NORMS OF TENSOR PRODUCTS AND ELEMENTARY OPERATORS

by

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ABSTRACT

In this thesis, we determine the norm of a two-sided symmetric operator in an algebra. More precisely, we investigate the lower bound of the operator using the injective tensor norm. Further, we determine the norm of the inner derivation on irreducible C*-algebra and confirm Stampfli's result for these algebras.

Chapter 1

INTRODUCTION

1.1 Introduction

In this section we give definitions of various mathematical concepts and examples that we intend to use in the subsequent chapters. We have also given some theorems and lemmas that we shall refer to in the subsequent chapters. We shall use the capital letters X, Y, U, V, W to denote vector spaces and small letters x, y, u, v, w to denote their elements.

1.1.1: Definition; Inner product space.

Let X be a vector space over the field of real or complex numbers. A mapping, denoted by $\langle .,. \rangle$ defined on $X \times X$ into the underlying field is called an inner product of any two elements x and y of X if the following conditions are satisfied;

(1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$, $\forall x, y \in X$.

(2) $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$, $\forall x, x', y \in X$

(3) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$, α belongs to the underlying field.

(4) $\overline{\langle x, y\rangle} = \langle y, x\rangle$

See [18] page 83 for verifications of 1-4.

If the inner product $\langle .,.\rangle$ is defined for every pair of elements $(x, y) \in X \times X$, then the vector space X together with the inner product $\langle .,.\rangle$ is called an inner product space or pre-Hilbert space usually denoted by$(X, \langle .,.\rangle)$.

1.1.2: Definition; Hilbert space.

An inner product space X is called a Hilbert space if the normed space induced by the inner product is a Banach space (complete normed space). That is, every Cauchy sequence $x_n \in X$ with respect to the norm induced by the inner product is convergent with respect to this norm.

1.2 Operators and Functionals

1.2.1: Definition; Operator.

Let X and Y be normed spaces. Then the mapping $T : X \longrightarrow Y$ is called an operator.

1.2.2: Definition; A linear operator.

Let X and Y be normed spaces. An operator T is said to be linear if the following conditions are satisfied;

$\forall x, y \in X$, α a scalar,

(i) $T(x + y) = Tx + Ty$

(ii) $T(\alpha x) = \alpha Tx$.

So, the map $T : X \longrightarrow Y$ is linear if $\forall x, y \in X$ and $\alpha, \beta \in \mathbb{K}$,

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

1.2.3: Definition; A bounded linear operator.

A linear operator $T : X \longrightarrow Y$ is said to be bounded if there exists a real constant $k > 0$ such that $\|Tx\| \leq k\|x\|$ $\forall x \in X$. We shall denote by $B(X, Y)$ the set of $T : X \longrightarrow Y$.

1.2.4: Definition; Norm of a bounded operator.

Let $T \in B(X, Y)$. Then the norm of T is defined as

$$\|T\| = sup\{\|Tx\| : x \in \mathfrak{D}(T), \|x\| < 1\}$$
$$= sup\{\|Tx\| : x \in \mathfrak{D}(T), x \neq 0\} < \infty.$$

The supremum being finite follows from the fact that $\|T(x)\| \leq k\|x\|, \forall x \in X$ and $k \geq 0$.

1.2.5: Theorem.

Let T be a linear operator then,

(a) The range of the operator T, $\mathfrak{R}(T)$ is a vector space.

(b) The dimension of the domain of T, $dim\mathfrak{D}(T)$ is finite.

(c) The null space of T, $\mathfrak{N}(T)$ is a vector space.

See [8] page 86 for proof.

1.2.6: Definition; Adjoint operator.

Let $T \in B(X, Y)$ where X, Y are Hilbert spaces, then the unique linear operator $T^* \in B(Y, X)$ satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in X$ and $y \in Y$ is called the adjoint (Hilbert adjoint) of T.

1.2.7: Definition; Self-adjoint, Positive, Normal and Unitary operators.

Let T be a bounded linear operator on a Hilbert space H into itself then,

(i) T is called self-adjoint or hermitian if $T = T^*$.

(ii) T is called normal if $TT^* = T^*T$.

(iii) T is called unitary if $T^*T = I = TT^*$ where I is the identity on H. This implies that, T preserves inner product on the Hilbert space, so that $\langle Tx, Ty\rangle = \langle x, y\rangle \forall x, y \in H$ and that T is a surjective isometry.

(iv) T is positive if $\langle Tx, Ty\rangle \geq 0$ for all $x \in H$.

1.2.8: Proposition.

Let $T \in B(H)$. Then the following statements are equivalent;

(i) T is self-adjoint.

(ii) $\langle Tx, x\rangle$ is a real number, $\forall x \in H$.

See [22] page 330 for proof.

1.2.9: Definition; Completely bounded operator.

Let H be a complex Hilbert space and $B(H)$ the set of all bounded linear operators on H. Any map $\phi : B(H) \longrightarrow B(H)$ induces a family of maps $\phi_n : M_n(B(H)) \longrightarrow M_n(B(H))$, $n \geq 1$ defined by $\phi_n([x_{i,j}])$ for any matrix $[x_{i,j}] \in M_n(B(H))$. If $sup\|\phi_n\|$ is finite then ϕ is said to be completely bounded and the supremum defines the completely bounded norm $\|\phi\|_{cb}$ of ϕ. (Here, of course the norm in $M_n(B(H))$ is given by the identification $M_n(B(H)) = B(H^n)$. "We refer to [4] and [14] for more on completely bounded mappings".

1.2.10: Definition; Elementary operator.

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. We call $T : B(H) \longrightarrow B(H)$ an elementary operator if T has a representation;

$$T(x) = \sum_{i=1}^{k} a_i x b_i \tag{1.1}$$

with $a_i, b_i \in B(H)$ for each i. The building blocks of such elementary operators of length one, that is if $k = 1$ in (1.1), has the form $T_{a,b}(x) = axb$.

Compact operators:

1.2.11: Definition; Compactness.

A metric space X is said to be compact (sequentially compact) if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, i.e. if every sequence in M has a convergent subsequence whose limit is an element of M.

1.2.12: Definition; Compact linear operators.

Let X and Y be normed spaces. An operator $T : X \longrightarrow Y$ is called a compact linear operator (or completely continuous linear operator) if T is linear and if for every bounded subset M of X the image $T(M)$ is relatively compact i.e. the closure $\overline{T(M)}$ is compact or totally bounded subset of Y.

1.2.13: Definition; Compact operators on Banach spaces.

An operator $T \in B(X, Y)$ is compact if TB_x, the image of the unit ball B_x under T, is relatively compact (i.e. totally bounded) subset of Y. Thus T is compact if and only if for every sequence $(x_n) \in X$ the sequence (Tx_n) has a convergent subsequence. In short, compact operators are "small" in the sense that they map the unit ball into a "small" set.

1.2.14: Theorem.

Let $T : H_1 \longrightarrow H_2$ be compact linear map between Hilbert spaces H_1 and H_2. Then the image of the closed ball of H_1 under T is compact.

Proof.

Let U be a closed unit ball of H_1. It is weakly compact, and T is weakly continuous. So $T(U)$ is weakly compact and therefore weakly closed. Hence $T(U)$ is norm closed, since the weak topology is weaker than the norm topology. Since T is a compact operator, this implies that $T(U)$ is norm compact.

1.2.15: Lemma.

Let X and Y be normed spaces. Then

(a) Every compact operator $T : X \longrightarrow Y$ is bounded hence continuous.

(b) If $dimX = \infty$, the identity operator $I : X \longrightarrow X$ (which is continuous) is not compact.

Proof.

(a) The unit ball $S \subset X$ such that $S = \{x \in X : \|x\| = 1\}$ is bounded. Since T is compact, $\overline{T(S)}$ is compact and is bounded by the fact that a compact subset M of a metric space is closed and bounded. So that $sup_{\|x\|=1}\|Tx\| < \infty$. Hence T is bounded since we have that T is continuous.

(b) The closed unit ball $S \in X$ such that $S = \{x \in X : \|x\| \leq 1\}$ is bounded. If $dimX = \infty$, then the fact that a normed space has a property that the closed unit ball S is compact, then X is finite dimensional, implies that S cannot be compact. Thus $I(S) = S = \overline{S}$ is not relatively compact.

1.2.16: Theorem.

Let X and Y be normed spaces and $T : X \longrightarrow Y$ a linear operator. Then T is compact if and only if it maps every bounded sequence $(x_n) \in X$ onto a sequence $(Tx_n) \in Y$ which has a convergent subsequence.

Proof.

If T is compact and (x_n) is bounded then $\overline{(Tx_n)} \in Y$ is compact and by definition of compactness, (Tx_n) contains a convergent subsequence.

Conversely,

Let every bounded sequence (x_n) contain a subsequence (x_{n_k}) such that (Tx_{n_k}) converges in Y. Consider any subset $S \subset X$ and let (y_n) be any sequence in $T(S)$. Then $y_n = Tx_n$ for some $(x_n) \in S$ and (x_n) is bounded since S is bounded. By assumption, (Tx_n) contain a subsequent sequence. Hence $\overline{T(S)}$ is compact because $y_n \in T(S)$ was arbitrary. Hence by definition, this shows that T is compact. *(compactness criterion.)*

1.2.17: Theorem.

Let X and Y be normed spaces and $T : X \longrightarrow Y$ a linear operator. Then,

(i) If T is bounded and $dim T(X) < \infty$, the operator T is compact.

(ii) If the $dim X < \infty$, the operator T is compact.

Proof.

(i) Let (x_n) be a bounded sequence in X. Then the inequality $\|Tx_n\| \leq \|T\| \|x_n\|$ shows that (Tx_n) is bounded. Hence (Tx_n) is relatively compact (in a finite dimensional normed space X, any subset $S \subset X$ is closed and bounded.) Since $dim T(X) < \infty$ it follows that

(Tx_n) has a convergent subsequence. Since (x_n) was arbitrary bounded sequence in X, the operator T is compact by theorem 1.2.16.

(ii) This follows from (i) by noting that $dimX < \infty$ implies boundedness of T and by the fact that if a normed space X is finite dimensional, then every linear operator on X is bounded. So $dimT(X) \leq dimX$ for any linear operator T. If $dim\mathfrak{D}(T) = n < \infty$ then $dim\mathfrak{R}(T) \leq n$.

1.2.18: Examples of compact operators.

(1) Every finite operator $T \in B(X, Y)$ is compact i.e. if $dimT = dimT(X) < \infty$ then $T \in B_o(X, Y)$. Indeed, the set $\mathbb{Z} = ImT$. Since $\mathbb{Z}$ is finite dimensional, $B_{\mathbb{Z}}$ is compact and so TB_X is a subset of the compact set $TB_{\mathbb{Z}}$.

(2) Every bounded linear functional $f \in X^*$ is a compact operator from X to $\mathbb{C}$.

(3) An operator T defined on the space ℓ^2 i.e. $T : \ell^2 \longrightarrow \ell^2$ defined by $y = (\eta_j) = Tx$ where $\eta_j = \varepsilon_j/j$ for $j = 1, 2, ...$

1.2.19: Definition; Uniform topology.

This is defined by the operator norm $\|T\|$ for $T \in B(H)$, where
$\|T\| = sup\{\|Tx\| : x \in H, \|x\| < 1\}$.

1.2.20: Definition; Strong-operator topology.

For $x \in H$, the map $T \longrightarrow \|Tx\|$ defines a semi-norm on $B(H)$. The family of all such semi-norms $\{\|Tx\| : x \in H\}$ defines a Hausdorff locally convex topology called the strong operator topology.

1.2.21: Definition; Weak-operator topology.

For $x, y \in H$, the map $T \longrightarrow |\langle Tx, y\rangle|$ defines a semi-norm on $B(H)$. The family of such semi-norms $\{|\langle Tx, y\rangle| : x, y \in H\}$ define a Hausdorff locally convex topology called the weak-operator topology.

1.2.22: Definition; The maximal numerical range.

Let H be a Hilbert space (complex). $T : H \longrightarrow H$, T bounded. Let $B(H)$ be the set of all bounded linear operators on H. For all $T \in B(H)$ we define a set $W(T)$ given by

$W(T) = \{\lambda : \langle Tx_n, x_n\rangle \longrightarrow \lambda, \|x_n\| = 1, \|Tx_n\| \longrightarrow \|T\|\}$.

When H is finite dimensional, $W(T)$ corresponds to the numerical range produced by the maximal vectors (vectors x such that $\|x\| = 1$ and $\|Tx\| = \|T\|$). Thus we have $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$

1.2.23: Lemma.

The set $W(T)$ is non-empty, closed, convex and contained in the closure of the numerical range [19].

1.2.24: Definition; Diagonal matrix.

A diagonal matrix is a square matrix in which the entries outside the main diagonal are zero. The diagonal entries themselves may or may not be zero. Thus the matrix $A = \delta(i, j)$ with n columns and n rows is diagonal if $\delta(i, j) = 0$, $i \neq j \forall i, j = \{1, 2, ..., n\}$.

1.2.25: Definition; Unitary diagonolizable operator.

A bounded operator T on a Hilbert space H is said to be unitary diagonolizable if it has diagonal matrix relative to some orthonormal basis i.e., if there is an orthonormal basis $\{e_n\}$ for H consisting of eigen vectors of T. We note that all normed operators on a finite dimensional space, and generally, all compact normed operators are unitarily diagonolizable.

1.2.26: Definition; Functionals.

A functional is an operator whose range lies on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$ while its domain lies in a vector space. It is usually denoted by f or F i.e. $f : X \longrightarrow \mathfrak{D}(f) \longrightarrow \mathbb{K}$, $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Functionals are said to be linear and bounded if for $f : X \longrightarrow \mathbb{K}$ there exists a real number $k \geq 0$ such that; $|f(x)| \leq k\|x\|$ for all $x \in X$. Further,

$$\|f\| = sup_{x \neq 0}\{\frac{|f(x)|}{\|x\|}, x \in X\}$$

1.2.27: Definition; Sesquilinear form.

