbert spaces. Then there is a unique inner product $\langle .,. \rangle$ on $H \otimes K$ such that*

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle \quad (x, x' \in H, \; y, y' \in K).$$

Proof. (From [13]) If τ and ρ are conjugate-linear maps from H and K, respectively, to $\mathbb{C}$, then there is a unique conjugate-linear map $\tau \otimes \rho$ from $H \otimes K$ to $\mathbb{C}$ such that

$$(\tau \otimes \rho)(x \otimes y) = \tau(x)\rho(y) \;\; (x \in H, \; y \in K).$$

If x is an element of a Hilbert space, let τ_x be the conjugate-linear functional defined by setting $\tau_x(y) = \langle x, y \rangle$.
Let X be the vector space of all conjugate-linear functionals on $H \otimes K$. The map $H \times K \rightarrow X$, $(x, y) \mapsto \tau_x \otimes \tau_y$, is bilinear, so there is a unique linear map $\pi : H \times K \rightarrow X$ such that

$$\pi(x \otimes y) = \tau_x \otimes \tau_y, \;\; \forall \, x, y.$$

The map $\langle .,. \rangle : (H \otimes K) \rightarrow \mathbb{C}, (z, z') \mapsto \pi(z)(z')$ is a sesquilinear form on

$H \otimes K$ such that

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle \quad (x, x' \in H, \ y, y' \in K).$$

If $z \in H \otimes K$, then $z = \sum_{j=1}^{n} x_j \otimes y_j$ for some $x_1, ..., x_n \in H$ and $y_1, ..., y_n \in K$. Let $e_1, ..., e_m$ be an orthonormal basis for the linear span of $y_1, ..., y_n$. Then $z = \Sigma_{j=1}^{m} x'_j \otimes e_j$ for some $x'_1, ..., x'_m \in H$, and therefore

$$\begin{aligned}\langle z, z \rangle &= \Sigma_{i,j=1}^{m} \langle x'_i \otimes e_i, x'_j \otimes e_j \rangle \\ &= \Sigma_{i,j=1}^{m} \langle x'_i, x'_j \rangle \langle e_i, e_j \rangle \\ &= \Sigma_{j=1}^{m} \|x'_j\|^2.\end{aligned}$$

Thus $\langle .,. \rangle$ is positive, and if $\langle z, z \rangle = 0$ then $x'_j = 0$ for $j = 1, ..., m$. So $z = 0$. Therefore, $\langle .,. \rangle$ is an inner product. □

Theorem 2.3.3. *Let H and K be Hilbert spaces and $H \otimes K$ be the tensor product between H and K such that $x \otimes y$ is an element of $H \otimes K$ where $x \in H$ and $y \in K$. Then $\|x \otimes y\| = \|x\|\|y\|$.*

Note*: This theorem was given in [13] as a note thus we have provided its proof below.*

Proof. We prove that $\|x \otimes y\|$ satisfy all the axioms of a norm.

(i) Clearly, $\|x \otimes y\| \geq 0$ and $\|x \otimes y\| = 0 \iff x \otimes y = 0$

(ii) $\|\alpha(x \otimes y)\| = |\alpha|\|x\|\|y\|, \ \forall \, x \in H, \ y \in K$ and $\alpha \in \mathbf{K}$.

We note that,

$$\begin{aligned}\|x \otimes y\|^2 &= \langle x \otimes y, x \otimes y\rangle \\ &= \langle x, x\rangle\langle y, y\rangle \\ &= \|x\|^2\|y\|^2\end{aligned}$$

and by algebraic properties of tensor products we have
$\alpha(x \otimes y) = (\alpha x \otimes y) = (x \otimes \alpha y)$. So,

$$\begin{aligned}\|\alpha(x \otimes y)\|^2 &= \langle \alpha x \otimes y, \alpha x \otimes y\rangle \\ &= \langle \alpha x, \alpha x\rangle\langle y, y\rangle \\ &= |\alpha|^2\|x\|^2\|y\|^2 \\ &= |\alpha|^2\|x \otimes y\|^2.\end{aligned}$$

Therefore, $\|\alpha(x \otimes y)\|^2 = |\alpha|^2\|x \otimes y\|^2$.
Taking square root of both sides we have, $\|\alpha(x \otimes y)\| = |\alpha|\|x\|\|y\|$.
(iii) $\forall\, x_1,\ x_2 \in H$ and $y_1,\ y_2 \in K$ we have

$$\|(x_1 \otimes y_1) + (x_2 \otimes y_2)\| \leq \|x_1 \otimes y_1\| + \|x_2 \otimes y_2\|.$$

Now, $\|(x_1 \otimes y_1) + (x_2 \otimes y_2)\|^2 = \langle x_1 \otimes y_1 + x_2 \otimes y_2, x_1 \otimes y_1 + x_2 \otimes y_2\rangle$

$$\begin{aligned}&= \langle x_1 \otimes y_1, x_1 \otimes y_1\rangle + \langle x_1 \otimes y_1, x_2 \otimes y_2\rangle + \langle x_2 \otimes y_2, x_1 \otimes y_1\rangle + \langle x_2 \otimes y_2, x_2 \otimes y_2\rangle \\ &= \langle x_1, x_1\rangle\langle y_1, y_1\rangle + \langle x_1, x_2\rangle\langle y_1, y_2\rangle + \langle x_2, x_1\rangle\langle y_2, y_1\rangle + \langle x_2, x_2\rangle\langle y_2, y_2\rangle \\ &= \|x_1\|^2\|y_1\|^2 + \|x_2\|^2\|y_2\|^2 + \langle x_1, x_2\rangle\langle y_1, y_2\rangle + (\overline{\langle x_1, x_2\rangle})(\overline{\langle y_1, y_2\rangle}) \\ &= \|x_1\|^2\|y_1\|^2 + \|x_2\|^2\|y_2\|^2 + \mathbf{2Re}\langle x_1, x_2\rangle\langle y_1, y_2\rangle.\end{aligned}$$

So by Cauchy-Schwarz inequality,

$$\begin{aligned}\|(x_1 \otimes y_1) + (x_2 \otimes y_2)\|^2 &\leq \|x_1\|^2\|y_1\|^2 + \|x_2\|^2\|y_2\|^2 + 2\|x_1\|\|x_2\|\|y_1\|\|y_2\| \\ &= (\|x_1\|\|x_2\| + \|y_1\|\|y_2\|)^2.\end{aligned}$$

i.e $(\|x_1 \otimes y_1) + (x_2 \otimes y_2)\|^2 \leq (\|x_1\|\|x_2\| + \|y_1\|\|y_2\|)^2$.
Taking square roots on both sides we obtain

$$\|(x_1 \otimes y_1) + (x_2 \otimes y_2)\| \leq \|x_1\|\|x_2\| + \|y_1\|\|y_2\|$$

Therefore, $\|(x_1 \otimes y_1) + (x_2 \otimes y_2)\| \leq \|x_1 \otimes y_1\| + \|x_2 \otimes y_2\|$. □

Remark 2.3.4. If H and K are as in Theorem (2.3.2), we shall regard $H \otimes K$ as a pre-Hilbert space with the above inner product. The Hilbert space completion of $H \otimes K$ is denoted by $H \hat{\otimes} K$, and called the the Hilbert space tensor product of H and K.

Lemma 2.3.5. *Let H, K be Hilbert spaces and suppose that $u \in B(H)$ and $v \in B(K)$. Then there is a unique operator $(u \hat{\otimes} v \in B(H \hat{\otimes} K)$ such that*

$$(u \hat{\otimes} v)(x \otimes y) = u(x) \otimes v(y) \quad (x \in H, y \in K).$$

Moreover, $\|u \hat{\otimes} v\| = \|u\|\|v\|$.