Let U and V be vector spaces over the same scalar field $\mathbb{K}$ ($\mathbb{R} or \mathbb{C}$). Then a sesquilinear form (functional) ℓ on $U \times V$ is a mapping $\ell : U \times V \longrightarrow \mathbb{K}$ such that $\forall u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{K}$;

(i) $\ell(u_1 + u_2, v) = \ell(u_1, v) + \ell(u_2, v)$

(ii) $\ell(u, v_1 + v_2) = \ell(u, v_1) + \ell(u, v_2)$

(iii) $\ell(\alpha u, v) = \alpha \ell(u, v)$

(iv) $\ell(u, \beta) = \overline{\beta}\ell(u, v)$.

Thus ℓ is linear in the first argument and conjugate linear in the second. If U and V are in $\mathbb{R}$ then (iv) is simply $\ell(u, \beta v) = \beta \ell(u, v)$ and ℓ is bilinear since it is linear in both arguments. If $k \geq 0$ such that $|\ell(u, v)| \leq k\|u\|\|v\|$ $\forall u, v$, then ℓ is bounded and the number

$$\|\ell\| = sup_{u \neq 0, v \neq 0}\{\frac{|\ell(u, v)|}{\|u\|\|v\|}, u \in U, v \in V\} = sup_{\|u\|=\|v\|=1}\{|\ell(u, v)| u \in U, v \in V\}$$

is the norm of ℓ.

1.2.28: Definition; Dual space.

The set of all functionals defined on a vector space X is called the dual of X and is denoted by X^*. It is also a vector space if addition and multiplication by vectors are pointwise defined.

1.2.29: Remark.

The dual space X^* of X is a Banach space whether X is a Banach space or not. See [18] page 26.

1.3 Algebra

1.3.0: Definition; An algebra.

A vector space X in which multiplication is defined having the following properties; $\forall x, y, z \in X$ and $\lambda \in \mathbb{K}$,

(a) $x(yz) = (xy)z$

(b) $x(y + z) = xy + xz$

(c) $(x + y)z = xz + yz$

(d) $\lambda(xy) = (\lambda x)y = x\lambda y$ is called an algebra.

An algebra X is called commutative (abelian) if $xy = yx$.

1.3.1: Definition; A Banach algebra.

(e) Given that X above is a Banach space (complete normed space) with respect to a norm that satisfies the **multiplicative inequality**

$$\|xy\| \leq \|x\|\|y\| \forall x, y \in X$$

then X is called a Banach algebra.

(f) Given that X contains a unit element e such that $xe = ex = x$, $\forall x \in X$ and $\|e\| = 1$. Then X is a unital Banach algebra if the properties (a) to (f) are satisfied by X.

1.3.2: Definition; Subalgebra.

A subspace S of X which is also an algebra with respect to the operation on X is a subalgebra of X.

1.3.3: Definition; Involution.

Let X be an algebra. A mapping from $X \longrightarrow X$ defined by $x \longrightarrow x^* \; \forall x, x^* \in X$ is an involution on X it satifies the following conditions; $\forall x, x^*, y \in X$ and λ a scalar,

(i) $(x+y)^* = x^* + y^*$

(ii) $(\lambda x)^* = \overline{\lambda} x^*$

(iii) $(xy)^* = y^* x^*$

(iv) $x^{**} = x$

1.3.4: Definition; *-algebra.

An algebra X with an involution $x \longrightarrow x^*$ is a *-algebra.

1.3.5: Definition; Banach *-algebra.

This is a normed algebra X with an involution, which is complete and has the property $\|x\| = \|x^*\|$. In this case, we define a normed algebra as follows: i.e. the algebra X is a normed algebra if for each element $x \in X$ there is an associated real number $\|x\|$, satisfying the axioms of a norm. Thus $\forall x, y \in X$,

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$

(2) $\|\alpha x\| = |\alpha|\|x\|$

(3) $\|x+y\| \leq \|x\| + \|y\|$

(4) $\|xy\| \leq \|x\|\|y\|$

1.3.6: Definition; C*-algebra.

A Banach *-algebra X with the property $\|x^*x\| = \|x\|^2$ $\forall x \in X$ is called a C*-algebra.

1.3.7: Examples of C*-algebra.

We refer to only one which is $B(H)$, the set of all bounded linear operators on a Hilbert space H. We prove that $B(H)$ is a C*-algebra.

$B(H)$ **is an algebra.**

Let $T \in B(H)$ where $T : H \longrightarrow H$. Multiplication is defined pointwise in $B(H)$. Thus

$$ST(x) = S(T(x)) \forall S, T \in B(H), x \in H$$

$B(H)$ **is a normed algebra.**

$B(H)$ is a normed space,consequently, a normed algebra. For if we let $T \in B(H)$ then $\|T\|$ satisfies the axioms of a norm i.e.,

(i) Clearly, $\|T\| \geq 0$ and $\|T\| = 0$ if and only if $T = 0$.

(ii) $\|\alpha T\| = sup\{\frac{\|(\alpha T)x\|}{\|x\|} : x \neq 0\}$
$=sup\{\frac{\|\alpha(Tx)\|}{\|x\|} : x \neq 0\}$
$= sup\{\frac{|\alpha|\|Tx\|}{\|x\|} : x \neq 0\}$
$= |\alpha| sup\{\frac{\|Tx\|}{\|x\|} : x \neq 0\}$
$= |\alpha|\|T\|$.

(ii) $\|T+S\| = sup\{\frac{\|(T+S)(x)\|}{\|x\|} : x \neq 0\}$

$= sup\{\frac{\|Tx+Sx\|}{\|x\|} : x \neq 0$

$\leq sup\{\frac{\|Tx\|}{\|x\|} + \frac{\|Sx\|}{\|x\|} : x \neq 0$

$\leq sup\{\frac{\|Tx\|}{\|x\|} : x \neq 0\} + sup\{\frac{\|Sx\|}{\|x\|} : x \neq 0\}$

$= \|T\|\|S\|.$

(iv) $\|TS\| = sup\{\frac{\|TS(x)\|}{\|x\|} : \|x\| = 1\}$

$= sup\{\frac{\|T(Sx)\|}{\|x\|} : \|x\| = 1\}$

$\leq \|T\| sup\{\frac{\|Sx\|}{\|x\|} : \|x\| = 1\}$

$= \|T\|\|S\|.$

$B(H)$ **is a *-algebra.**

Since $B(H)$ is an algebra and $T \in B(H)$, it has an involution from $B(H) \longrightarrow B(H)$ define by $T \longrightarrow T^*$ i.e. since T is a bounded linear operator,

(i) $(T+S)^* = T^* + S^*$.

$\langle (T+S)z, x\rangle = \langle z, (T+S)^*x\rangle \ \forall x, z \in H.$

Also,

$\langle (T+S)z, x\rangle = \langle Tz + Sz, x\rangle$

$= \langle Tz, x\rangle + \langle Sz, x\rangle$

$= \langle z, T^*x\rangle + \langle z, S^*x\rangle$. Thus

$\langle z, (T+S)^*x\rangle = \langle z, T^*x + S^*x\rangle.$

(ii) $(\alpha T)^* = \overline{\alpha}T^*$. Clearly,

$$\langle (\alpha T)z, x\rangle = \langle z, (\alpha T)^*x\rangle \tag{1.2}$$

Also,

$$\langle (\alpha T)z, x\rangle = \alpha\langle T(z), x\rangle = \alpha\langle z, T^*(x)\rangle = \langle z, \overline{\alpha}T^*(x)\rangle \tag{1.3}$$

From equations (1.2) and (1.3), $\langle z, (\alpha T)^* x\rangle = \langle z, \overline{\alpha} T^*(x)\rangle$.

(iii) $(TS)^* = S^*T^*$

Clearly, $\langle (TS)x, y\rangle = \langle x, (TS)^* y\rangle$

Since $(TS)x = T(S(x))$,

$$\begin{aligned}\langle (TS)(x), y\rangle &= \langle T(S(x)), y\rangle \\ &= \langle Sx, T^*y\rangle \\ &= \langle x, S^*(T^*(y))\rangle \\ &= \langle x, (S^*T^*)(y)\rangle\end{aligned}$$

(iv) $T^{**} = T$

$\langle Tx, y\rangle = \langle x, T^*y\rangle = \langle (T^*)^* x, y\rangle \; \forall x, y \in H.$

Since $B(H)$ satisfy (i) to (iv), it is an involution and hence a *-algebra.

$B(H)$ **is a Banach *-algebra.**

For all $T \in B(H)$, $\|T\| = \|T^*\|$.

$$\begin{aligned}\|T^*(x)\|^2 &= \langle T^*x, T^*x\rangle \\ &= \langle T(T^*(x)), x\rangle \\ &\leq \|T(T^*(x))\|\|x\| \\ &\leq \|T^*x\|\|T\|\|x\|\end{aligned}$$

$\|T^*(x)\| \leq \|T\|\|x\|$ i.e. $\|T^*\| \leq \|T\|$.

Conversely, applying this relation to T^{**} we have, $\|T^{**}\| \leq \|T^*\|$. But $T^{**} = T$. Therefore, $\|T\| \leq \|T^*\|$.

$B(H)$ **is a C*-algebra.**

Since $B(H)$ is a *-algebra, we need to show that it has the property $\|T^*T\| = \|T\|^2$, $\forall T \in B(H)$.

$\|T^*T(x)\| \leq \|T^*\|\|T\|\|x\| = \|T\|^2\|x\|$ hence

$$\|T^*T\| \leq \|T\|^2 \tag{1.4}$$

On the other hand, $\|Tx\|^2 = \langle Tx, Tx\rangle$

$$= \langle T^*Tx, x\rangle$$

$$\leq \|T^*T\|\|x\|^2 \text{ hence}$$

$$\|T\|^2 \leq \|T^*T\|. \tag{1.5}$$

From equations (1.4) and (1.5), $\|T^*T\| = \|T\|^2$. Hence $B(H)$ is a C*-algebra.

1.3.8: Definition; Positive functionals.

This is a linear functional f on a Banach algebra $\mathcal{A}$ with an involution that satisfies the condition $f(xx^*) \geq 0$ for all $x \in \mathcal{A}$.

1.3.9: Definition; Complex Homomorphism.

Suppose $\mathcal{A}$ is a complex algebra and f is a linear functional on $\mathcal{A}$ which is not identically zero. If $f(xy) = f(x)f(y)$ for all $x \in \mathcal{A}$ then f is a complex homomorphism on $\mathcal{A}$ i.e. a multiplicative linear mapping from one Banach algebra into another.

An element $x \in \mathcal{A}$ is invertible if it has an inverse in $\mathcal{A}$ i.e. if there exists an element $x^{-1} \in \mathcal{A}$ such that $x^{-1}x = xx^{-1} = e$, e is the unit element in$\mathcal{A}$.

1.3.10: Definition; *-morphism (homomorphism)

Suppose $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, a mapping $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a C*-homomorphism if for any $a, b \in \mathbb{C}$ and $x, y \in \mathcal{A}$ the following four conditions are satisfied.

(i) $\phi(ax + by) = a\phi(x) + b\phi(y)$

(ii) $\phi(xy) = \phi(x)\phi(y)$

(iii) $\phi(x^*) = (\phi(x))^*$

(iv) ϕ maps a unit in $\mathcal{A}$ to a unit in $\mathcal{B}$

If further ϕ is $1-1$ and onto, then it is a C*-isomorphism i.e.$\forall x, y \in \mathcal{A}$ and $x \neq y$, $\phi(x) \neq \phi(y)$.

1.3.11: Definition; State.

Let $\mathcal{A}$ be an algebra with involution. A linear functional f on $\mathcal{A}$ is self-adjoint or hermittian if $f(x^*) = f(\overline{x})$ $\forall x \in \mathcal{A}$. If further, $\|f\| = f(e) = 1$, then f is called a state.

1.3.12: Example.

A functional f on $B(H)$ for example is a state if $x \in H$, $\|x\| = 1$ and $f(T) = \langle Tx, x\rangle$ for all $T \in B(H)$.

Proof.

For all $T_1, T_2 \in B(H)$ and $\alpha_1, \alpha_2 \in \mathbb{K}$

$$\begin{aligned} f(\alpha_1 T_1 + \alpha_2 T_2) &= \langle(\alpha_1 T_1 + \alpha_2 T_2)x, x\rangle \\ &= \langle \alpha_1 T_1 x, x\rangle + \langle \alpha_2 T_2 x, x\rangle \\ &= \alpha_1\langle T_1 x, x\rangle + \alpha_2\langle T_2 x, x\rangle \\ &= \alpha_1 f(T_1) + \alpha_2 f(T_2). \end{aligned}$$

Also,

$|f(T)| = |\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2$

i.e. $\|f\| \leq \|x\|^2$ but $\|x\| = 1$. So

$$\|f\| \leq 1 \tag{1.6}$$

$f(I) = \langle Ix, x\rangle = \langle x, x\rangle = \|x\|^2 = 1$

$$1 = |f(I)| \leq \|f\|\|I\| = \|f\|. \tag{1.7}$$

From equations (1.7) and (1.8), $f(I) = \|f\| = 1$. The functional f on $B(H)$ is positive since $f(T^*T) = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0$. Hence $f$is a state on $B(H)$.

1.3.13: Definition; Representation.

A representation of a C*-algebra $\mathcal{A}$ is defined to be the pair (H, ϕ), where H is a complex Hilbert space and ϕ a *-morphism of $\mathcal{A}$ into $B(H)$. The representation (H, ϕ) is said to be faithful if and only if ϕ is a *-isomorphism between $\mathcal{A}$ and $\phi(\mathcal{A})$ i.e. if and only if $ker(\phi) = \{0\}$.

The space H is called the representation space, the operators $\phi(x)$ are called the representatives of $\mathcal{A}$. By implicit identification of ϕ and the set of representatives, one also says that ϕ is a representation of $\mathcal{A}$ on H.

1.3.14: Gelfand-Naimark Segal Representation.