Proof. (From [13]) The map $(u, v) \mapsto u \otimes v$ is bilinear, so to show that $u \otimes v : H \otimes K \mapsto H \otimes K$ is bounded, we may assume that u and v are unitaries [13], since the unitaries span the C*-algebras $B(H)$ and $B(K)$. If $z \in H \otimes K$, then we may write $z = \Sigma_{i=1}^{n} x_i \otimes y_i$ where $y_1, ..., y_n$ are orthogonal. Hence,

$$\begin{aligned}
\|(u \otimes v)(z)\|^2 &= \left\| \sum_{i=1}^{n} u(x_i) \otimes v(y_i) \right\|^2 \\
&= \sum_{i=1}^{n} \|u(x_i) \otimes v(y_i)\|^2 \text{ (since } v(y_1), ..., v(y_n) \text{ are orthogonal)} \\
&= \sum_{i=1}^{n} \|(x_i)\|^2 \|(y_i)\|^2 \\
&= \|z\|^2.
\end{aligned}$$

Consequently, $\|u \otimes v\| = 1$.

Thus, for all operators u, v on H, K respectively, the linear map $u \otimes v$ is bounded on $H \otimes K$ and hence has an extension to a bounded linear map $u \hat{\otimes} v$ on $H \hat{\otimes} K$.

The maps $B(H) \to B(H\hat{\otimes}K)$ defined by $u \mapsto u \otimes id_k$ (where id_K is identity in K) and $B(K) \to B(H\hat{\otimes}K)$ defined by $v \mapsto id_k \otimes v$ (where id_H is identity in H) are *-homomorphisms and therefore isometric. Hence $\|u\hat{\otimes} id\| = \|u\|$ and $\|id\hat{\otimes} v\| = \|v\|$. Therefore,

$$\begin{aligned}
\|u\hat{\otimes}v\| &= \|(u\hat{\otimes}id)(id\hat{\otimes}v)\| \\
&\leq \|u\hat{\otimes}id\|\|id\hat{\otimes}v\| \\
&= \|u\|\|v\|.
\end{aligned}$$

If ϵ is a sufficiently small positive number, and if $u, v \neq 0$, then there are unit vectors x and y such that

$$\|u(x)\| > \|u\| - \epsilon > 0$$

and

$$\|v(y)\| > \|v\| - \epsilon > 0.$$

Hence,

$$\begin{aligned} \|(u\hat{\otimes}v)(x\otimes y)\| &= \|u(x)\|\|v(y)\| \\ &> (\|u\|-\epsilon)(\|v\|-\epsilon) \\ \Rightarrow \|u\hat{\otimes}v\| &> (\|u\|-\epsilon)(\|v\|-\epsilon). \end{aligned}$$

As $\epsilon \to 0$ we obtain $\|u\hat{\otimes}v\| \geq \|u\|\|v\|$. □

Theorem 2.3.6. *Let $T : H_1 \to H_2$ and $S : K_1 \to K_2$ be bounded operators between Hilbert spaces. Then there exists a unique bounded operator $T\hat{\otimes}S : H_1\hat{\otimes}K_1 \to H_2\hat{\otimes}K_2$ such that $(T\hat{\otimes}S)(x\otimes y) = T(x)\otimes S(y)\ \forall\ x \in H_1$ and $\forall\ y \in K_1$. Moreover, $\|T\hat{\otimes}S\| = \|T\|\|S\|$. (original proof in [8])*

Proof. Since the algebraic tensor product $H_1 \otimes K_1$ is dense in $H_2 \otimes K_2$, there may exist at most one bounded operator satisfying the desired condition. Further, by the identity $\|x \otimes y\| = \|x\|\|y\|$ for the norm in the Hilbert tensor product, for this hypothetical operator $T \otimes S$ we would have from the definition of norm,

$$\begin{aligned} \|T\hat{\otimes}S\| &\geq \sup\{\|(T\hat{\otimes}S)(x\otimes y)\| : x \in B_{H_1}, y \in B_{K_1}\} \\ &= \sup\{\|T(x)\|\|S(y)\| : x \in B_{H_1}, y \in B_{K_1}\} \\ &= \|T\|\|S\|. \end{aligned}$$

We must show that this operator indeed exists and $\|T\hat{\otimes}S\| \leq \|T\|\|S\|$. We state the following lemma which gives a solution.

Lemma 2.3.7. *There exists a bounded operator* $T\hat{\otimes}1 : H_1\hat{\otimes}K_1 \to H_2\hat{\otimes}K_1$ *such that* $(T\hat{\otimes}1)(x \otimes y) = T(x) \otimes y$ *for all* $x \in H_1$ *and* $y \in K_1$. *Moreover,* $\|T\hat{\otimes}1\| \leq \|T\|$.

Proof. Consider the bilinear operator $\mathbf{R} : H_1 \times K_1 \to H_2\hat{\otimes}K_1 : (x, y) \mapsto T(x) \otimes y$. Suppose $R' : H_1 \otimes K_1 \to H_2\hat{\otimes}K_1$. Take $u \in H_1 \times K_1$, and a representation $u = \sum_{i=1}^{n} x_i \otimes y_i$. Without loss of generality, we can assume that the system $y_1, ..., y_n \in K_1$ is orthonormal.

The system $x_1 \otimes y_1, ..., x_n \otimes y_n \in H_1 \otimes K_1$ and $T(x_1) \otimes y_1, ..., T(x_n) \otimes y_n \in H_1 \otimes K_1$ is orthogonal in $H_2\hat{\otimes}K_1$. Therefore, using the Pythagorean equality we have

$$\begin{aligned}
\|R'(u)\|^2 &= \left\| \sum_{i=1}^{n} T(x_i) \otimes y_i \right\|^2 \\
&= \sum_{i=1}^{n} \|T(x_i) \otimes y_i\|^2 \\
&= \sum_{i=1}^{n} \|T(x_i)\|^2 \\
&\leq \|T\|^2 \sum_{i=1}^{n} \|x_i\|^2 \\
&= \sum_{i=1}^{n} \|x_i \otimes y_i\|^2 \\
&= \|T\|^2 \|u\|^2.
\end{aligned}$$

Thus, R' is a bounded operator from the pre-Hilbert space $H_1 \otimes K_1$ to the Hilbert space $H_2\hat{\otimes}K_1$, and $\|\mathbf{R}\| \leq \|T\|$. Extending this by continuity to the whole $H_1\hat{\otimes}K_1$, we obtain the operator $T\hat{\otimes}1$ with required properties.

□

Now we complete the proof of the theorem. Similarly to the lemma, we obtain a bounded linear operator $1\hat{\otimes}S : H_2\hat{\otimes}K_1 \to H_2\hat{\otimes}K_2$ such that $(1\hat{\otimes}S)(x \otimes y) = x \otimes S(y)$ for all $x \in H_2$ and $y \in K_1$ and $\|1\hat{\otimes}S\| \leq \|T\|$. Put $T\hat{\otimes}S := (1\hat{\otimes}S)(T\hat{\otimes}1) : H_1\hat{\otimes}K_1 \to H_2\hat{\otimes}K_2$.
By the multiplicative inequality for the operator norm, this operator is bounded and $\|T\hat{\otimes}S\| \leq \|T\|\|S\|$ but from the definition, $\|T\hat{\otimes}S\| \geq \|T\|\|S\|$ so $\|T\hat{\otimes}S\| = \|T\|\|S\|$. □

2.4 Tensor products of operators

Let X, X', Y and Y' be vector spaces over the same field and $T : X \to X'$, $S : Y \to Y'$ be operators. Then there is a unique linear operator

$$T \odot S : X \otimes Y \to X' \otimes Y'$$

defined by

$$(T \odot S)(x \otimes y) = T(x) \otimes S(y), \;\; \forall\, x \in X,\; y \in Y. \tag{2.4.1}$$

The function $f : X \times Y \to X' \otimes Y'$ defined by $f(x, y) = T(x) \otimes S(y)$ is bilinear and so by the universal property of tensor products, there exists a unique linear operator $T \odot S$ for which equation (2.4.1) holds. The map $T \odot S$ is called the **tensor product** of T and S.
Thus, we have a map $\tau : \mathfrak{L}(X, Y) \times \mathfrak{L}(X', Y') \to \mathfrak{L}(X \otimes Y, X' \otimes Y')$ defined by

$$\tau(T, S) = T \odot S. \tag{2.4.2}$$

This map is bilinear so there is a unique linear operator

$$\theta : \mathfrak{L}(X, Y) \otimes \mathfrak{L}(X', Y') \to \mathfrak{L}(X \otimes Y, X' \otimes Y')$$

satisfying

$$\theta(T \otimes S) = T \odot S.$$

Lemma 2.4.1. *Let θ be as defined above, then θ is injective.*

Proof. (From [23]) First we note that any nonzero vector $\eta \in \mathfrak{L}(X, Y) \otimes \mathfrak{L}(X', Y')$ has the form

$$\eta = \sum_{i=1}^{n} T_i \otimes S_i$$

where both $T_i's$ and $S_i's$ are linearly independent. It suffices to show that $ker(\theta) = \{0\}$.

Suppose

$$\theta(\eta) = \theta\left(\sum_{i=1}^{n} T_i \otimes S_i\right) = 0.$$

Then

$$\sum_{i=1}^{n} T_i(x) \otimes S_i(y) = 0, \;\; \forall\, x \in X, \; y \in Y. \qquad (2.4.3)$$

Let us choose $x \in X$ so that $T_i(x) \neq 0$, and suppose that $T_1(x), ..., T_m(x)$ is a maximal linearly independent set among $T_1(x), ..., T_n(x)$. Thus, for scalars $r_{u,j}$,

$$T_u(x) = \sum_{j=1}^{m} r_{u,j} T_j(x) \;\; for \; u = m+1, ..., n.$$

Hence equation (2.4.3) gives

$$\begin{aligned}
0 &= \sum_{i=1}^{m} T_i \otimes S_i(y) + \sum_{u=m+1}^{n} \left(\sum_{j=1}^{m} r_{u,j} T_j(x) \right) \otimes S_u(y) \\
&= \sum_{i=1}^{m} T_i(x) \otimes S_i(y) + \sum_{j=1}^{m} T_j(x) \otimes \left(\sum_{u=m+1}^{n} r_{u,j} S_u(y) \right) \\
&= \sum_{i=1}^{m} T_i(x) \otimes \left(S_i(y) + \sum_{u=m+1}^{n} r_{u,j} S_u(y) \right)
\end{aligned}$$

and since $T_1(x), ..., T_m(x)$ are linearly independent, we must have

$$S_i(y) + \sum_{u=m+1}^{n} r_{u,j} S_u(y) = 0, \;\; \forall\, i = 1, ..., m \;\; \textit{and} \;\; \forall\, y \in Y.$$

Hence

$$S_i + \sum_{u=m+1}^{n} r_{u,j} S_u = 0$$

which contradicts the fact that $S_i^{\prime}s$ are linearly independent.

Hence $\theta(\eta) \neq 0$ and so θ is injective. □

Remark 2.4.2. We note that if all vector spaces are finite dimensional, then θ is also surjective, and hence is an isomorphism [23].

Theorem 2.4.3. *Let* $T \in \mathfrak{L}(X, X')$ *and* $S \in \mathfrak{L}(Y, Y')$. *There is a unique linear operator* $T \odot S \in \mathfrak{L}(X \otimes Y, X' \otimes Y')$, *called the tensor product of* T *and* S *satisfying* $(T \odot S)(x \otimes y) = T(x) \otimes S(y)$. *Moreover, there is a unique injective linear operator*

$$\theta : \mathfrak{L}(X, Y) \otimes \mathfrak{L}(X', Y') \to \mathfrak{L}(X \otimes Y, X' \otimes Y')$$

satisfying

$$\theta(T \otimes S) = T \odot S.$$

See [23] for proof.

Properties of the operator $T \odot S$.

The operator $T \odot S$ is both linear and bounded.

(i) **Linearity.**

The map $T \odot S : X \otimes Y \to X' \otimes Y'$ is defined by

$$T \odot S \left(\sum_{i=1}^{n} x_i \otimes y_i \right) = \sum_{i=1}^{n} T(x_i) \otimes S(y_i), \quad \forall\, x \in X,\ y \in Y.$$

Let $\alpha,\ \beta \in \mathbf{K}$ and $\sum_{i=1}^{n} x_i \otimes y_i,\ \sum_{i=1}^{n} x_i' \otimes y_i' \in X \otimes Y$. Then

$$\begin{aligned} T \odot S \left(\alpha \sum_{i=1}^{n} x_i \otimes y_i + \beta \sum_{i=1}^{n} x_i' \otimes y_i' \right) &= T \odot S \left(\alpha \sum_{i=1}^{n} x_i \otimes y_i \right) + T \odot S \left(\beta \sum_{i=1}^{n} x_i' \otimes y_i' \right) \\ &= \alpha \sum_{i=1}^{n} T(x_i) \otimes S(y_i) + \beta \sum_{i=1}^{n} T(x_i') \otimes S(y_i') \\ &= \alpha T \odot S \left(\sum_{i=1}^{n} x_i \otimes y_i \right) + \beta T \odot S \left(\sum_{i=1}^{n} x_i' \otimes y_i' \right). \end{aligned}$$

(ii) **Boundedness.**

We need to show that there exists a constant $M > 0$ such that,

$$\| T \odot S \left(\sum_{i=1}^{n} x_i \otimes y_i \right) \| \leq M \left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|.$$

Now,

$$\begin{aligned}
\left\|T\odot S\left(\sum_{i=1}^{n}x_i\otimes y_i\right)\right\| &= \left|\sum_{i=1}^{n}T(x_i)\otimes S(y_i)\right| \\
&\leq \left\|\sum_{i=1}^{n}T(x_i)\otimes S(y_i)\right\| \\
&\leq \sum_{i=1}^{n}\|T(x_i)\|\|S(y_i)\| \\
&\leq \sum_{i=1}^{n}\|T\|\|x_i\|\|S\|\|y_i\| \\
&\leq \|T\|\|S\|\sum_{i=1}^{n}\|x_i\|\|y_i\| \\
&\leq \|T\|\|S\|\|\sum_{i=1}^{n}x_i\otimes y_i\|.
\end{aligned}$$

(iii) **The norm property of $T\odot S$**

By definition,

$$\begin{aligned}
\|T\odot S\| &= \sup_{\|\sum_{i=1}^{n}x_i\otimes y_i\|=1}\left\|T\odot S\left(\sum_{i=1}^{n}x_i\otimes y_i\right)\right\| \\
&\leq \sup_{\|\sum_{i=1}^{n}x_i\otimes y_i\|=1}\frac{\|T\|\|S\|\|(\sum_{i=1}^{n}x_i\otimes y_i)\|}{\|\sum_{i=1}^{n}x_i\otimes y_i\|} \\
&= \|T\|\|S\|.
\end{aligned}$$

Therefore,

$$\|T\odot S\|\leq\|T\|\|S\|. \tag{2.4.4}$$

On the other hand,

$\|T\odot S\|=\sup\frac{\|T\odot S(\sum_{i=1}^{n}x_i\otimes y_i)\|}{\|\sum_{i=1}^{n}x_i\otimes y_i\|}$, $\forall\ \sum_{i=1}^{n}x_i\otimes y_i\in X\otimes Y$ and $\sum_{i=1}^{n}x_i\otimes$

$y_i \neq 0$. It follows that

$$\|T \odot S\| \geq \frac{\|T \odot S(\sum_{i=1}^{n} x_i \otimes y_i)\|}{\|\sum_{i=1}^{n} x_i \otimes y_i\|}$$

$\forall \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$ and $\sum_{i=1}^{n} x_i \otimes y_i \neq 0$.