With each positive linear functional, there is associated representation. Suppose that f is a positive linear functional on a C*-algebra $\mathcal{A}$, setting $N_f = \{a \in \mathcal{A} : f(a^*a) = 0\}$ where N_f is a left ideal on $\mathcal{A}$. N_f is closed [11] and the map $(\mathcal{A}/N_f)^2 \longrightarrow \mathbb{C}$ defined by $(a + N_f, b + N_f) = f(b^*a)$ is a well defined inner product on $\mathcal{A}/N_f$. We denote H_f the Hilbert completion of $\mathcal{A}/N_f$. If $a \in \mathcal{A}$, we define an operator $\phi(a) \in B(\mathcal{A}/N_f)$ by setting $\phi(a)(b + N_f) = ab + N_f$. The inequality $\phi(a) \leq \|a\|$ holds since we have $\|\phi(a)(b + N_f)\|^2 = f(b^*a^*ab)) \leq \|a\|^2 f(b^*b) \leq \|a\|^2\|b + N_f\|^2$. The operator $\phi(a)$ has a unique extension to a bounded operator $\phi_f(a)$ on H_f. The map $\phi_f : \mathcal{A} \longrightarrow B(H_f)$ defined by $a = \phi_f(a)$ is a *-homomorphism. The representation (H_f, ϕ_f) of $\mathcal{A}$ is called the Gelfand Naimark-Segal representation associated to f (GNS representation).

1.3.15: Definition; Calkin algebra.

Calkin algebra denoted by $B(H)/K(H)$is the quotient of $B(H)$, the algebra of all bounded linear operators on separable infinite dimensional Hilbert space H, by the ideal $K(H)$ of compact operators. Since the compact operator $K(H)$ is norm closed, minimal ideal in $B(H)$, the Calkin algebra is simple. As a C*-algebra, the Calkin algebra is remarkable because it is not isomorphic to an algebra of operators on a separable Hilbert space; instead, a larger Hilbert space has to be chosen. (By GNS theorem, every C*-algebra is isomorphic to an algebra of operators on a Hilbert space, for many other simple C*-algebras, there are explicit descriptions of such Hilbert spaces, but for the Calkin algebra this is not the case).

1.3.16: Remark.

If $K(H)$ is an ideal of $B(H)$, then $B(H)/K(H)$ is a C*-algebra with the multiplication given by

$(T + K(H))(S + K(H)) = TS + K(H)\ \forall T, S \in B(H)$.

Calkin algebra is a vector space if we define addition as below;

For $B(H)/K(H) = \{T + K(H) : T \in B(H)\}$,

$(T + K(H)) + (S + K(H)) = (T + S) + K(H)\ \forall T, S \in B(H)$.

1.3.17: Lemma.

Let $K(H)$ be a subspace of $B(H)$. Then the set of all cosets $B(H)/K(H) = \{T + K(H) : T \in B(H)\}$ is abelian under coset addition; $(T + K(H)) + (S + K(H)) = (T + S) + K(H)$. In order for the product $(T + K(H))(S + K(H)) = TS + K(H)$ to be well defined, we must have, $S + K(H) = S' + K(H) \Longrightarrow TS + K(H) = TS' + K(H)$ or equivalently, $S - S' \in K(H) \Longrightarrow T(S - S') = (S - S')T \in K(H)$. But $S - S'$ may be any *element of $K(H)$ and T any element of $B(H)$ and so this condition implies that $K(H)$ must be ideal.*

Conversely, if $K(H)$ is an ideal, then the coset multiplication is well defined.

1.3.18: Theorem.

Let $B(H)$ be the set of all bounded operators on H and $K(H)$ the set of compact operators on H. Then

$$\|T+S\| = inf\{\|T+S\| : S \in K(H)\} \tag{1.8}$$

defines a norm on the Calkin algebra $B(H)/K(H)$ $\forall T \in B(H)$.

Proof.

(i) $\|T+S\| \geq 0$ is clear since

$\|T+S\| = inf\{\|T+S\| : \|T\| = 1, S \in K(H)\}$.

Also,

$\|T+S\| = 0$ if and only if $\|T+S\| = 0$ implies that $\|T\| = 0$ since the zero element in $B(H)/K(H)$ is the coset $0 + K(H) = K(H)$ i.e. $0 + S = S, S \in K(H)$.

(ii) $$\begin{aligned}\|\alpha(T+S)\| &= inf\{\|\alpha(T+S)\| : S \in K(H)\} \\ &= inf\{|\alpha|\|T+S\| : S \in K(H)\} \\ &= |\alpha| inf\{\|T+S\| : S \in K(H)\} \\ &= |\alpha|\|T+S\|.\end{aligned}$$

(iii) $$\begin{aligned}\|(T+R)+S\| &= inf\{\|(T+R)+S\| : S \in K(H)\} \text{ for all } T, R \in B(H) \\ &= inf\{\|(T+S_1)+(R+S_2) : S_1, S_2 \in K(H)\} \\ &\leq inf\{\|T+S_1\| : S_1 \in K(H)\} + inf\{\|R+S_2\| : S_2 \in K(H)\} \\ &= \|T+S_1\| + \|R+S_2\|\end{aligned}$$

1.3.19: Definition;Span of S.

Let S be a non-empty subset of a linear space X over the field $\mathbb{K}$. The set of all linear combinations of elements of S is called the space spanned by S and is represented by $[S]$ i.e. $[S] = \{\alpha_1 x_1 + ... + \alpha_n x_n\}$: $n \in \mathbb{N}, x_i \in S$ and $\alpha_i \in \mathbb{K}$ for $i = 1, ..., n$.

1.3.20: Definition; Convex set.

Let X be a linear space. A subset M of the linear space X is convex if for all $x, y \in M$, and for any positive real number t satisfying $0 < t < 1$, $tx + (1 - t)y \in M$.

1.3.21: Lemma.

Let $x_1, x_2, ..., x_n$ be points in the convex set M and let $a_1, a_2, ..., a_n$ be non-negative scalars with $a_1 + a_2 + ... + a_n = 1$; then $a_1 x_1 + a_2 x_2 + ... + a_n x_n \in M$.

1.3.22: Definition; Convex hull.

If M is a subset of a linear space X, then a convex hull of M, represented by $(C_o M)$ is the smallest convex subset of X containing M i.e. the intersection of all the convex subsets of X that contain M.

1.3.23: Remark.

The intersection of any convex subsets of X is also convex.

1.4 Tensor products

Tensor product, denoted by $\otimes$, may be applied in different contexts to vectors, matrices, tensors, vector spaces, algebras, topological vector spaces and modules. In each case the significance of the symbol $\otimes$ is the same; the most general, bilinear map.

Let U and V be vector spaces over the same field F, and let T be the subspace of the free vector space $\pounds_{u\times v}$ on $U \times V$ generated by all vectors of the form;

(i) $r(u,v) + s(u',v) - (ru + su',v)$

(ii) $r(u,v) + s(u,v') - (u, rv + sv')$ $\forall r, s \in F, u, u' \in U$ and $v, v' \in V$

The quotient space $\pounds_{u\times v}/T$ is called the tensor product of U and V denoted by $U \otimes V$. An element of $U \otimes V$ has the form $\Sigma r_i(u_i, v_i) + T$. The coset $(u,v) + T$ is denoted by $u \otimes v$ and therefore any element μ of $U \otimes V$ has the form $\mu = \Sigma_i u_i \otimes v_i$. We note that by (i) and (ii), any element of T is equal to the zero vector.
Given bases $\{u_i\}$ and $\{v_i\}$ for U and V respectively, the tensors of the form $u_i \otimes v_i$ forms a basis for $U \otimes V$. The dimensions of the tensor product therefore is the product of the dimensions of the original spaces, for example, $R^m \otimes R^n$ will have dimension mn.

1.4.1 Bilinear maps and tensor products

A mapping f from the cartesian product$X \times Y$ of vector spaces into a vector space Z is bilinear if it is linear in each variable i.e.
$f(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 f(x_1, y) + \alpha_2 f(x_2, y)$ and
$f(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 f(x, y_1) + \beta_2 f(x, y_2)$ $\forall x, x_1, x_2 \in X,\ y, y_1, y_2 \in Y$ and scalars $\alpha_i, \beta_i, (i = 1, 2)$. We write $B(X, Y; Z)$ to denote the vector space of bilinear mappings from the product $X \times Y$ into Z; (the set of all bilinear functions from $X \times Y$ to Z). When Z is a scalar field we denote the corresponding space of bilinear forms simply by $B(X \times Y)$ i.e. bilinear

function $f : X \times Y \longrightarrow F$ with values in the base field F is a bilinear form on $X \times Y$.

1.4.1.1: Lemma.

Let f be a mapping from a cross product space to the tensor product space $f : X \times Y \longrightarrow X \otimes Y$ defined by $f(x, y) = x \otimes y$. Then f is a bilinear map.

Proof.

Let $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$. Also let $\alpha, \beta \in \mathbb{K}$.

Linearity in X

$$\begin{aligned} f(\alpha x_1 + \beta x_2, y) &= (\alpha x_1 + \beta x_2) \otimes y \\ &= (\alpha x_1 \otimes y) + (\beta x_2 \otimes y) \\ &= \alpha(x_1 \otimes y) + \beta(x_2 \otimes y) \\ &= \alpha f(x_1, y) + \beta f(x_2, y) \end{aligned}$$

Linearity in Y.

$$\begin{aligned} f(x, \alpha y_1 + \beta y_2) &= x \otimes (\alpha y_1 + \beta y_2) \\ &= (x \otimes \alpha y_1) + (x \otimes \beta y_2) \\ &= \alpha(x \otimes y_1) + \beta(x \otimes y_2) \\ &= \alpha f(x, y_1) + \beta f(x, y_2). \end{aligned}$$

1.4.1.2: Remark.

The tensor product $X \otimes Y$, of vector spaces X and Y can be constructed as a space of linear functionals on $B(X \times Y)$ in the following ways. For $x \in X$, $y \in Y$, we denote $x \otimes y$ the functional given by evaluation at the point (x, y) i.e. $(x \otimes y)(f) = \langle f, x \otimes y \rangle = f(x, y)$ for each bilinear form on $X \times Y$. The tensor product $X \otimes Y$ is the subspace of the dual $B(X \times Y)^*$ spanned by these elements. Thus a typical tensor in $X \otimes Y$ has the form $\sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ $\forall n \in \mathbb{N}$, $\lambda_i \in \mathbb{K}$, $x \in X$ and $y \in Y$. We also note that the space $(X \times Y)^*$

(the dual space of $X \times Y$ containing all linear functionals on that space) corresponds naturally to the space of all bilinear functionals on $X \times Y$ i.e. every bilinear functional is a functional on the tensor product and vise versa. Whenever X and Y are finite dimensional, there is a natural isomorphism between $X^* \otimes Y^*$ and $(X \otimes Y)^*$. For vector spaces of arbitrary dimension we only have an inclusion $X^* \otimes Y^* \subset (X \otimes Y)^*$. So the the tensors of linear functionals are bilinear functionals. This gives us a new way to look at the space of bilinear functionals as a tensor product itself.

1.4.2 Algebraic properties of tensor products.

Tensor products obey a number of nice rules. For matrices A, B, C, D, vectors U, V, W and scalars a, b, c, d, the following hold;

(1) $(A \otimes B)(C \otimes D) = AC \otimes BD$

(2) $(A \otimes B)(U \otimes V) = AU \otimes BV$

(3) $(U + V) \otimes W = U \otimes W + V \otimes W$

(4) $U \otimes (V + W) = U \otimes V + U \otimes W$

(5) $aU \otimes bV = ab(U \otimes V)$

(6) $(U \otimes V)^* = U^* \otimes V^*$

(7) $(U \otimes V) \otimes W = U \otimes (V \otimes W)$

(8) $(\beta U) \otimes V = U \otimes (\beta V)$

(9) $(V \otimes U)^{-1} = V^{-1} \otimes U^{-1}$

i.e. $(U \otimes V)(U^{-1} \otimes V^{-1}) = UU^{-1} \otimes VV^{-1} = I \otimes I = I$

This shows that $(U \otimes V)^{-1} = U^{-1} \otimes V^{-1}$.

(10) Thus for matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes U = \begin{pmatrix} A \otimes U & B \otimes U \\ C \otimes U & D \otimes U \end{pmatrix}$$

which specializes for scalars too

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes U = \begin{pmatrix} aU & bU \\ cU & dU \end{pmatrix}$$

We note that the conjugate transpose distributes over tensor products such that $(A \otimes B)^t = A^t \otimes B^t$.

1.4.3 Universal property of tensor products

The space of all bilinear maps from $X \times Y$ to another vector space Z is naturally isomorphic to the space of all linear maps from $X \otimes Y$ to Z. This is built into the construction; $X \otimes Y$ has all relations that are necessary to ensure that a homomorphism from $X \otimes Y$ to Z will be linear.

1.4.3.1: Lemma.

Let X and Y be vector spaces over the same field F. There exists $X \otimes Y$ called tensor product of X and Y with a canonical bilinear homomorphism $f : X \times Y \longrightarrow X \otimes Y$ distinguished up to isomorphism, by the following universal property; Every bilinear homomorphism $\phi : X \times Y \longrightarrow Z$ lifts to a unique homomorphism $\widetilde{\phi} : X \otimes Y \longrightarrow Z$ such that $\phi(x, y) = \widetilde{\phi}(x \otimes y)$ for all $x \in X$ and $y \in Y$.

Proof.