Hence

$$\|T \odot S\| \geq \|T\|\|S\|. \tag{2.4.5}$$

So by equations (2.4.4) and (2.4.5), we obtain

$$\|T \odot S\| = \|T\|\|S\|.$$

2.5 Tensor product of normed spaces

Like in vector spaces, maps between normed spaces are bilinear. If X, Y, Z are normed spaces over a field $\mathbf{K}$, then $B(X, Y; Z)$ is the set of bounded linear mappings from $X \times Y$ to Z.

Definition 2.5.1. Let X, Y be normed spaces over $\mathbf{K}$ with dual spaces X', Y'. Given $x \in X$ and $y \in Y$, let $x \otimes y$ be the element of $B(X', Y'; \mathbf{K})$ defined by

$$x \otimes y = f(x)g(y) \quad (f \in X',\, g \in Y').$$

The algebraic tensor product of X and Y is defined to be the linear span of $\{x \otimes y : x \in X,\ y \in Y\} \in B(X', Y'; \mathbf{K})$.

2.5.1 Projective tensor norm

Definition 2.5.2. Given normed spaces X, Y, the **projective tensor norm** p on $X \otimes Y$ is defined by

$$p(u) = \inf\{\sum_i \|x_i\|\|y_i\| : u = \sum_i x_i \otimes y_i\}$$

where the infimum is taken over all (finite) representations of u.

Lemma 2.5.3. *The projective tensor norm p is a norm on $X \otimes Y$ and*
(i) $p(u) \geq w(u)$ $(u \in X \otimes Y)$, w is weak tensor norm,
(ii) $p(x \otimes y) = \|x\|\|y\|$, $x \in X, y \in Y$. For proof see [12].

Remark 2.5.4. The completion of $(X \otimes Y, p)$ is called the projective tensor product of X and Y and is denoted by $X \otimes_p Y$.

2.5.2 Haagerup norm

The Haagerup norm is a very important operator space cross-norm. The motivation was the consideration of operators of the form $\phi(a) = \Sigma_{i=1}^n u_i a v_i$ for $a \in A$ where $u_1, ..., u_n, v_1, ..., v_n$ are some fixed elements in A [1, 6]. These operators result from the action of $\Sigma_{i=1}^n u_i \otimes v_i \in A \otimes A^{op}$ on A(where A^{op} is the C*-algebra A with the reversed product). If $A \subseteq B(H)$ then for

$\xi, \eta \in H$ where $\|\xi\| = 1,\ \|\eta\| = 1$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}
|\langle \phi(a)\xi, \eta\rangle| &= \left|\left\langle \sum_{i=1}^{n} u_i a v_i \xi, \eta \right\rangle\right| \\
&= \left|\left\langle \sum_{i=1}^{n} a v_i \xi, u_i^* \eta \right\rangle\right| \\
&\leq \left(\sum_{i=1}^{n} \|a v_i \xi\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \|u_i^* \eta\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Further, $\|a v_i \xi\| \leq \|a\| \|v_i \xi\|$ and

$$\begin{aligned}
\sum_{i=1}^{n} \|v_i \xi\|^2 &= \sum_{i=1}^{n} \langle v_i \xi, v_i \xi \rangle \\
&= \sum_{i=1}^{n} \langle \xi, v_i^* v_i \xi \rangle \\
&\leq \left\| \sum_{i=1}^{n} v_i^* v_i \right\| \|\xi\|^2.
\end{aligned}$$

Similarly,

$$\sum_{i=1}^{n} \|u_i^* \eta\|^2 \leq \left\| \sum_{i=1}^{n} u_i u_i^* \right\| \|\eta\|^2.$$

So,

$$|\langle \phi(a)\xi, \eta\rangle| \leq \|a\| \left\| \sum_{i=1}^{n} u_i u_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^{n} \|v_i^* v_i \right\|^{\frac{1}{2}} \|\xi\| \|\eta\|.$$

Hence, $\|\phi\| \leq \| \sum_{i=1}^{n} u_i u_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^{n} \|v_i^* v_i\|^{\frac{1}{2}}$.

For the reverse inclusion, we may also allow infinite (countable) sequences of u_i and v_i provided that $\sum_{i=1}^{n} u_i u_i^*$ and $\sum_{i=1}^{n} v_i^* v_i$ are norm convergent.

Therefore, the natural definition following from these considerations is

$$\|t\|_h = \inf\left\{\left\|\sum_{i=1}^{n} u_i u_i^*\right\|^{\frac{1}{2}} \left\|\sum_{i=1}^{n} v_i^* v_i\right\|^{\frac{1}{2}} : n \in \mathbf{N},\ t = \sum_{i=1}^{n} u_i \otimes v_i \in A \otimes B\right\}.$$

We show that Haagerup norm is actually a norm. To do this, we show that it satisfies the properties of a norm.

(i) We note that $\|t\|_h = \inf\{\|\sum_{i=1}^{n} a_i \otimes b_i\|\}$. So clearly, $\|t\|_h \geq 0$ and $\|t\|_h = 0$ if and only if $t = 0$.

(ii) We show that $\forall\, \alpha \in \mathbf{K}$, $\|\alpha t\|_h = |\alpha|\|t\|_h$.

Now,

$$\begin{aligned}
\|\alpha t\|_h &= \inf\left\{\left\|\sum_{i=1}^{n}(\alpha a_i)(\alpha a_i)^*\right\|\left\|\sum_{i=1}^{n} b_i^* b_i\right\|\right\}^{\frac{1}{2}} \\
&= \inf\left\{\left\|\sum_{i=1}^{n} a_i a_i^*\right\|\left\|\sum_{i=1}^{n}(\alpha b_i^*)(\alpha b_i)\right\|\right\}^{\frac{1}{2}} \\
&= \inf\left\{|\alpha|^2\left\|\sum_{i=1}^{n} a_i a_i^*\right\|\left\|\sum_{i=1}^{n} b_i^* b_i\right\|\right\}^{\frac{1}{2}} \\
&= |\alpha| \inf\left\{\left\|\sum_{i=1}^{n} a_i a_i^*\right\|\left\|\sum_{i=1}^{n} b_i^* b_i\right\|\right\}^{\frac{1}{2}} \\
&= |\alpha|\|t\|_h.
\end{aligned}$$

(iii) If $t,\ t' \in B(H) \otimes B(H)$ then $\forall\, t = \sum_{i=1}^{n} a_{i1} \otimes b_{i1}$ and $t' = \sum_{i=1}^{n} a_{i2} \otimes b_{i2}$, $\|t + t'\|_h \leq \|t\|_h + \|t'\|_h$.

Now,

$$\begin{aligned}
\|t+t'\|_h &= \inf\left\{\left\|\left(\sum_{i=1}^{n} a_{i1}\otimes b_{i1}\right)+\left(\sum_{i=1}^{n} a_{i2}\otimes b_{i2}\right)\right\|\right\} \\
&\leq \inf\left\{\left\|\sum_{i=1}^{n} a_{i1}\otimes b_{i1}\right\|+\left\|\sum_{i=1}^{n} a_{i2}\otimes b_{i2}\right\|\right\} \\
&\leq \inf\left\{\left\|\sum_{i=1}^{n} a_{i1}\otimes b_{i1}\right\|\right\}+\inf\left\{\left\|\sum_{i=1}^{n} a_{i2}\otimes b_{i2}\right\|\right\} \\
&= \|t\|_h+\|t'\|_h.
\end{aligned}$$

(iv) If $t,\ t' \in B(H)\otimes B(H)$ then $\|tt'\|_h \leq \|t\|_h\|t'\|_h$.

Now,

$$\begin{aligned}
\|tt'\|_h &= \inf\left\{\left\|\left(\sum_{i=1}^{n} a_{i1}\otimes b_{i1}\right)\left(\sum_{i=1}^{n} a_{i2}\otimes b_{i2}\right)\right\|\right\} \\
&\leq \inf\left\{\left\|\sum_{i=1}^{n} a_{i1}\otimes b_{i1}\right\|\left\|\sum_{i=1}^{n} a_{i2}\otimes b_{i2}\right\|\right\} \\
&\leq \inf\left\{\left\|\sum_{i=1}^{n} a_{i1}\otimes b_{i1}\right\|\right\}\inf\left\{\left\|\sum_{i=1}^{n} a_{i2}\otimes b_{i2}\right\|\right\} \\
&= \|t\|_h\|t'\|_h.
\end{aligned}$$

The upper bound therefore, is given by $\|T\| \leq \|\sum_{i=1}^{n} a_i\otimes b_i\|$ in terms of the Haagerup norm $\|.\|_h$ on $B(H)\otimes B(H)$. The equality holds when the operators $a_ia_i^*$ commute and $b_i^*b_i$ commute [27].

In the next theorem we use the following notations. For $\eta,\ \xi\in H$ we use $\eta\otimes\xi^*$ for the rank one operator on H with $(\eta\otimes\xi^*)(\theta)=\langle\theta,\xi\rangle\eta$.

Theorem 2.5.5. *For $T\in \varepsilon\ell(B(H))$, $Tx=\sum_{i=1}^{n} a_ixb_i$, we have*

$$\|T\| = \sup_{p_1,p_2}\left\|\sum_{i=1}^{n}(p_1a_i)\otimes(b_ip_2)\right\|$$

where $p_1, p_2 \in B(H)$ *are rank one projections* $(p_i^2 = p_i = p_i^* \quad (i = 1, 2))$ *(original proof in [27]).*

Proof. Let $p_1 = \xi \otimes \xi^*$ and $p_2 = \eta \otimes \eta^*$ be one dimensional projections (where η, $\xi \in H$ are unit vectors). We look at the operator

$$T_{p_1,p_2}(x) = \sum_{i=1}^{n} (p_1 a_i) \otimes (b_i p_2),$$

an operator with a one dimensional range. Specifically it is the operator

$$x \mapsto \langle (Tx)\eta, \xi \rangle \xi \otimes \eta^*$$

and thus a linear functional.

For this operator, $(p_1 a_i)(p_1 a_j)^*$ are commuting and so are $(b_i p_2)^*(b_j p_2)$. Hence

$$\|T_{p_1,p_2}\| = \left\| \sum_{i=1}^{n} (p_1 a_i) \otimes (b_i p_2) \right\|_h .$$

Alternatively, the norm of a linear functional is the same as its completely bounded norm, hence $\|T_{p_1,p_2}\| = \|T_{p_1,p_2}\|_{cb}$ = the Haagerup tensor norm for T of the form $T = \sum_{i=1}^{n} a_i \otimes b_i$.

Clearly,

$$\begin{aligned} \|T\| &= \sup\{\|Tx\| : x \in B(H),\ \|x\| \leq 1\} \\ &= \sup\{\mathbf{R}\langle (Tx)\eta, \xi\rangle : x \in B(H),\ \|x\| \leq 1,\ \eta,\ \xi \in H, \|\xi\| = \|\eta\| = 1\} \\ &= \sup\{\mathbf{R}\langle (T_{p_1,p_2}x)\eta, \xi\rangle : x \in B(H),\ \|x\| \leq 1,\ \eta,\ \xi \in H, \|\xi\| = \|\eta\| = 1\} \\ &= \sup_{p_1,p_2} \|T_{p_1,p_2}\|. \end{aligned}$$

Since $\|T_{p_1,p_2}\| \leq \|T\|$, then

$$\|T\| = \sup_{p_1,p_2} \left\| \sum_{i=1}^{n} (p_1 a_i) \otimes (b_i p_2) \right\|.$$

□

2.6 Tensor product of C*-algebras

Theorem 2.6.1. *Let A, B be normed algebras over $\boldsymbol{K}$. There exists a unique product on $A \otimes B$ with respect to which $A \otimes B$ is an algebra and $(a \otimes b)(c \otimes d) = ac \otimes bd$ $(a, c \in A,\ b, d \in B)$. See [13] for proof.*

We note that $A \otimes B$ endowed with multiplication is called the **algebra tensor product** and $A \otimes B$ together with an involution the ***-algebra tensor product** [12].
The norm of a C*-algebra is unique in the sense that on a given *-algebra A there is at most one norm which makes A into a C*-algebra [13]. We consider two types of norms and we determine the relationship between them.

2.6.1 Spatial norm

The norm $\|.\|_\pi$ defined by the inclusion $A \otimes B \subseteq B(H) \otimes B(K) \subseteq B(H \hat{\otimes} K)$ is called the spatial norm, assuming that A and B are faithfully represented on Hilbert spaces H and K respectively. This norm was introduced by T. Turumaru in 1953. The definition does not depend on particular representations of A and B, that is, $\forall\, t \in A \otimes B$

$$\|t\|_\pi = \|\theta \otimes \vartheta(t)\|_{B(H\otimes K)}$$

for any two faithful representations θ of A on H and ϑ of B on K.

First,we show that the spatial norm is actually a norm. $\forall\, t \in A \otimes B$, we have $\|t\|_\pi = \|\theta \otimes \vartheta(t)\|_{B(H\otimes K)}$ which defines a norm.

(i) Clearly, $\|t\|_\pi \geq 0$ and $\|t\|_\pi = 0$ if and only if $t = 0$. i.e

$\|(\theta \otimes \vartheta)t\|_{B(H\otimes K)} \geq 0$ and

$\|(\theta \otimes \vartheta)t\|_{B(H\otimes K)} = 0$ if and only if $t = 0$.

(ii) We show that $\|\alpha t\|_\pi = |\alpha|\|t\|_\pi, \ \ \forall\, \alpha \in \mathbf{K}$.

Now,

$$\begin{aligned}
\|\alpha t\|_\pi &= \|(\theta \otimes \vartheta)(\alpha t)\|_{B(H\otimes K)} \\
&= \|\alpha(\theta \otimes \vartheta)(t)\|_{B(H\otimes K)} \\
&= |\alpha|\; \|(\theta \otimes \vartheta)(t)\|_{B(H\otimes K)} \\
&= |\alpha|\|t\|_\pi.
\end{aligned}$$

(iii) We show that $\|t + s\|_\pi \leq \|t\|_\pi + \|s\|_\pi$.

$$\begin{aligned}
\|t + s\|_\pi &= \|(\theta \otimes \vartheta)(t + s)\|_{B(H\otimes K)} \\
&= \|(\theta \otimes \vartheta)(t) + (\theta \otimes \vartheta)(s)\|_{B(H\otimes K)} \\
&\leq \|(\theta \otimes \vartheta)(t)\|_{B(H\otimes K)} + \|(\theta \otimes \vartheta)(s)\|_{B(H\otimes K)} \\
&= \|t\|_\pi + \|s\|_\pi.
\end{aligned}$$

(iv) Lastly, we show that $\|ts\|_\pi \leq \|t\|_\pi \|s\|_\pi$.

$$\begin{aligned}
\|ts\|_\pi &= \|(\theta \otimes \vartheta)(ts)\|_{B(H\otimes K)} \\
&= \|(\theta \otimes \vartheta)(t)(\theta \otimes \vartheta)(s)\|_{B(H\otimes K)} \\
&\leq \|(\theta \otimes \vartheta)(t)\|_{B(H\otimes K)} \|(\theta \otimes \vartheta)(s)\|_{B(H\otimes K)} \\
&= \|t\|_\pi \|s\|_\pi.
\end{aligned}$$

2.6.2 Maximal C*-norm

The second natural norm on $A \otimes B$ was introduced in 1965 by A. Guichardet. It is the *maximal C*-norm* $\|.\|_\nu$ defined as:

$$\|t\|_\nu = \sup\{\|\tau t\| : \tau \textit{ is a subtensor representation of } A \otimes B\}, \textit{ for } t \in A \otimes B.$$

Next, we need to show that maximal C*-norm is actually a norm i.e it must satisfy all the properties of a norm.