Since $f(x, y) = x \otimes y = (x, y) + T$, the map $f : X \times Y \longrightarrow X \otimes Y$ is a canonical injection $j : X \times Y \longrightarrow \pounds_{X\times Y}$ followed by a canonical projection $\pi : \pounds_{X\times Y} \longrightarrow X \otimes Y = \pounds_{X\times Y}/T$ i.e. $f = \pi o j$

The universal property of free vector spaces implies that there is a unique linear transformation $\sigma : \pounds_{X\times Y} \longrightarrow Z$ for which $\sigma o j = \phi$. Since ϕ is bilinear, it sends any of the vectors

(i) $r(x, y) + s(x', y) - (rx + sx', y)$

(ii) $r(x, y) + s(x, y') - (x, sy + sy')$

that generates T to the zero vector, so $T \subset ker(\sigma)$. Hence there exists a unique linear transformation $\widetilde{\phi} : X \times Y \longrightarrow Z$ for which $\widetilde{\phi} o \pi = \sigma$. Thus $\widetilde{\phi} o f = \widetilde{\phi} o \pi o j = \sigma o j = \phi$. Moreover, if $\widetilde{\phi} o f = \phi$, then $\sigma' = \widetilde{\phi}' o \pi : \pounds_{X\times Y} \longrightarrow Z$ is a linear transformation for which $\sigma' o j(x, y) = \widetilde{\phi} o \pi o j(x, y) = \widetilde{\phi} o f(x, y) = \phi(x, y) = \sigma o j(x, y)$ and so $\sigma' o j = \sigma o j \Rightarrow \sigma' = \sigma \Rightarrow \widetilde{\phi}' = \phi$. Hence $\widetilde{\phi}$ is unique.

1.4.3.2: Remark.

The universal property of tensor products says that for each bilinear function $\phi : X \times Y \longrightarrow Z$, there corresponds a unique linear function $\widetilde{\phi} : X \otimes Y \longrightarrow Z$ through which the function $f : X \times Y \longrightarrow X \otimes Y$ is factored i.e. $\phi = \widetilde{\phi} o f$. This establishes a map $\psi : B(X, Y; Z) \longrightarrow \mathfrak{L}(X \otimes Y, Z)$ defined by $\psi(\phi) = \widetilde{\phi}$ where $\psi(\phi)$ is a unique linear map $\psi(\phi) : X \otimes Y \longrightarrow Z$ defined by $\psi(\phi)(x \otimes y) = \phi(x, y)$.

We observe that ψ is *linear*, since if $\phi, t \in B(X, Y; Z)$, then $\forall r, s \in F$ $[r\psi(\phi) + s\psi(t)](x \otimes y) = r(\phi)(x, y) + s(t)(x, y) = (r\phi + st)(x, y)$ and so the uniqueness part of the universal property implies that $r\psi(\phi) + s\psi(t) = \psi(r\phi + st)$.

Also, ψ is *surjective* since if $\widetilde{\phi} : X \otimes Y \longrightarrow Z$ is any linear map, then $\phi = \widetilde{\phi} o f : X \times Y \longrightarrow Z$ is bilinear, and by the uniqueness part of the universal property, $\psi(\phi) = \widetilde{\phi}$.

Finally, ψ is *injective,* for if $\psi(\phi) = 0$ then $\phi = \psi(\phi) o f = 0$.

This implies that for X, Y, Z vector spaces over the same field F the map $\psi : B(X, Y; Z) \longrightarrow \mathfrak{L}(X \otimes Y; Z)$ defined by the fact that $\psi(\phi)$ is the unique linear map for which $\phi = \psi(\phi) o f$ is an isomorphism. Thus

$B(X, Y; Z) \simeq \mathfrak{L}(X \otimes Y; Z)$.

1.4.4 Tensor norm

Proposition 1

Let X and Y be Hilbert spaces. We denote $X \otimes Y$ the tensor product space between X and Y. The elements of $X \otimes Y$ are denoted by $x \otimes y$ where $x \in X$ and $y \in Y$. Then $\|x \otimes y\| = \|x\| \|y\|$ defines a norm.

Proof.

We shall prove that $\|x \otimes y\|$ satisfy all the axioms of a norm.

(i) $\|x \otimes y\| \geq 0$ and that $\|x \otimes y\| = 0 \Leftrightarrow x \otimes y = 0$ is clear.

(ii) $\|\alpha(x \otimes y)\| = |\alpha| \|x \otimes y\|$ $\alpha \in \mathbb{K}$.

Now, $\|x \otimes y\|^2 = \langle x \otimes y, x \otimes y \rangle = \langle x, x \rangle \langle y, y \rangle = \|x\|^2 \|y\|^2$ and by the algebraic properties of tensor products,

$\alpha(x \otimes y) = (\alpha x \otimes y) = (x \otimes \alpha y)$, so

$$\begin{aligned}
\|\alpha(x \otimes y)\|^2 &= \langle \alpha x \otimes y, \alpha x \otimes y \rangle \\
&= \langle x \otimes \alpha y, x \otimes \alpha y \rangle \\
&= \langle \alpha x, \alpha x \rangle \langle y, y \rangle \\
&= |\alpha|^2 \|x\|^2 \|y\|^2
\end{aligned}$$

$= |\alpha| \|x \otimes y\|$

$\|\alpha(x \otimes y)\| = |\alpha| \|x \otimes y\|.$

(iii) For all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have that

$\|(x_1 \otimes y_1) + (x_2 \otimes y_2) \leq \|x_1 \otimes y_1\| + \|x_2 \otimes y_2\|$. Now,

$\|(x_1 \otimes y_1) + (x_2 \otimes y_2)\|^2 = \langle x_1 \otimes y_1 + x_2 \otimes y_2, x_1 \otimes y_1 + x_2 \otimes y_2 \rangle$

$= \langle x_1 \otimes y_1, x_1 \otimes y_1 \rangle + \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle + \langle x_2 \otimes y_2, x_1 \otimes y_1 \rangle + \langle x_2 \otimes y_2, x_2 \otimes y_2 \rangle$

$= \langle x_1, x_1 \rangle \langle y_1, y_1 \rangle + \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle + \langle x_2, x_1 \rangle \langle y_2 y_1 \rangle + \langle x_2, x_2 \rangle \langle y_2, y_2 \rangle$

$= \|x_1\|^2 \|y_1\|^2 + \|x_2\|^2 \|y_2\|^2 + \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle + \overline{\langle x_1, x_2 \rangle \langle y_1, y_2 \rangle}$

$= \|x_1\|^2 \|y_1\|^2 + \|x_2\|^2 \|y_2\|^2 + 2Re\langle x_1, x_2 \rangle \langle y_1, y_2 \rangle.$

$\leq \|x_1\|^2 \|y_1\|^2 + \|x_2\|^2 \|y_2\|^2 + 2\|x_1\| \|x_2\| \|y_1\| \|y_2\|$

$= \{\|x_1\| \|y_1\| + \|x_2\| \|y_2\|\}^2$ by Cauchy-Schwarz inequality.

$\Rightarrow \|(x_1 \otimes y_1) + (x_2 \otimes y_2)\| \leq \|x_1\| \|y_1\| + \|x_2\| \|y_2\|.$

Proposition 2

Let X and Y be vector spaces, let E and F be linearly independent subsets of X and Y respectively. Then $\{x \otimes y : x \in E, y \in F\}$ is a linearly independent subset of $X \otimes Y$.

Proof.

Suppose we have that $\mu = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i = 0$ where $x_i \in E$ and $y_i \in F$. Let f, g be linear functionals on X and Y respectively and consider the bilinear form defined by $\phi(x, y) = f(x)g(y)$. We have $\mu(\phi) = 0$ and so $\sum_{i=1}^{n} \lambda_i f(x_i) g(y_i) = g(\sum_{i=1}^{n} \lambda_i f(x_i) y_i) = 0$. Since this holds for every $g \in Y^*$, we can conclude that $\sum_{i=1}^{n} \lambda_i f(x_i) y_i = 0$ and so by linear independence of F, we have $\lambda_i f(x_i) = 0$ for all $f \in X^*$. But by linear independence of E, each x_i is non zero and it follows that $\lambda_i = 0$ for all i. Thus if X and Y are finite dimensional spaces then $dim(X \otimes Y) = dim(X)dim(Y)$.

1.4.3.3: Definition; Haagerup norm.

The Haagerup norm on the algebraic tensor product $B(H) \otimes B(H)$ is defined by $\|\phi_n\| = inf \| \sum_{i=1}^{k} a_i a_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^{k} b_i^* b_i \|^{\frac{1}{2}}$, where the infimum is taken over all possible representations of ϕ in the form $\phi = \sum_{i=1}^{k} a_i \otimes b_i$. By natural map; $B(H) \otimes B(H) \longrightarrow \mathcal{CB}(B(H))$ is defined by $\theta(\sum_i a_i \otimes b_i)(x) = \sum_i a_i x b_i$. We may algebraically identify $B(H) \otimes B(H)$ with the space of all elementary operators on $B(H)$. For each ϕ in $B(H) \otimes B(H)$ the completely bounded norm of $\theta(\phi)$ is equal to the Haagerup norm of ϕ [9]. $\theta(\sum_i a_i \otimes b_i)(x) = \sum_i a_i x b_i$. We may algebraically identify $B(H) \otimes B(H)$ with the space of all elementary operators on $B(H)$. For each ϕ in $B(H) \otimes B(H)$ the completely bounded norm of $\theta(\phi)$ is equal to the Haagerup norm of ϕ [9].

1.4.5 Statement of The Problem

Let H be a complex Hilbert space, $T : H \to H$ a bounded linear operator and $B(H)$ the set of bounded linear operators on H. Clearly $B(H)$ is an algebra. Our main result shall concern the operator $T_{a,b} : B(H) \to B(H)$ defined by $T_{a,b}(x) = axb + bxa$ for all $x \in H$ and a, b fixed in $B(H)$. No formula is known for computing the norm of $T_{a,b}$. Clearly,
$\|T_{a,b/B(H)}\| \leq 2\|a\|\|b\|$. But in estimating the norm of $T_{a,b}$ in the opposite direction, the largest possible c such that $\|T_{a,b/B(H)}\| \geq c\|a\|\|b\|$ for all $a, b \in B(H)$ and $c \in \mathbb{R}$ is not known. Nyamwala [12] proved $c = 2$ in $B(\mathbf{C}^2)$. We shall extend our research to investigate the norm of derivation of the elementary operator and the corresponding tensor norm. We shall further establish the relationship between the norm of derivation of the elementary operator $T_{a,b}$ and the corresponding tensor norm.

1.4.6 Objectives of the study

(i) To investigate the lower bound of the operator $T_{a,b}$.

(ii) To investigate the derivation of the operator $T_{a,b}$.

Chapter 2

TENSOR PRODUCT OPERATOR.

In this chapter we show that the tensor products $T \otimes S$ and $T \odot S$ are normed operators. We have also shown the relationship between the C*-norms; spatial, projective and Haagerup. Consequently, we prove that $\|T_{a,b}\| \geq 2\|a\|\|b\|$ on the injective tensor norm.

The standard tensor product of Hilbert spaces H and K shall be denoted by $H\tilde{\otimes}K$ i.e. the tensor product $H \otimes K$ completed with respect to the norm induced by the inner product given on elementary tensors by

$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle_H \langle y, y' \rangle_K$, so that $B(H) \otimes B(K) \subseteq B(H\tilde{\otimes}K)$ via $(T \otimes S)(x \otimes y) = T(x) \otimes S(y)$ for all $T \in B(H)$, $S \in B(K)$.

2.0.0: Theorem.

Let H and K be Hilbert spaces, $B(H)$ and $B(K)$ be sets of bounded linear operators on H and K respectively. Suppose that $T \in B(H)$ and $S \in B(K)$, then there is a unique linear bounded operator $T\tilde{\otimes}S \in B(H\tilde{\otimes}K)$ defined by

$(T\tilde{\otimes}S)(x\otimes y)=T(x)\otimes S(y)$ *for all* $x\in H$ *and* $y\in K$. *Moreover,* $\|T\tilde{\otimes}S\|=\|T\|\|S\|$.

Proof.

The map $\phi: T\times S\longrightarrow T\otimes S$ defined by $\phi(T,S)=T\otimes S$ is bilinear.

Linearity in T

Let $\alpha,\beta\in\mathbb{K}$, $T_1,T_2\in B(H)$ and $S\in B(K)$. Then

$$\begin{aligned}\phi(\alpha T_1+\beta T_2,S)&=(\alpha T_1+\beta T_2)\otimes S\\&=(\alpha T_1\otimes S)+(\beta T_2\otimes S)\\&=\alpha(T_1\otimes S)+\beta(T_2\otimes S)\\&=\alpha\phi(T_1,S)+\beta\phi(T_2,S).\end{aligned}$$

Linearity in S.

$$\begin{aligned}\phi(T,\alpha S_1+\beta S_2)&=T\otimes(\alpha S_1+\beta S_2)\ \forall S_1,S_2\in B(K).\\&=(T\otimes\alpha S_1)+(T\otimes\beta S_2)\\&=\alpha(T\otimes S_1)+\beta(T\otimes S_2)\\&=\alpha\phi(T,S_1)+\beta\phi(T,S_2).\end{aligned}$$

The operator $T\otimes S: H\otimes K\longrightarrow H\otimes K$ is bounded. We may assume that T and S are unitaries, since unitaries span the C*-algebras $B(H)$ and $B(K)$. Now, $\sum_{i=1}^{n}x_i\otimes y_i\in H\otimes K$ where $y_1,....,y_n$ are orthogonal. Hence

$$\begin{aligned}\|(T\otimes S)(\textstyle\sum_{i=1}^{n}x_i\otimes y_i)\|^2&=\|\textstyle\sum_{i=1}^{n}T(x_i)\otimes S(y_i)\|^2\\&=\textstyle\sum_{i=1}^{n}\|T(x_i)\otimes S(y_i)\|^2\\&\text{(since } S(y_1,...,S(y_n) \text{ are orthogonal).}\\&=\textstyle\sum_{i=1}^{n}\|T(x_i)\|^2\|S(y_i)\|^2\\&=\textstyle\sum_{i=1}^{n}\|x_i\|^2\|y_i\|^2\\&=\|\textstyle\sum_{i=1}^{n}x_i\otimes y_i\|^2\end{aligned}$$

Consequently, $\|T\otimes S\|=1$.

Thus, for all operators T, S on H, K respectively, the linear map $T \otimes S$ is bounded on $H \otimes K$ and hence has an extension to a bounded linear map $T\widetilde{\otimes}S$ on $H\widetilde{\otimes}K$. The maps $B(H) \longrightarrow B(H\widetilde{\otimes}K)$ defined by $T \longrightarrow T\widetilde{\otimes}e_K$ and $B(K) \longrightarrow B(H\widetilde{\otimes}K)$ defined by $S \longrightarrow e_H\widetilde{\otimes}S$ are injective *-homomorphism.