(i) Clearly, $\|t\|_\nu = \sup\{\|\tau t\|_{B(H)}\} \geq 0$ and $\|t\|_\nu = 0$ if and only if $t = 0$, $\forall\, t \in A \otimes B$.

(ii) We show that $\|\alpha t\|_\nu = |\alpha|\|t\|_\nu, \ \ \forall\, \alpha \in \mathbf{K}$.

$$\begin{aligned}
\|\alpha t\|_\nu &= sup\{\|\alpha\tau t\| : \tau \text{ is a subtensor representation of } A\otimes B\}, \text{ for } t \in A\otimes B. \\
&= \sup\left\{\left\|\alpha\tau\sum_{i=1}^{n} x_i\otimes y_i\right\|\right\} \\
&= \sup\left\{\left\|\sum_{i=1}^{n} \alpha\tau_1(x_i)\otimes\tau_2(y_i)\right\|\right\} \\
&= |\alpha|\sup\left\{\left\|\sum_{i=1}^{n} \tau_1(x_i)\otimes\tau_2(y_i)\right\|\right\} \\
&= |\alpha|\|t\|_\nu, \ \ \forall\, \alpha \in \mathbf{K}.
\end{aligned}$$

(iii) We let $x_i,\ x_i' \in A,\ y_i,\ y_i' \in B$, then for $t = \sum_{i=1}^n x_i\otimes y_i$ and $s = \sum_{i=1}^n x_i'\otimes y_i'$, we have,

$$\begin{aligned}
\|t+s\|_\nu &= \sup\{\|\tau(t+s)\| : \tau \text{ subtensor representation of } A\otimes B\}, \text{ for } t\in A\otimes B. \\
&= \sup\{\|\tau(t)+\tau(s)\|_{B(H)}\} \\
&= \sup\left\{\left\|\left[\sum_{i=1}^{n}\tau_1(x_i)\otimes\tau_2(y_i)\right]+\left[\sum_{i=1}^{n}\tau_1(x_i')\otimes\tau_2(y_i')\right]\right\|\right\} \\
&\leq \sup\left\{\left\|\sum_{i=1}^{n}\tau_1(x_i)\otimes\tau_2(y_i)\right\|\right\}+\sup\left\{\left\|\sum_{i=1}^{n}\tau_1(x_i')\otimes\tau_2(y_i')\right\|\right\} \\
&= \|t\|_\nu+\|s\|_\nu.
\end{aligned}$$

(iv) Lastly, we show that $\|ts\|_\nu \leq \|t\|_\nu\|s\|_\nu$. We let $x_i,\ x_i' \in A,\ y_i,\ y_i' \in B$, then for $t = \sum_{i=1}^{n} x_i \otimes y_i$ and $s = \sum_{i=1}^{n} x_i' \otimes y_i'$, we have,

$$\begin{aligned}
\|ts\|_\nu &= \sup\{\|\tau(ts)\| : \tau \textit{ subtensor representation of } A \otimes B\}, \textit{ for } t \in A \otimes B. \\
&= \sup\{\|\tau(t)\tau'(s)\|_{B(H)}\} \\
&= \sup\{\|\tau\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\tau'\left(\sum_{i=1}^{n} x_i' \otimes y_i'\right)\|_{B(H)}\} \\
&= \sup\left\{\left\|\left[\sum_{i=1}^{n} \tau_1(x_i) \otimes \tau_2(y_i)\right]\left[\sum_{i=1}^{n} \tau_1'(x_i') \otimes \tau_2'(y_i')\right]\right\|\right\} \\
&\leq \sup\left\{\left\|\sum_{i=1}^{n} \tau_1(x_i) \otimes \tau_2(y_i)\right\|\right\} \sup\left\{\left\|\sum_{i=1}^{n} \tau_1'(x_i') \otimes \tau_2'(y_i')\right\|\right\} \\
&= \|t\|_\nu \|s\|_\nu.
\end{aligned}$$

2.6.3 Relationship between spatial norm and maximal C*-norm

Theorem 2.6.2. *Let A, B be C*-algebras. There is a minimal C*-norm ($\|t\|_\pi$) and maximal norm ($\|t\|_\nu$) such that any C*-norm ($\|t\|$) on $A \otimes B$ must satisfy $\|t\|_\pi \leq \|t\| \leq \|t\|_\nu$. (This is a known result but no proof has been found).*

Proof. We denote by $A\hat{\otimes}_\pi B$ (resp. $A\hat{\otimes}_\nu B$) the completion of $A \otimes_\pi B$ for the norm ($\|t\|_\pi$) (resp. $A \otimes_\nu B$ for the norm ($\|t\|_\nu$)).
The maximal norm is described as $\|t\|_\nu = \sup \|\phi(t)\|_{B(H)}$ where the supremum is taken over all possible Hilbert spaces H of all possible *-homomorphisms;

$$\phi : A \otimes B \to B(H).$$

For any such ϕ there is a pair of *-homomorphisms $\phi_i : A \to B(H)$

$(i = 1, 2)$ with commuting ranges such that,

$$\phi\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} \phi_1(x_i)\phi_2(y_i).$$

Conversely, any such pair $\phi_i : A \to B(H)$, $\phi_i : B \to B(H)$ $(i = 1, 2)$ of *-homomorphisms with commuting ranges determine uniquely a *-homomorphism $\phi : A \otimes B \to B(H)$ by setting $\phi(x_i \otimes y_i) = \phi_1(x_i)\phi_2(y_i)$. Thus we can write for $t = \sum_{i=1}^{n} x_i \otimes y_i \in A \otimes B$, $||t||_\nu = \sup\{\sum_{i=1}^{n} \phi_1(x_i)\phi_2(y_i)\}$ where the supremum runs over all possible such pairs. The inequality

$$||t|| \leq ||t||_\nu.$$

follows by considering Gelfand- Naimark embedding of the completion of $(A \otimes B, ||t||)$ into $B(H)$ for some H [13]. The minimal norm can be described as follows; embedding A and B as C*-subalgebras of $B(H_1)$ and $B(H_2)$ respectively. Then for any $t = \sum_{i=1}^{n} x_i \otimes y_i \in A \otimes B$, $||t||_\pi$ coincides with the norm induced by the space $B(H_1 \otimes_{||\cdot||} H_2)$ that is, we have an embedding (an isometric *-homomorphism) of the completion denoted by $A\hat{\otimes}_\pi B$ into $B(H_1 \otimes H_2)$.

In other words, the minimal tensor product operator spaces, when restricted to two C*-algebras coincides with the minimal C*-tensor product.

Let (C_1, D_2) be another pair of C*-algebras and consider completely bounded maps $f_1 : A \to C$ and $f_2 : B \to D$. Then $f_1 \otimes f_2$ defines a completely bounded map from $A\hat{\otimes}_\pi B$ to $C \otimes D$ with $||f_1 \otimes f_2||_{cb} = ||f_1||_{cb}||f_2||_{cb}$. In sharp contrast, the analogous property does not hold for maximal tensor products. However, it does hold if we assume further, that f_1 and f_2

are positive and then the resulting map $f_1 \otimes f_2$ is also completely positive (on the maximal tensor product) and we have

$$\|f_1 \otimes f_2(t)\|_{C\hat{\otimes}_\nu D} \leq \|f_1\|\|f_2\|\|t\|_{A\hat{\otimes}_\nu B}, \quad \forall\, t \in A \otimes B.$$

□

Chapter 3

NORMS OF ELEMENTARY OPERATORS

3.1 Introduction

In this chapter we concentrate on the norms of elementary operators, especially on the lower estimate of these norms. We refer the reader back to the introductory chapter for historical background on elementary operators and other important definitions used in this chapter.