For example,

$\phi(T) = T\widetilde{\otimes}e_K = \phi(T)$,

$\phi(T_1T_2) = \phi(T_1)\phi(T_2)$ and

$\phi(T^*) = \phi(T)^*$ for all $T_1, T_2 \in B(H)$.

Consequently, the maps are isometric for if $T_1 \neq T_2$, then $\phi(T_1) \neq \phi(T_2)$.

Hence, $\|T\widetilde{\otimes}e\| = \|T\|$ and $\|e\widetilde{\otimes}S\| = \|S\|$.

So, $\|T\widetilde{\otimes}S\| = \|(T\widetilde{\otimes}e)(e\widetilde{\otimes}S)\| \leq \|T\|\|S\|$.

If ϵ is a sufficiently small positive number, and if $T, S \neq 0$, then there are unit vectors x and y such that $\|T(x)\| > \|T\| - \epsilon > 0$ and $\|S(y)\| > \|S\| - \epsilon > 0$. Hence, $\|(T\widetilde{\otimes}S)(x \otimes y)\| = \|T(x)\|\|S(y)\| > (\|T\| - \epsilon)(\|S\| - \epsilon)$. So $\|T\widetilde{\otimes}S\| > (\|T\| - \epsilon)(\|S\| - \epsilon)$ and as $\epsilon \longrightarrow 0$ we get $\|T\widetilde{\otimes}S\| \geq \|T\|\|S\|$. See [11].

2.0.1: Lemma.

Let H and K be Hilbert spaces and suppose that $T, T' \in B(H)$ and $S, S' \in B(K)$. Then

(i) $(T\widetilde{\otimes}S)(T'\widetilde{\otimes}S') = TT'\widetilde{\otimes}SS'$ *and*

(ii) $(T\widetilde{\otimes}S)^* = T^*\widetilde{\otimes}S^*$.

Proof.

(i) The proof follows from the following theorem.

2.0.2: Theorem.

If $\mathcal{A}$ and $\mathcal{B}$ are algebras, then there is a unnique associative multiplication M on the vector space $\mathcal{A} \otimes \mathcal{B}$ for which the equation $M(a_1 \otimes b_1, a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$ holds for all $a_1, a_2 \in \mathcal{A}$ and $b_1 b_2 \in \mathcal{B}$

Proof.

Let$\varrho : \mathcal{A} \longrightarrow \mathfrak{L}(\mathcal{A})$ and $\rho : \mathcal{B} \longrightarrow \mathfrak{L}(\mathcal{B})$ be left regular representations of $\mathcal{A}$ and $\mathcal{B}$ respectively. Consider a bilinear map $\phi : \mathcal{A} \times \mathcal{B} \longrightarrow \mathfrak{L}(\mathcal{A} \otimes \mathcal{B})$ defined by $\phi(a, b) = \varrho(a)\rho(b)$ where $\varrho(a) \otimes \rho(b)$ is the unique linear transformation on $\mathcal{A} \otimes \mathcal{B}$ where, $\varrho(a) \otimes \rho(b)[c \otimes d] = (ac) \otimes (bd)$ for all $c \otimes d \in \mathcal{A} \otimes \mathcal{B}$. By the universal property of tensor products, there is a unique linear transformation $\widetilde{\phi} : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathfrak{L}(\mathcal{A} \otimes \mathcal{B})$ where $\widetilde{\phi}(a \otimes b) = \phi(a, b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. We define $M((\mathcal{A} \otimes \mathcal{B}) \times (\mathcal{A} \otimes \mathcal{B})) \longrightarrow \mathcal{A} \otimes \mathcal{B}$ by $M(\xi, \eta) = \widetilde{\phi}\xi[\eta]$ for all $\xi, \eta \in \mathcal{A} \otimes \mathcal{B}$. Since $\widetilde{\phi}$ and $\widetilde{\phi}\xi$ are linear transformations, M is a bilinear function, thus it remains to show that M is an associative multiplication. To do so, we note that by bilinearity of M, it is sufficient to show that M, it is sufficient to show that M is associative on the spanning set of elementary tensors.

Verification.

$$
\begin{aligned}
M((a_1 \otimes b_1), M(a_2 \otimes b_2, a_3 \otimes b_3)) &= \varrho(a_1) \otimes \rho(b_1)[\varrho(a_2) \otimes \rho(b_2)(a_3 \otimes b_3)] \\
&= \varrho(a_1) \otimes \rho(b_1[(a_2 a_3) \otimes (b_2 b_3)] \\
&= a_1(a_2 a_3) \otimes (b_1 b_2) b_3 \\
&= (a_1 a_2) a_3 \otimes (b_1 b_2) b_3 \\
&= M(M(a_1 \otimes b_1, a_2 \otimes b_2) a_3 \otimes b_3).
\end{aligned}
$$

Thus M is an associative multiplication on $\mathcal{A} \otimes \mathcal{B}$. Suppose now

M' is another such multiplication and similarly we can show that $M'(a_1 \otimes b_1, a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$ holds $\forall a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. Then M and M' have identical values on spanning set of $\mathcal{A} \otimes \mathcal{B}$ and therefore $M = M'$, which proves that M is unique.

(ii) $(T\tilde{\otimes}S)^* = T^*\tilde{\otimes}S^* \Leftrightarrow (T\tilde{\otimes}S)^*(x \otimes y) = T^*(x)\tilde{\otimes}S^*(y)\ \forall x \otimes y \in H \otimes K$.
By definition of $(T\tilde{\otimes}S)^*$,
$\langle (T\tilde{\otimes}S)x' \otimes y', x \otimes y \rangle = \langle x' \otimes y', (T\tilde{\otimes}S)^* x \otimes y \rangle\ \forall x \otimes y, x' \otimes y' \in H \otimes K$.
Also, $\langle (T\tilde{\otimes}S)x' \otimes y', x \otimes y \rangle = \langle T(x')\tilde{\otimes}S(y'), x \otimes y \rangle$

$$= \langle T(x'), x \rangle \tilde{\otimes} \langle S(y'), y \rangle$$
$$= \langle (x', T^* x \rangle \tilde{\otimes} \langle (y', S^* y \rangle$$
$$= \langle x' \tilde{\otimes} y', T^* x \tilde{\otimes} S^* y \rangle$$

i.e.$\langle x' \otimes y', (T\tilde{\otimes}S)^* x \otimes y \rangle = \langle x' \otimes y', T^* x \tilde{\otimes} S^* y \rangle$.

2.0.3:Theorem.

Let H and K be Hilbert spaces. Then there is a unique inner product $\langle ., . \rangle$ on $H \otimes K$ such that $\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle\ \forall x, x' \in H$ and $y, y' \in K$.

Proof.

Let $\tau : H \longrightarrow \mathbb{C}$ and $\rho : K \longrightarrow \mathbb{C}$ be conjugate linear maps. Then there is a unique conjugate linear map, $\tau \otimes \rho : H \otimes K \longrightarrow \mathbb{C}$ defined by $\tau \otimes \rho(x \otimes y) = \tau(x) \otimes \rho(y)\ \forall x \in H, y \in K$. We note that $\overline{\tau}$ and $\overline{\rho}$ are linear and set $\tau \otimes \rho = (\overline{\tau} \otimes \overline{\rho})^-$. Now, τx is a conjugate linear functional defined by setting $\tau x(y) = \langle x, y \rangle\ \forall x \in H$. If X is the space of all conjugate linear functionals on $H \otimes K$, then the map from $H \times K$ defined by $(x, y) = \tau x \otimes \tau y$ is *bilinear*, i.e. $\forall \alpha, \beta \in \mathbb{K}, x, x' \in H$,

$$(\alpha x + \beta x', y) = \tau(\alpha x + \beta x') \otimes \tau y$$
$$= (\tau \alpha x \otimes \tau y) + (\tau \beta x' \otimes \tau y)$$

$$= \alpha(\tau x \otimes \tau y) + \beta(\tau x' \otimes \tau y)$$
$$= \alpha(x, y) + \beta(x', y).$$

Also, $(x, \alpha y + \beta y') = \tau x \otimes \tau(\alpha y + \beta y')$
$$= (\tau x \otimes \tau \alpha y) + (\tau x \otimes \tau y')$$
$$= \alpha(\tau x \otimes \tau y) + \beta(\tau x \otimes \tau y')$$
$$= \alpha(x, y) + \beta(x, y')\ \forall y, y' \in K.$$

And so by the universal property of tensor products, there exists a unique linear map $f : H \otimes K \longrightarrow X$ defined by $f(x \otimes y) = \tau x \otimes \tau y\ \forall x \in H$ and $y \in K$. The map $\langle ., . \rangle : (H \otimes K)^2 \longrightarrow \mathbb{C}$ defined by $(z, z') = f(z)(z')$ is a sesquilinear form on $H \otimes K\ \forall z \in H \otimes K$ such that $\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle$ for all $x, x' \in H$ and $y, y' \in K$. If $z \in H \otimes K$, then $z = \sum_{i=1}^{n} x_i \otimes y_i$ for some $x_1, ..., x_n \in H$ and $y_1, ..., y_n \in K$. Let $e_i, .., e_m$ be an orthonormal basis for the linear span of $y_1, .., y_n$. Then $z = \sum_{i=1}^{n} x_i \otimes e_i$ for some $x_1, .., x_m \in H$. So, $\langle z, z' \rangle = \sum_{i,j=1}^{m} \langle x_i' \otimes e_i, x_j' \otimes e_j \rangle = \sum_{i,j=1}^{m} \langle x_i', x_j' \rangle \langle e_i, e_j \rangle = \sum_{i=1}^{m} \|x_i'\|^2$. Thus $\langle ., . \rangle$ is positive, and therefore if $\langle z, z \rangle = 0$ then for $x_i's$ we have $z = 0$ for $i = 1, ..., m$. Hence $\langle ., . \rangle$ is an inner product.

2.0.4: Lemma.

If E_1 and E_2 are orthonormal basis for H and K as above respectively, then $E_1 \otimes E_2 = \{x \otimes y : x \in E_1, y \in E_2\}$ is an orthonormal basis for $H \widetilde{\otimes} K$. See [12] and [20] page 301 for the proof.

2.1 Norm of tensor product operator.

Let V, V', W, W' be vectors over the same field. Let $T : V \longrightarrow V'$ and $S : W \longrightarrow W'$ be operators. Then there is a unique linear operator

$$T \odot S : V \otimes W \longrightarrow V' \otimes W' \tag{2.1}$$

defined by $T \odot S(x \otimes y) = T(x) \otimes S(y)$ for all $x \in V, y \in W$. The function $\widetilde{\phi} : V \times W \longrightarrow V' \otimes W'$ defined by $\widetilde{\phi}(x, y) = T(x) \otimes S(y)$ is bilinear and so by the universal property of tensor products, there exists a unique linear operator for which (2.1) holds. The map $T \odot S$ is called the tensor product of T and S. The map $\phi : \mathfrak{L}(V, W) \otimes \mathfrak{L}(V', W') \longrightarrow \mathfrak{L}(V \otimes W, V' \otimes W')$ defined by $\phi(T, S) = T \odot S$ is also bilinear and so there is a linear transformation $\psi : \mathfrak{L}(V, W) \otimes \mathfrak{L}(V', W') \longrightarrow \mathfrak{L}(V \otimes W, V' \otimes W')$ defined by $\psi(T \otimes S) = T \odot S$.

ψ is injective.

We observe that any non zero product $\xi \in \mathfrak{L}(V, W) \otimes \mathfrak{L}(V', W')$ has the form $\xi = \sum_{i=1}^{n} T_i \otimes S_i$ where $T_i's$ and $S_i's$ are linearly independent. It suffices therefore to show that $ker(\psi) = \{0\}$.

Suppose $\psi(\xi) = \psi(\sum_{i=1}^{n} T_i \otimes S_i) = 0$ then $\forall v \in V, y \in W$,

$$\sum_{i=1}^{n} T_i(x) \otimes S_i(y) = 0 \tag{2.2}$$

Let us choose $x \in V$ so that $T_i(x) \neq 0$ and suppose that $T_1(x), ..., T_k(x)$ is a maximal linearly independent set among $T_1(x), .., T_n(x)$. Then

$T_1(x) = \sum_{i=1}^{k} r_{l,j} T_j(x)$ for $l = k+1, .., n$. Hence equation (2.2) gives

$0 = \sum_{i=1}^{k} T_i(x) \otimes S_i(y) + \sum_{i=k+1}^{n} (\sum_{j=1}^{n} r_{i,j} T_j(x)) \otimes S_l(y)$

$= \sum_{i=1}^{k} T_i(x) \otimes S_i(y) + \sum_{j=1}^{k} T_i(x) \otimes \sum_{i=k+1}^{n} r_{i,j} S_l(y)$

$= \sum_{i=1}^{k} T_i(x) \otimes S_i(y) + \sum_{i=k+1}^{n} r_{i,j} S_l(y)$ and since $T_1(x), .., T_k(x)$ are linearly independent, we must have $S_i(y) + \sum_{i=k+1}^{n} r_{i,j} S_l(y) = 0$ for all $i = 1, .., k$ and

$y \in W$. So $S_i + \sum_{i=k+1}^{n} r_{i,j} S_l = 0$ which is a contradiction to the fact that the $S_i's$ are linearly independent. Hence $\psi(\xi) \neq$ and so ψ is injective. See [20] page 303.

2.1.1: The operator $T \odot S$ is both linear and bounded.