Definition 3.1.1. Let H be a Hilbert space and $B(H)$ the algebra of bounded linear operators on H. Then $T : B(H) \to B(H)$ is an elementary operator if T has a representation $T(x) = \sum_{i=1}^{n} a_i x b_i$ where a_i, b_i are fixed in $B(H)$.

Remark 3.1.2. An elementary operator is a bounded linear operator.

To see this, let x, $y \in B(H)$ and α, $\beta \in \mathbf{K}$ then for a_i, b_i fixed in $B(H)$

we have,

$$\begin{aligned}
T(\alpha x+\beta y) &= \sum_{i=1}^{n} a_i(\alpha x+\beta y)b_i \\
&= \sum_{i=1}^{n}(\alpha a_i x+\beta a_i y)b_i \\
&= \sum_{i=1}^{n}(\alpha a_i x b_i+\beta a_i y b_i) \\
&= \sum_{i=1}^{n}\alpha a_i x b_i+\sum_{i=1}^{n}\beta a_i y b_i \\
&= \alpha\sum_{i=1}^{n} a_i x b_i+\beta\sum_{i=1}^{n} a_i y b_i \\
&= \alpha T(x)+\beta T(y).
\end{aligned}$$

Hence the operator is linear. To prove that T is bounded, we need to show that there exists a constant $M>0$ such that $\|T(x)\|\leq M\|x\|,\ \ \forall\, x\in H$. Now,

$$\begin{aligned}
\|T(x)\| &= \left\|\sum_{i=1}^{n} a_i x b_i\right\| \\
&\leq \sum_{i=1}^{n}\|a_i x b_i\| \\
&\leq \sum_{i=1}^{n}\|a_i\|\|b_i\|\|x\|.
\end{aligned}$$

So

$$\|T(x)\|\leq\sum_{i=1}^{k}\|a_i\|\|b_i\|\|x\|. \tag{3.1.1}$$

Let $\sum_{i=1}^{n}\|a_i\|\|b_i\|$ in equation (3.1.1) be M, then the equation reduces to $\|T(x)\| \leq M\|x\| \ \forall\, x \in H$. Therefore, T is a bounded linear operator.

3.2 Overview of the norm problem

The norm problem for elementary operators involves finding a formula which describes the norm of an elementary operator in terms of its coefficients. Therefore, finding the norm of elementary operators has been considered by many authors (see [3, 4, 22, 26]). Timoney [27] came up with a formula for the norm of an elementary operator on a C*-algebra, involving matrix valued numerical ranges and a kind of tracial geometric mean. Our concern has been be to investigate the lower estimate of these norms since the upper estimates are easy to find as we observe in the next lemma.

3.3 Main results

Lemma 3.3.1. *Let $T : B(H) \to B(H)$ be the elementary operator such that T has a representation $T(x) = \sum_{i=1}^{n} a_i x b_i$ where a_i, b_i are fixed in $B(H)$ and $x \in B(H)$. Then $\|T\| \leq \sum_{i=1}^{n} \|a_i\| \|b_i\|$.*

Proof. We have

$$\begin{aligned} \|T(x)\| &= \left\|\sum_{i=1}^{n} a_i x b_i\right\| \\ &\leq \sum_{i=1}^{n} \|a_i x b_i\| \\ &\leq \sum_{i=1}^{n} \|a_i\|\|b_i\|\|x\|. \end{aligned}$$

Clearly, $\|T\| \leq \sum_{i=1}^{n} \|a_i\|\|b_i\|$. □

Example 3.3.2. Let $\mathcal{U}_{a,b}(x) = axb + bxa$ be an elementary operator where $n = 2$ then $\|\mathcal{U}\| \leq 2\|a\|\|b\|$.

To see this, we note from Lemma (3.3.1) that,

$$\begin{aligned} \|\mathcal{U}(x)\| &= \|axb + bxa\| \\ &\leq \|axb\| + \|bxa\| \\ &\leq \|a\|\|x\|\|b\| + \|b\|\|x\|\|a\|. \end{aligned}$$

Clearly, $\|\mathcal{U}\| \leq 2\|a\|\|b\|$.

3.3.1 A general norm inequality

For two C*-algebras A and B a linear operator $T : A \to B$ is called *positive* if $Ta \geq 0$ whenever $a \in A$. For other conditions on positivity of T, see [28]. The following lemma introduces us to a general norm inequality.

Lemma 3.3.3. *Let a and b be positive operators on A. If T is the operator matrix on $A \oplus A$ defined by* $T = \begin{bmatrix} a & c^* \\ c & b \end{bmatrix}$ *then*

$$\|T\| \leq \max\{\|a\|, \|b\|\} + \|c\|.$$

Proof. We write

$$T = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & c^* \\ c & 0 \end{bmatrix} \tag{3.3.1}$$

By the definition of maximal norm of matrices [16, 17, 18], if $M_{n,m}$ is the set of all $n \times m$ matrices over ($\mathbb{C}$ *or* $\mathbb{B}$), then for $D \in M_{n,m}$,

$\|D\|_{\max} = \max_{i,j} |a_{i,j}|$ where $a_{i,j} \in D$ $(i = 1, ..., n,\ j = 1, ..., m)$.

Now,

$$\|T\| = \left\| \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & c^* \\ c & 0 \end{bmatrix} \right\| \tag{3.3.2}$$

$$\|T\| \leq \left\| \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & c^* \\ c & 0 \end{bmatrix} \right\| \tag{3.3.3}$$

Therefore, $\left\| \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right\| = \max\{\|a\|, \|b\|\}.$

Similarly, $\left\| \begin{bmatrix} 0 & c^* \\ c & 0 \end{bmatrix} \right\| = \max\{\|c\|, \|c^*\|\} = \|c\|$ (*since* $\|c\| = \|c^*\|$).

Hence $\|T\| \leq \max\{\|a\|, \|b\|\} + \|c\|$. □

Remark 3.3.4. The norm of the operator $M_{a,b} + M_{c,d}$ is usually very difficult to compute (see [25, 26, 28]). The following theorem gives a more useful insight.

Theorem 3.3.5. *If a, b, c and d are operators in $B(H)$, then*

$$M_{a,b} + M_{c,d} \leq \left[(\max\{\|b\|^2, \|d\|^2\} + \|bd^*\|)(\max\{\|a\|^2, \|c\|^2\} + \|c^*a\|)\right]^{\frac{1}{2}}.$$

See [5] for proof.

Theorem (3.3.5) leads to the following important properties of operators.

Corollary 3.3.6. *If $a, b \in B(H)$, then the following properties hold:*
(1) $\|a + b\|^2 \leq 2\,(\max\{\|a\|^2, \|b\|^2\} + \|b^*a\|)$,
(2) $\|aa^* + bb^*\| \leq (\max\{\|a\|^2, \|b\|^2\} + \|b^*a\|)$.