Linearity

The map $T \odot S : V \otimes W \longrightarrow V' \otimes W'$ is defined by

$T \odot S(\sum_{i=1}^{n} x_i \otimes y_i) = \sum_{i=1}^{n} T(x_i) \otimes S(y_i)$ $\forall x \in V$, $y \in W$. Let $\alpha, \beta \in \mathbb{K}$ and $\sum_{i=1}^{n} x_i \otimes y_i, \sum_{i=1}^{n} x_i' \otimes y_i' \in V \otimes W$. Then

$$\begin{aligned} T \odot S(\alpha \textstyle\sum_{i=1}^{n} x_i \otimes y_i + \beta \sum_{i=1}^{n} x_i' \otimes y_i') &= \\ &= T \odot S(\alpha \textstyle\sum_{i=1}^{n} x_i \otimes y_i) + T \odot S(\beta \sum_{i=1}^{n} x_i' \otimes y_i') \\ &= \alpha \textstyle\sum_{i=1}^{n} T(x_i) \otimes S(y_i) + \beta \sum_{i=1}^{n} T(x_i') \otimes S(y_i') \\ &= \alpha T \odot S(\textstyle\sum_{i=1}^{n} x_i \otimes y_i) + \beta T \odot S(\sum_{i=1}^{n} x_i' \otimes y_i'). \end{aligned}$$

Boundedness

$$\begin{aligned} \|T \odot S(\textstyle\sum_{i=1}^{n} x_i \otimes y_i)\| &= \| \textstyle\sum_{i=1}^{n} T(x_i) \otimes S(y_i)\| \\ &\leq \textstyle\sum_{i=1}^{n} \|Tx_i\| \|Sy_i\| \\ &\leq \textstyle\sum_{i=1}^{n} \|T\| \|x_i\| \|S\| \|y_i\| \\ &= \|T\| \|S\| \textstyle\sum_{i=1}^{n} \|x_i\| \|y_i\| \\ &= \|T\| \|S\| \| \textstyle\sum_{i=1}^{n} x_i \otimes y_i\|. \end{aligned}$$

2.1.2: The norm of $T \odot S$.

$$\|T \odot S\| = sup_{\| \sum_{i=1}^{n} x_i \otimes y_i\| = 1} \{\|T \odot S(\textstyle\sum_{i=1}^{n} x_i \otimes y_i)\|, x \in V, y \in W\}$$

$$\leq sup_{\| \sum_{i=1}^{n} x_i \otimes y_i\| = 1} \{\frac{\|T\| \|S\| \| \sum_{i=1}^{n} x_i \otimes y_i\|}{\| \sum_{i=1}^{n} x_i \otimes y_i\|}, x \in V, y \in W\} = \|T\| \|S\|$$

$$\|T \odot S\| \leq \|T\| \|S\| \tag{2.3}$$

Conversely,since

$$\|T \odot S\| = sup\{\frac{|T \odot S(\sum_{i=1}^{n} x_i \otimes y_i)|}{\|\sum_{i=1}^{n} x_i \otimes y_i\|}, x \in V, y \in W\}$$

It follows that

$$\|T \odot S\| \geq \frac{|T \odot S(\sum_{i=1}^{n} x_i \otimes y_i)|}{\|\sum_{i=1}^{n} x_i \otimes y_i\|}$$

for all $\sum_{i=1}^{n} x_i \otimes y_i \in V \otimes W$ and $\sum_{i=1}^{n} x_i \otimes y_i \neq 0$

$$\|T\|\|S\| \geq \|T\|\|S\| \tag{2.4}$$

So by equations (2.3) and (2.4) we have $\|T \odot S\| = \|T\|\|S\|$.

2.2 Tensor products of Banach spaces

The obvious way to define the tensor product of two Banach spaces **A** and **B** is to copy the method for Hilbert spaces; define a norm on the algebraic tensor product, then take the completion in this norm. The problem is that there are more than one natural way to define a norm on the tensor product. A cross norm $\|.\|$ on the algebraic tensor product of **A** and **B** is a norm satisfying the conditions $\|a \otimes b\| = \|a\|\|b\|$ and $\|a^* \otimes b^*\| = \|a^*\|\|b^*$. Here a^* and b^* are the duals of a and b respectively and $\|.\|^*$ is the dual of $\|.\|$. There is the smallest cross norm $\|.\|_\vee$ called the **injective** cross norm given by $\|\mu\|_\vee = Sup|(a^* \otimes b^*)(\mu)|$ where the supremum is taken over all pairs a^* and b^* of norm at most one. The largest cross norm $\|.\|_\wedge$ is called the **projective** cross norm given by $\|\mu\|_\wedge = inf \sum_i \|a_i\|\|b_i\|$ where the infimum is taken over all finite decompositions $\mu = \sum_i a_i \otimes b_i$. The completion of the algebraic tensor products in these two norms are called injective and projective tensor products, denoted by $\mathbf{A} \otimes_\vee \mathbf{B}$ and $\mathbf{A} \otimes_\wedge \mathbf{B}$ respectively.

The norm used for the Hilbert space tensor product is not equal to either of these norms in general.

2.3 Tensor products of operator spaces

The operator space injective tensor product also known as spatial tensor product is defined as follows; if X and Y are operator spaces contained in $B(H)$ and $B(K)$ respectively, then $B(H \otimes K)$ assigns an operator space structure to $X \otimes Y$ which is independent of the particular Hilbert spaces on which X and Y are represented. We write this operator space as
$X \otimes_{\vee} Y$. The operator space projective tensor product $X \otimes_{\wedge} Y$, is defined by specifying $\mathcal{CB}(\mathcal{X} \otimes_{\wedge} Y, B(H))$ for any arbitrary Hilbert space. A map $\phi : X \otimes_{\wedge} Y \longrightarrow B(H)$ is completely contractive iff $\|[\phi(x_{i,j} \otimes y_{k,l}]\|_{n,m} \leq \|[x_{i,j}]\|_n \|[y_{k,l}]\|_m$ whenever $[x_{i,j}] \in M_n(X)$ and $[y_{k,l}] \in M_m(Y)$. The Haagerup tensor product $X \otimes_h Y$ of operator spaces X and Y may also be defined by specifying $\mathcal{CB}(X \otimes_h Y, B(H))$ for an arbitrary Hilbert space. The map $\phi : X \otimes_h Y \longrightarrow B(H)$ is completely contractive iff $\|[\sum_k \phi(x_{i,k} \otimes y_{k,j})]\|_h \leq \|[x_{i,j}\|_h \|[y_{i,j}]\|_h$ whenever $[x_{i,j}] \in M_n(X)$ and $[y_{i,j}] \in M_n(Y)$.

2.3.1; Theorem.

The largest reasonable operator space norm (cross norm) is maximal and minimal is the smallest operator space tensor norm $\|.\|$ such that $\|.\|^*$ is also reasonable. Also, $\|.\|_{\vee} = \|.\|_{\wedge}$

This theorem is the precise analogue of the Banach case.

2.4 Tensor product of C*-algebras.

From [5] the norm of C*-algebra is unique in the sense that on a given algebra $\mathcal{A}$, there exists at most one norm which make $\mathcal{A}$ into a C*-algebra. Also, on a *-algebra $\mathcal{A}$ there may exist different norms satisfying the C*-property. The completion with respect to any of such norms results in a C*-algebra which contain $\mathcal{A}$ as a dense subalgebra. This is precisely what happens when tensor product of C*-algebras is considered; in general case, there are many norms on the algebraic tensor $\mathcal{A} \otimes \mathcal{B}$ (which is a *-algebra) with the C*-property i.e. C*-tensor norms and tensor products of C*-algebras completed with respect to C*-norm shall be refered to as C*-tensor product. These are **spatial** (minimal) and **maximal** norms.

2.4.1 Spatial norm.

2.4.1.1: Lemma.

Suppose that f is a positive linear functional on a C-algebra $\mathcal{A}$ then*

(i) For each $a \in \mathcal{A}$, $f(a^*a) = 0$ if and only if $f(ba) = 0$ for all $b \in \mathcal{A}$.

(ii) The linearity of $f(b^*a^*ab) \leq \|a^*a\| f(b^*b)$ holds for all $a, b \in \mathcal{A}$.

Proof.

Condition (i) follows from the Cauchy Schwarz inequality i.e $|f\langle x, y\rangle| \leq f(x,x)^{\frac{1}{2}} f(y,y)^{\frac{1}{2}}$ for all $x, y \in H$ holds for any positive sesquilinear form f. It implies that the function $p : x \longrightarrow \sqrt{f(x,x)}$ is a semi-norm on H; p satisfies the axioms of a norm except that the implication $p(x) = 0 \Rightarrow x = 0$ may not hold.

To show condition (ii) we may suppose, using (i) that $f(b^*b) > 0$. The function $p : \mathcal{A} \longrightarrow \mathbb{C}$ defined by $f(b^*cb)/f(b^*b)$ is positive and linear. If $(\mu_\lambda)_{\lambda\in\wedge}$ is any approximate unit for $\mathcal{A}$, then $\|p\| = lim_\lambda p(\mu_\lambda) = lim_\lambda f(b^*\mu_\lambda b)/f(b^*b) = f(b^*b)/f(b^*b) = 1$. Hence we have $p(a^*a) \leq (a^*a)$, therefore $f(b^*a^*ab) \leq \|a^*a\|f(b^*b)$.

2.4.1.2: Theorem (GNS).

If $\mathcal{A}$ is a C^*-algebra, then it has a faithful representation. Specifically, its universal representation is faithful. [11]

Proof.

Let (H, ϕ) be the universal representation of $\mathcal{A}$ and suppose that a is an element of $\mathcal{A}$ such that $\phi(a) = 0$, then since if a is a normal element of a non-zero C*-algebra $\mathcal{A}$, then there is a state f of $\mathcal{A}$ such that $\|a\| = |f(a)|$. We have $\|a^*a\| = f(a^*a)$. If $b = (a^*a)^{\frac{1}{4}}$, then $\|a\|^2 = f(a^*a) = f(b)^4 = \|\phi f(b)(b + N_f)\|^2 = 0$ since $\phi f(b^4) = \phi f(a^*a) = 0$ so $\phi(f)(b) = 0$. Hence $b = 0$ and thus ϕ is injective.

2.4.1.3: Theorem.

Suppose that (H, ϕ) and $K, \psi)$ are representations of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, then there exists $\pi : \mathcal{A} \otimes \mathcal{B} \longrightarrow B(H\tilde{\otimes}K)$ such that $\pi(a \otimes b) = \phi(a)\tilde{\otimes}\psi(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Moreover, if ϕ and ψ are injective, so is π. See [11] and [12] for the proof.

2.4.1.4: Definition.

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras with faithful representations (H, ϕ) and (K, ψ) respectively. The norm $\|.\|_\vee$ defined by the inclusion $\mathcal{A} \otimes \mathcal{B} \subseteq B(H) \otimes B(K) \subseteq B(H\tilde{\otimes}K)$ is called the **Spatial norm** i.e. for all $t \in \mathcal{A} \otimes \mathcal{B}$ we have $\|t\|_\vee = \|(\phi \otimes \psi)(t)\|_{B(H\otimes K)}$.

$\|t_\vee = \|(\phi\otimes\psi)(t)\|_{B(H\otimes K)}$ *defines a norm.*

(i) $\|t\|_\vee \geq 0$ and $\|t\|_\vee = 0$ if and only if $t = 0$. i.e.

$\|(\phi\otimes\psi)(\sum_{i=1}^n x_i \otimes y_i)\|_{B(H\otimes K)} \geq 0$ and

$\|(\phi\otimes\psi)(\sum_{i=1}^n x_i \otimes y_i)\|_{B(H\otimes K)} = 0$ iff $\sum_{i=1}^n x_i \otimes y_i = 0\ \forall x \in H, y \in K$.

(ii) $\|\alpha t\| = \|\alpha(\phi\otimes\psi)(t)\|$

$= \|\alpha(\phi\otimes\psi)(\sum_{i=1}^n x_i \otimes y_i)\|$

$= \|\alpha(\sum_{i=1}^n \phi(x_i)\otimes\psi(y_i))\|\ \forall\alpha \in \mathbb{K}$. So,

$\|\alpha(\sum_{i=1}^n \phi(x_i)\otimes\psi(y_i))\|^2 = \langle\alpha(\sum_{i=1}^n \phi(x_i)\otimes\psi(y_i)), \alpha(\sum_{i=1}^n \phi(x_i)\otimes$
$\psi(y_i))\rangle$

$= \langle\alpha\sum_{i=1}^n \phi(x_i), \alpha\sum_{i=1}^n \phi(x_i)\rangle\langle\psi(y_i), \psi(y_i)\rangle$

$= |\alpha|^2\sum_{i=1}^n \|\phi(x_i)\|^2\|\psi(y_i)\|^2$

$= |\alpha|^2\sum_{i=1}^n \|\phi(x_i)\otimes\psi(y_i)\|^2$

$= |\alpha|^2\|(\phi\otimes\psi)(\sum_{i=1}^n x_i\otimes y_i)\|^2$. So,

$\|\alpha(\sum_{i=1}^n \phi(x_i)\otimes\psi(y_i))\| = |\alpha|\|(\phi\otimes\psi)(\sum_{i=1}^n x_i\otimes y_i)\|$

$= |\alpha|\|t\|$.

(iii) Let $x_i, x_i^{'} \in \mathcal{A}$, $y_i, y_i^{'} \in \mathcal{B}$ and $\alpha \in \mathbb{K}$. Then for $t = \sum_{i=1}^n x_i \otimes y_i$, $s = \sum_{i=1}^n x_i^{'} \otimes y_i^{'}$,

$\|(\phi\otimes\psi)(t)+(\phi\otimes\psi)(s)\|^2 = \langle(\phi\otimes\psi)(t)+(\phi\otimes\psi)(s), (\phi\otimes\psi)(t)+(\phi\otimes\psi)(s)\rangle$

$= \langle(\phi\otimes\psi)(t), (\phi\otimes\psi)(t)\rangle + \langle(\phi\otimes\psi)(t), (\phi\otimes$
$\psi)(s)\rangle + \langle(\phi\otimes\psi)(s), (\phi\otimes\psi)(t)\rangle + \langle(\phi\otimes\psi)(s), (\phi\otimes$
$\psi)(s)\rangle$

$= \sum_{i=1}^n \|\phi(x_i)\|^2\|\psi(y_i)\|^2 + \sum_{i=1}^n \|\phi(x_i^{'})\|^2\|\psi(y_i^{'})\|^2 +$
$2Re\langle\sum_{i=1}^n \phi(x_i), \sum_{i=1}^n \phi(x_i^{'})\rangle\langle\psi(y_i), \psi(y_i^{'})\rangle$

$\leq \{\sum_{i=1}^n \|\phi(x_i)\|\|\psi(y_i)\| + \|\sum_{i=1}^n \|\phi(x_i^{'})\|\|\psi(y_i^{'})\|\}^2$.