Proof. The inequality in (1) follows from theorem(3.3.5) by letting $b = d = I$. The second inequality follows by letting $b = a^*$ and $d = c^*$ in the same theorem. □

Theorem 3.3.7. *If $a, b \in B(H)$, and let $a \otimes b$ denote the tensor product of a and b then $\|a \otimes b + b \otimes a\| \leq \sqrt{2\|a\|^2\|b\|^2 + 2\|b^*a\|^2}$.*

Proof.

$$\begin{aligned}
\|a \otimes b + b \otimes a\|^2 &= \langle a \otimes b + b \otimes a, a \otimes b + b \otimes a\rangle \\
&= \langle a \otimes b, a \otimes b\rangle + \langle a \otimes b, b \otimes a\rangle + \langle b \otimes a, a \otimes b\rangle + \langle b \otimes a, b \otimes a\rangle \\
&= \langle a, a\rangle\langle\, b, b\rangle + \langle a, b\rangle\langle b, a\rangle + \langle b, a\rangle\langle a, b\rangle + \langle b, b\rangle\langle a, a\rangle \\
&= \|a\|^2\|b\|^2 + \|b\|^2\|a\|^2 + \langle a, b\rangle\langle b, a\rangle + (\overline{\langle a, b\rangle})(\overline{\langle b, a\rangle}),\ for\ \overline{\langle a, b\rangle} = \langle b, a\rangle \\
&= \|a\|^2\|b\|^2 + \|b\|^2\|a\|^2 + 2\mathbf{Re}\langle a, b\rangle\langle b, a\rangle
\end{aligned}$$

So by Cauchy-Schwarz inequality,

$$\begin{aligned}\|(a \otimes b) + (b \otimes a)\|^2 &\leq \|a\|^2\|b\|^2 + \|b\|^2\|a\|^2 + 2\|a\|\|b\|\|b\|\|a\| \\ &= 2\|a\|^2\|b\|^2 + 2\|a\|\|b\|\|b\|\|a\|.\end{aligned}$$

Therefore,

$$\|(a \otimes b) + (b \otimes a)\|^2 \leq 2\|a\|^2\|b\|^2 + 2\|a\|\|b\|\|b\|\|a\|. \tag{3.3.4}$$

But $\|b\| = \|b^*\|$ so replacing $\|b\|$ by $\|b^*\|$ in the second summand on the right hand side of equation (3.3.4), we get

$$\|a \otimes b + b \otimes a\|^2 \leq 2\|a\|^2\|b\|^2 + 2\|b^*a\|^2$$

Taking the positive square root on both sides yields

$$\|a \otimes b + b \otimes a\| \leq \sqrt{2\|a\|^2\|b\|^2 + 2\|b^*a\|^2}.$$

Alternatively,

$$\begin{aligned}\|a \otimes b + b \otimes a\|^2 &= 2\left(\max\{\|a \otimes b\|^2, \|b \otimes a\|^2\} + \|(b \otimes a)^*(a \otimes b)\|\right) \\ &\leq 2\left(\max\{\|a\|^2\|b\|^2, \|b\|^2\|a\|^2\} + \|(b^* \otimes a^*)(a \otimes b)\|\right) \\ &\leq 2\|a\|^2\|b\|^2 + \|b^*a \otimes a^*b\| \\ &\leq 2\|a\|^2\|b\|^2 + \|b^*a\|\|a^*b\| \\ &\leq 2\|a\|^2\|b\|^2 + 2\|b^*a\|^2.\end{aligned}$$

Taking square root on both sides we have,

$$\|a \otimes b + b \otimes a\| \leq \sqrt{2\|a\|^2\|b\|^2 + 2\|b^*a\|^2}.$$

□

Lemma 3.3.8. *If $a,\ b \in B(H)$, then*

$$\|\mathcal{U}_{a,\,b}\| \leq [(\|a\|\|b\| + \|ab^*\|)(\|a\|\|b\| + \|b^*a\|)]^{\frac{1}{2}}. \tag{3.3.5}$$

In particular, if $ab^* = b^*a = 0$, then $\|\mathcal{U}_{a,b}\| = \|a\|\|b\|$. See [5] for proof.

3.3.2 The Complex Hilbert space case.

In this subsection we concentrate on a complex Hilbert space over the field $\mathbf{K}$. We show that for a basic elementary operator M, $\|M\| = \|a\|\|b\|$.

Definition 3.3.9. Let $\phi \in H^*$ and $\xi \in H$. We define $(\phi \otimes \xi) \in B(H)$ by

$$(\phi \otimes \xi)\eta = \phi(\eta)\xi, \ \forall \ \in \ H.$$

Theorem 3.3.10. *Let H be a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H. Let $M_{a,b} : B(H) \to B(H)$ be defined by $M_{a,b}(x) = axb,\ \forall x \in B(H)$ where $a,\ b$ are fixed in $B(H)$. Then $\|M_{a,b}\| = \|a\|\|b\|$.*

Proof. By definition, $\|M_{a,b}|B(H)\| = \sup\{\|M_{a,b}(x)\| :\ x \in B(H),\ \|x\| = 1\}$.
This implies that $\|M_{a,b}|B(H)\| \geq \|M_{a,b}(x)\|,\ \forall x \in B(H),\ \|x\| = 1$.
So $\forall \epsilon > 0,\ \|M_{a,\,b}|B(H)\| - \epsilon < \|M_{a,b}(x)\|,\ \forall x \in B(H),\ \|x\| = 1$.

But, $\|M_{a,b}|B(H)\| - \epsilon < \|axb\| \leq \|a\|\|x\|\|b\| = \|a\|\|b\|$.

Since ϵ is arbitrary, this implies that

$$\|M_{a,b}|B(H)\| \leq \|a\|\|b\|. \tag{3.3.6}$$

On the other hand, let $\xi,\ \eta \in H,\ \ \|\xi\| = \|\eta\| = 1,\ \ \phi \in H^*$.

Now,

$$\|M_{a,b}|B(H)\| \geq \|M_{a,b}(x)\| : \ \forall x \in B(H),\ \|x\| = 1.$$

But,

$$\begin{aligned}
\|M_{a,b}(x)\| &= \sup\{\|(M_{a,b}(x))\eta\| : \ \forall \eta \in H,\ \|\eta\| = 1\} \\
&= \sup\{\|(axb)\eta\| : \ \eta \in H,\ \|\eta\| = 1\}.
\end{aligned}$$

Setting $a = (\phi \otimes \xi_1),\ \forall \xi_1 \in H,\ \|\xi_1\| = 1$ and
$b = (\varphi \otimes \xi_2),\ \forall \xi_2 \in H,\ \|\xi_2\| = 1$, we have,

$$\begin{aligned}
\|M_{a,b}|B(H)\| &\geq \|M_{a,b}(x)\| \geq \|(M_{a,b}(x))\eta\| \\
&= \|(axb)\eta\| \\
&= \|((\phi \otimes \xi_1)x(\varphi \otimes \xi_2))\eta\| \\
&= \|(\phi \otimes \xi_1)x(\varphi(\eta)\xi_2)\| \\
&= \|(\phi \otimes \xi_1)\varphi(\eta)x(\xi_2)\| \\
&= |\varphi(\eta)|\|(\phi \otimes \xi_1)x(\xi_2)\| \\
&= |\varphi(\eta)|\|\phi(x(\xi_2))\xi_1\| \\
&= |\varphi(\eta)||\phi(x(\xi_2))|\|\xi_1\| \\
&= \|a\|\|b\|.
\end{aligned}$$

Therefore,

$$\|M_{a,b}|B(H)\| \geq \|a\|\|b\|. \tag{3.3.7}$$

Hence by inequalities (3.3.6)and (3.3.7),

$$\|M_{a,b}|B(H)\| = \|a\|\|b\|.$$

This completes the proof. □

Chapter 4

CONCLUSION AND RECOMMENDATION

In this last chapter, we draw conclusions and make recommendations based on our objective of study and the results obtained.

4.1 Conclusion

We summarize our work by highlighting the results obtained in our study pertaining to the problem stated in section 1.5.
Our objective was to determine the lower estimate of the norm of the basic elementary operator through tensor products as stated in section 1.6. In chapter one, we gave basic definitions and concepts which were essential to our study. In chapter two, we considered the spatial norm, projective norm, Haagerup norm and the maximal norm. We have shown the relationship between spatial and the maximal norm.
Lastly, we have shown that for the basic elementary operator M,
$\|M\| = \|a\|\|b\|$.

4.2 Recommendation.

Norms of elementary operators is a very interesting area of study in mathematics and has not been exhausted. In our case we considered a basic elementary operator. Efforts thus can be directed on determining the lower estimate of the norm of the Jordan elementary operator ($\mathcal{U}_{a,b}(x) = axb + bxa$) acting on two or higher dimensional Hilbert spaces.

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