Taking square roots on both sides,

$\|(\phi \otimes \psi)(t) + (\phi \otimes \psi)(s)\| \leq \|t\|\|s\|$.

2.4.1.5: Remark.

The spatial norm is the least reasonable C*-norm on the tensor product of C*-algebras and is often referred to as "the minimal C*-norm" [5].

2.4.2 Projective norms on tensor products.

Let U, V, W be normed spaces and $\phi : U \times V \longrightarrow U \otimes V$ the tensor map. Then every continuous bilinear map $f : U \times V \longrightarrow W$ factors through a linear map $g : U \otimes V \longrightarrow W$ i.e. $f = g(\phi)$. The identification $T : \mathfrak{L}(U, V : W) \longrightarrow \mathfrak{L}(U \otimes V : W)$ defined by $T(f) = g$ is an algebraic isomorphism. Here the norm is defined on $U \otimes V$ so that T becomes an isometry. A norm on the algebraic tensor product $U \otimes V$ is called a tensor norm or cross norm if $\|x \otimes y\| = \|x\|\|y\|$ for all decomposable tensors $x \otimes y$. [See proposition 1 page 28]. Clearly, if U, V contain non zero vectors, then $\|\phi\| = 1$ for every tensor norm on $U \otimes V$. For every $\mu \in U \otimes V$, we shall write $\mu = \sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in U$ and $y_i \in V$. We note that x_i, y_i may be zero vectors and hence μ may be zero tensor. Let $\|\mu\| = inf \sum_{i=1}^{n} \|x_i\|\|y_i\|$ where the infimum is taken over all representations of μ as a sum of decomposable tensors. This is called the **projective norm.**

In theorems 2.4.2.1 and 2.4.2.2. we assume that our results holds for finite tensor products of normed spaces without further specifications: see [21].

2.4.2.1: Theorem.

The projective norm is the largest tensor norm on $U \otimes V$.

Proof.

Clearly, the projective norm is positive and projective norm of zero tensor is zero. Let $c = \sum_{i=1}^{n} x_i^{'} \otimes y_i^{'}$ and $\mu = \sum_{i=1}^{n} x_i \otimes y_i$ be tensors in $U \otimes V$. We have $\|c+\mu\| \leq \sum_{i=1}^{n} \|x_i^{'}\|\|y_i^{'}\| + \sum_{i=1}^{n} \|x_i\|\|y_i\|$. Taking infimum over the representations of c, μ we obtain $\|c+\mu\| \leq \|c\|\|\mu\|$. Similarly, $\|\lambda\mu\| \leq |\lambda|\|\mu\|$ $\forall \lambda \in \mathbb{K}$. If $\lambda \neq 0$, then $|\lambda|\|\mu\| = |\lambda|\|\frac{1}{\lambda}\lambda\mu\| \leq |\lambda||\frac{1}{\lambda}|\|\lambda\mu\| = \|\lambda\mu\|$. i.e. $|\lambda|\|\mu\| \leq \|\lambda\mu\|$. Therefore, $|\lambda|\|\mu\| = \|\lambda\mu\|$ which can be verified directly if $\lambda = 0$.

Suppose $\mu \neq 0$, we write $\mu = \sum_{i=1}^{n} x_i \otimes y_i$ where $\{x_i\}$ and $\{y_i\}$ are linearly independent sets. There are continuous linear forms f, g on U, V respectively such that $f(x_i) = g(y_i) = 1$ and $f(x_i) = g(y_i) = 0$ for all $i \geq 2$. Hence, $(f \otimes g)(\mu) =_{i=1}^{n} f(x_i)g(y_i) = 1$. But for any representation $\mu = \sum_{i=1}^{n} x_i \otimes y_i$,

$1 = (f \otimes g)(\sum_{i=1}^{n} x_i \otimes y_i)$

$= \sum_{i=1}^{n} f(x_i)g(y_i)$

$= |\sum_{i=1}^{n} f(x_i)g(y_i)|$

$\leq \sum_{i=1}^{n} |f(x_i)||g(y_i)|$

$\leq \|f\|\|g\| \sum_{i=1}^{n} \|x_i\|\|y_i\|$. Taking infimum over all representations of μ, we have $1 \leq \|f\|\|g\|\|\mu\|$. Therefore $\mu \neq= 0$. This proves that the projective norm is a norm on $U \otimes V$.

To show that the projective norm is a tensor norm, we suppose that $\mu = E \otimes F \neq 0$ is a decomposable tensor. Then both E and F are continuous linear forms on U and V respectively such that $\|f\| = \|g\| = 1$,$f(E) = \|E\|$ and $g(F) = \|F\|$. Thus $(f \otimes g)(\mu) = f(E)g(F) = \|E\|\|F\|$. For any representation $\mu = \sum_{i=1}^{n} x_i \otimes y_i$, calculation as above gives $\|E\|\|F\| \leq \|\mu\|$. This together with the definition shows that $\|E\|\|F\| = \|\mu\|$. Therefore projective norm is a tensor norm on $V \otimes U$. Finally, let $|.|$ be any tensor norm on $U \otimes V$. Then for every $\mu = \sum_{i=1}^{n} x_i \otimes y_i$, we have $|\mu| \leq \sum_{i=1}^{n} |x_i \otimes y_i| \leq \sum_{i=1}^{n} \|x_i\| \|y_i\|$. Taking the infimum over all representations of μ, $|\mu| \leq \|\mu\|$ hence projective norm is the largest tensor norm.

2.4.2.2: Theorem.

For every continuous bilinear map $f : U \times V \longrightarrow W$, there is a unique continuous linear map $g : U \otimes V \longrightarrow W$ such that $f = g\phi$ where $\phi : U \times V \longrightarrow U \otimes V$ is a tensor map. If we let $T : \mathfrak{L}(U, V; W) \longrightarrow \mathfrak{L}(U \otimes V; W)$ defined by $Tf = g$. Then T is an isometric isomorphism.

Proof.

Let $\mu = \sum_{i=1}^{n} x_i \otimes y_i$ be any tensor in $U \otimes V$ for all $x_i \in U$ and $y_i \in V$. Then, $\|g(\mu)\| \leq \sum_{i=1}^{n} \|g(x_i \otimes y_i)\| = \sum_{i=1}^{n} \|f(x_i, y_i)\| \leq \sum_{i=1}^{n} \|f\| \|x_i\| \|y_i\|$. Taking the infimum over all representations of μ, we have $\|g(\mu)\| \leq \|f\| \|\mu\|$. Hence g is continuous under the projective norm on $U \otimes V$. Furthermore, $\|g\| \leq \|f\|$.

Conversely, if $g \in U \otimes V \longrightarrow W$ is continuous linear, then the composite $f = g\phi$ is continuous bilinear, i.e.

$f(x, y) = \|f(x \otimes y)\| \leq \|g\| \|x \otimes y\| = \|g\| \|x\| \|y\|$ so that

$\|f\| \leq \|g\|$. Thus $\|f\| = \|g\|$. Tf is linear in f. $\square$

2.4.2.3: Maximal C*-norm.

This norm has good properties, the most important being that the representation defined by $\sum_{i=1}^{n} a_i \otimes b_i \longrightarrow \sum_{i=1}^{n} \phi(a_i)\psi(b_i)$ can be continuously extended to a representation on the C*-algebra $\mathcal{A} \otimes_{\wedge} \mathcal{B}$ for any pair of commuting representations ϕ and ψ of $\mathcal{A}$ and $\mathcal{B}$ respectively, on the same Hilbert space. A pair (ϕ, ψ) of representations is called commuting if $\phi(a)\psi(b) = \psi(b)\phi(a)$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. An algebraic representation $\phi : \mathcal{A} \otimes \mathcal{B} \longrightarrow B(H)$

which satisfies $\|\phi(a \otimes b)\| \leq \|a\|\|b\| \; \forall a \in \mathcal{A},\ b \in \mathcal{B}$ is called a **subtensor representation.** Since for every subtensor representation π of $\mathcal{A} \otimes \mathcal{B}$ there exists a pair of commuting representation ϕ and ψ of $\mathcal{A}$ and $\mathcal{B}$ such that $\pi(a \otimes b) = \phi(a)\psi(b) = \psi(b)\phi(a)$ and every representation of $\mathcal{A} \otimes_{\|.\|} \mathcal{B}$ is a subtensor (for every C*-norm $\|.\|$), then

$\|t\|_\wedge = sup\{\|\phi(t)\| : \phi$ subtensor representation of $A \otimes B\}$ for $t \in \mathcal{A} \otimes \mathcal{B}$.

This is the original Guichardet's definition of the maximal C*-norm for the tensor product of C*-algebras [5].

Proposition 4.

Let $\mathcal{A}$ and $\mathcal{B}$ be C-algebras. ϕ and ψ be faithful representations of $\mathcal{A}$ and $\mathcal{B}$ respectively on Hilbert spaces H and K also respectively. Then there is a maximal C*norm $\|.\|_\wedge$ on $A \otimes B$ defined by $\|t\|_\wedge = sup\{\|\phi(t)\|_{B(H)}\}$.*

Proof.

(i) Clearly, $\|t\|_\wedge = sup\{\|\phi(t)\|_{B(H)}\} \geq 0$ and $\|t\|_\wedge = 0$ if and only if $t = 0$ for all $t \in \mathcal{A} \otimes \mathcal{B}$.

(ii) $\|\alpha t\|_\wedge = sup\{\|\alpha\phi(t)\| : \phi$ subtensor representation of $A \otimes B\}$

$$= sup\{\alpha\phi(\textstyle\sum_{i=1}^n x_i \otimes y_i)\|\}$$
$$= sup\{\| \textstyle\sum_{i=1}^n \alpha\phi_1(x_i) \otimes \phi_2(y_i)\|\}$$
$$= |\alpha| sup\{\|_{i=1}^n \phi_1(x_i) \otimes \phi_2(y_i)\|\}$$
$$= |\alpha|\|t\|_\wedge \text{ for all } \alpha \in \mathbb{K}.$$

(iii) Let $t = \sum_{i=1}^n x_i \otimes y_i$ and $s = \sum_{i=1}^n x_i^{'} \otimes y_i^{'}$, then

$\|t + s\| = sup\{\|\phi(t + s)\|: \phi$ subtensor representation of $\mathcal{A} \otimes \mathcal{B}\}$

$$= sup\{\|\phi t + \phi s\|_{B(H)}\}$$
$$= sup\{\|[\textstyle\sum_{i=1}^n \phi_1(x_i) \otimes \phi_2(y_i] + [\sum_{i=1}^n \phi_1(x_i^{'}) \otimes \phi_2(y_i^{'})]\|\}$$

$\leq sup\{\| \sum_{i=1}^{n} \phi_1(x_i) \otimes \phi_2(y_i)\|\} + sup\{\| \sum_{i=1}^{n} \phi_1(x_i^{'}) \otimes \phi_2(y_i^{'})\|\}$

$= \|t\|_{\wedge} + \|s\|_{\wedge}$.

2.4.2.4: Theorem.

Let $\mathcal{A}$ and $\mathcal{B}$ be C-algebras. There is a minimal C*-norm $\|.\|_{\vee}$ and maximal norm $\|.\|_{\wedge}$, so that any C*-norm on $\mathcal{A} \otimes \mathcal{B}$ must satisfy, $\|t\|_{\vee} \leq \|t\| \leq \|t\|_{\wedge}$ for all $t \in \mathcal{A} \otimes \mathcal{B}$.*

Proof.

We denote by $\mathcal{A}\widetilde{\otimes}_{\vee}\mathcal{B}$ (respectively $\mathcal{A}\widetilde{\otimes}_{\wedge}\mathcal{B}$) the completion of $\mathcal{A} \otimes_{\vee} \mathcal{B}$ for the norm $\|t\|_{\vee}$ (respectively $\|t\|_{\wedge}$). The maximal norm is described as $\|t\|_{\wedge} = sup\|\phi(t)\|_{B(H)}$ where the supremum runs over all possible Hilbert spaces H of all possible *-homomorphisms; $\phi : \mathcal{A} \otimes \mathcal{B} \longrightarrow B(H)$. For any such ϕ, there is a pair of (necessary contractive) *-homomorphisms $\phi_i : A \longrightarrow B(H)$ $(i = 1, 2)$ with commuting ranges such that, $\phi(\sum_{i=1}^{n} x_i \otimes y_i) = \sum_{i=1}^{n} \phi_1(x_i)\phi_2(y_i)$.

Conversely, any such pair $\phi_i : \mathcal{A} \longrightarrow B(H)$, $\phi_i : \mathcal{B} \longrightarrow B(H)$ $(i = 1, 2)$ of *-homomorphisms of commuting ranges determines uniquely a *homomorphism $\phi : \mathcal{A} \otimes \mathcal{B} \longrightarrow B(H)$ by setting $\phi(x_i \otimes y_i) = \phi_1(x_i)\phi_2(y_i)$. Thus we can write for $t = \sum_{i=1}^{n} x_i \otimes y_i \in \mathcal{A} \otimes \mathcal{B}$, $\|t\|_{\wedge} = sup\{\| \sum_{i=1}^{n} \phi_1(x_i)\phi_2(y_i)\|\}$ where the supremum runs over all possible such pairs. The inequality $\|t\| \leq \|t\|_{\wedge}$ follows by considering Gelfand Naimark embedding of the completion of $(\mathcal{A} \otimes \mathcal{B}, \|t\|)$ into $B(H)$ for some H [11].

The minimal norm can be described as follows; embedding $\mathcal{A}$ and $\mathcal{B}$ ac C*-sub-algebras of $B(H_1)$ and $B(H_2)$ respectively. Then for any $t = \sum_{i=1}^{n} x_i \otimes y_i$ in $\mathcal{A} \otimes \mathcal{B}$, $\|t\|_{\vee}$ coincides with the norm induced by the space $B(H_1 \otimes_{\|.\|} H_2)$, i.e. we have an embedding (an isometric *-homomorphism) of the completion denoted by $\mathcal{A}\widetilde{\otimes}_{\vee}\mathcal{B}$ into $B(H_1 \otimes H_2)$. In other words, the minimal tensor product operator spaces, when restricted to two C*-algebras coincides with the minimal C*-tensor product.

Let $(\mathcal{C}, \mathcal{D})$ be another pair of C*-algebras and consider completely bounded maps $f_1 : \mathcal{A} \longrightarrow \mathcal{C}$ and $f_2 : \mathcal{B} \longrightarrow D$. Then $f_1 \otimes f_2$ defines a completely bounded map from $\mathcal{A}\widetilde{\otimes}_{\vee}\mathcal{B}$ to $\mathcal{C} \otimes \mathcal{D}$ with $\|f_1 \otimes f_2\|_{cb} = \|f_1\|_{cb}\|f_2\|_{cb}$. In sharp contrast, the analogous property does not hold for the maximal tensor products. However, it does hold if we moreover assume that f_1 and f_2 are positive and then the resulting map $f_1 \otimes f_2$ is also completely positive (on the maximal tensor product) hence

$\|f_1 \otimes f_2(t)\|_{\mathcal{C}\widetilde{\otimes}_{\wedge}\mathcal{D}} \leq \|f_1\|\|f_2\|\|t\|_{\mathcal{A}\widetilde{\otimes}_{\wedge}\mathcal{B}}$ for all $t \in \mathcal{A} \otimes \mathcal{B}$.

2.4.3 Haagerup norm.

Besides the minimal and the maximal norm, there is another important operator space cross-norm: the Haagerup norm. Generally, the Haagerup norm on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras is defined by

$$\|t\|_h = inf \|\sum_{i=1}^{n} x_i x_i^*\|^{\frac{1}{2}} \|\sum_{i=1}^{n} y_i^* y_i\|^{\frac{1}{2}}$$

for $t \in \mathcal{A} \otimes \mathcal{B}$. The proof that $\|.\|_h$ is a norm is not completely trivial since the proof of the triangle inequality and the definiteness are non-trivial. We also note that the Haagerup norm is not a C*-norm, but if the definition is repeated for $n \in \mathbb{N}$ and $t \in M_n(X \otimes Y)$ for operator spaces X and Y, it turns out that the Haagerup norm is an operator space cross-norm with a number of good properties [5].

The motivation for the Haagerup norm was the consideration of elementary operator $\phi : B(H) \longrightarrow B(H)$ defined by $\phi(a) = \sum_{i=1}^{n} x_i a y_i$ for $a \in B(H)$ and x_i, y_i fixed in $B(H)$. These operators result from the action of $\sum_{i=1}^{n} x_i \otimes y_i \in B(H) \otimes B(H)^{op}$ on $B(H)$ (where $B(H)^{op}$ is the C*-algebra with the reversed product). For some $\xi, \eta \in H$ where $\|\xi\| = \|\eta\| = 1$, the Cauchy-Shwarz inequality implies;

$$\begin{aligned} |\langle \phi(a)\xi, \eta\rangle| &= |\langle \sum_{i=1}^{n} x_i a y_i \xi, \eta\rangle| \\ &= |\langle \sum_{i=1}^{n} a y_i \xi, x_i^* \eta\rangle| \\ &\leq (\sum_{i=1}^{n} \|a y_i \xi\|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} \|x_i^* \eta\|^2)^{\frac{1}{2}}. \text{ But} \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{n} \|x_i^* \eta\|^2 &= \sum_{i=1}^{n} \langle x_i^* \eta, x_i^* \eta\rangle \\ &= \sum_{i=1}^{n} \langle \eta, x_i x_i^* \eta\rangle \\ &\leq \| \sum_{i=1}^{n} x_i x_i^* \| \|\eta\|^2 \end{aligned}$$

Also, $\|a y_i \xi\| \leq \|a\| \|y_i \xi\|$,

$$\begin{aligned} \sum_{i=1}^{n} \|y_i \xi\|^2 &= \sum_{i=1}^{n} \langle y_i \xi, y_i \xi\rangle \\ &= \sum_{i=1}^{n} \langle \xi, y_i^* y_i \xi\rangle \\ &\leq \| \sum_{i=1}^{n} y_i^* y_i \| \|\xi\|^2. \end{aligned}$$

So, $|\langle \phi(a)\xi, \eta\rangle \leq \|a\| \| \sum_{i=1}^{n} x_i x_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^{n} y_i^* y_i \|^{\frac{1}{2}} \|\xi\| \|\eta\|$.

Hence, $\|\phi\| \leq \| \sum_{i=1}^{n} x_i x_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^{n} y_i^* y_i \|^{\frac{1}{2}}$

From these considerations we obtain the natural definition;

$\|t\|_h = inf\{\| \sum_{i=1}^{n} x_i x_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^{n} y_i^* y_i \|^{\frac{1}{2}} : n \in \mathbb{N}, t \in B(H) \otimes B(H)\}$.

2.4.3.1: Theorem.

Let $a, b \in B(H)$ and let $T_{a,b} = a \otimes b + b \otimes a$. Then $\|T_{a,b}\|_{\vee} \geq 2\|a\| \|b\|$.

Proof.

Let $\underline{a} = [a, b]$, $\underline{b} = [b, a]^t$. We shall use the notation $\underline{a} \odot \underline{b} = a \otimes b + b \otimes a$ and recall that the Haagerup norm of $\|a \otimes b + b \otimes a\|_h \geq \|a\| \|b\|$ [9]. We assume that $\|a\| = \|b\| = 1$. Let $a, b \in \Delta = (B(H)^*)_1$ where $\Delta = \{f \in B(H)^* : \|f\| \leq 1\}$

and $T_{a,b} \in \Delta \times \Delta$. We let $s_o, t_o \in \Delta$ be some scalars of modulae 1 and that $a(s_o) = 1$, $b(t_o) = 1$. If $a_1 = a(t_o)$ and $b_1 = b(s_o)$, then

(i) $$\begin{aligned} T_{a,b}(s_o, s_o) &= (a \otimes b + b \otimes a)(s_o, s_o) \\ &= a \otimes b(s_o, s_o) + b \otimes a(s_o, s_o) \\ &= a(s_o)b(s_o) + b(s_o)a(s_o) \\ &= b_1 + b_1 \\ &= 2b_1. \end{aligned}$$

(ii) $$\begin{aligned} T_{a,b}(t_o, t_o) &= a \otimes b + b \otimes a)(t_o, t_o) \\ &= a \otimes b(t_o, t_o) + b \otimes a(t_o, t_o) \\ &= a(t_o)b(t_o) + b(t_o)a(t_o) \\ &= a_1 + a_1 \\ &= 2a_1. \end{aligned}$$

(ii) $$\begin{aligned} T_{a,b}(s_o, t_o) &= a \otimes b(s_o, t_o) + b \otimes a(s_o, t_o) \\ &= a \otimes b(s_o, t_o) + b \otimes a(s_o, t_o) \\ &= a(s_o)b(t_o) + b(s_o)a(t_o) \\ &= 1.1 + b_1 a_1 \\ &= 1 + a_1 b_1. \end{aligned}$$

If $|a_1|$ or $|b_1|$ is greater or equal to 1, then the proof is completed.

Chapter 3

THE NORM OF A DERIVATION

3.1 Introduction.

In this chapter, we determine the norm of the inner derivation $\Delta_T : TA - AT$ acting on $B(H)$ which is irreducible.More precisely, we show that $\|T_{A,A}\| = 2inf\{\|A - \lambda\| \lambda \in \mathbb{C}\}$.

A derivation Δ on a C*-algebra $\mathcal{A}$ is a linear mapping $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ satisfying the usual Leibniz product rule i.e. $\Delta(x, y) = x(\Delta y) + (\Delta x)y \ \forall x, y \in \mathcal{A}$. Such a mapping is bounded as was first shown by Sakai [16]. If there is an element a such that $\Delta x = xa - ax \ \forall x \in \mathcal{A}$, then the derivation is **inner**. In most cases such an element doesn't exist in $\mathcal{A}$. Therefore one tries to extend the derivation Δ to a bigger C*-algebra which may contain an implementing element.

Since Δ is inner, it is easier to estimate its norm which of course, is important from the analytic point of view. It is easy to see that if $\Delta x = xa - ax \ \forall x \in \mathcal{A}$, then $\|\Delta\| \leq 2dist(a, Z(\mathcal{A}))$ where $Z(\mathcal{A})$ is the center of $\mathcal{A}$.

3.2 Preliminary results.

"We say that a state f of a C*-algebra $B(H)$ is definite on the self-adjoint operator A in $B(H)$ when $f(A^2) = f(A)^2$. In this case, f is multiplicative on the C*-subalgebra of $B(H)$ generated by A. The following lemma is a combination of Singer's argument that the derivations of commutative C*-algebras are 0 and results on the multiplicative properties of definite states". See [7].

3.2.1: Lemma.

If Δ is a derivation of the C-algebra $B(H)$ and f is definite on A in $B(H)$, then $f(\Delta(A)) = 0$.*

Proof.

We note that $\Delta(I) = \Delta(I^2) = 2\Delta(I)$, so that $\Delta(I) = 0$. Thus $\Delta(A) = \Delta(A - f(A)I)$; and we may assume that $f(A) = 0$. In this case $0 = f(A^+) = f(A^-)$, where $A = A^+ - A^-$, A^+ and A^- are "positive" and "negative" parts of A; for $A^+A = {A^+}^2$, so that

$0 = f(A^+)f(A) = f(A^+A) = f({A^+}^2) = f(A^+)^2$.

Since $\Delta(A) = \Delta(A^+) - \Delta(A^-)$, it will suffice to show that $f(\Delta(A^+)) = f(\Delta(A^-)) = 0$. We may assume that $A > 0$ and $f(A) = 0$. Let $T = A^{\frac{1}{2}}$. Then $f(T) = 0$. Hence

$f(\Delta(A)) = f[\Delta(T)T] + f[T\Delta(T)] = f[\Delta(T)]f(T) + f(T)f[\Delta(T)] = 0$. The substance of the foregoing lemma is that each derivation of a C*-algebra maps each self-adjoint operator in the algebra onto an operator that has 0 diagonal relative to a diagonalization which diagonalizes A [7].

3.2.2: Theorem.

Each derivation of a C-algebra annihilates its center [7].*

Proof.

LetΔ be a derivation of the C*-algebra $B(H)$ with center $Z(B(H))$. Let f be a pure state of $B(H)$, and z an element of $Z(B(H))$. The representation of $B(H)$ associated with f is irreducible [23] and therefore maps $Z(B(H))$ into scalars. Together with the Schwarz inequality, this yields that f is multiplicative on $Z(B(H))$. From the preceding lemma,$f(\Delta(z)) = 0$. Since the pure states of $B(H)$ separate $B(H)$, $\Delta(z) = 0$.

3.2.3: Lemma.

If Δ is a derivation of the C-algebra $B(H)$ acting on the space H, then Δ has a unique ultra weakly continuous extension which is a derivation of $B(H)^-$.*

Proof.

We show that for each x, y in H, $\omega_{x,y} o\Delta$ is strongly continuous at 0 on ϑ_1^+, the positive operators in the unit ball ϑ_1 of $B(H)$. Now

$$A \longrightarrow ([A\Delta(A) + \Delta(A)A]x, y) \qquad (= (\Delta A^2)x, y))$$

is strongly continuous at 0 on ϑ_{1^*}, the set of self-adjoint operators in the unit ball of $B(H)$, since $|\langle(A\Delta(A) + \Delta(A)A)x, y\rangle| \leq \|\Delta\|(\|Ax\|\|y\| + \|x\|\|Ay\|)$ where $\|\Delta\| < \infty$ by Sakai's theorem [21]. Moreover, $A \longrightarrow A^{\frac{1}{2}}$ is strongly continuous at 0 on positive operators, since $\|A^{\frac{1}{2}}\| = |\langle Ax, x\rangle| \leq \|Ax\|\|x\|$. Thus $A \longrightarrow A^{\frac{1}{2}} \longrightarrow (\Delta(A)x, y)$ is strongly continuous at 0 on ϑ_1. We note next that Δ is weakly continuous on ϑ_1 to $B(H)$ in the weak operator topology. Since $Ax = A^+x - A^-x$ with A^+ and A^- orthogonal, $\|A^+\| \leq \|Ax\|$ and $\|A^-x\| \leq \|Ax\|$; so that $A \longrightarrow A^+$ and $A \longrightarrow A^-$ are strongly continuous mappings on the self-adjoint operators in $B(H)$ at 0. Thus, $A \longrightarrow (\Delta(A^+)x, y) - (\Delta(A^-)x, y) = (\Delta(A)x, y)$ is strongly continuous at 0 on ϑ_{1^*}. By linearity this mapping is strongly continuous at 0 on $2\vartheta_{1^*}$ and from this, everywhere on ϑ_{1^*}. Hence the inverse image of a closed convex subset of the complex numbers under $A \longrightarrow (\Delta(A)x, y)$ has an intersection with ϑ_{1^*} which is strongly closed relative to ϑ_{1^*}. This intersection being convex, each weak limit point is a strong limit point [3,15], so that it is weakly closed relative to ϑ_{1^*}. Since the closed convex subsets of the complex numbers form a subbase for the closed subsets, $A \longrightarrow (\Delta(A)x, y)$ is weakly continuous on ϑ_{1^*}. Now $A \longrightarrow (A + A^*)/2$ and $A \longrightarrow (A - A^*)/2i$ are weakly continuous mappings of ϑ_1 into ϑ_{1^*}; so that $A \longrightarrow (\Delta(\frac{A+A^*}{2})x, y) + i(\Delta(\frac{A-A^*}{2i})x, y) = (\Delta(A)x, y)$ is weakly continuous on ϑ_1. Thus Δ is weakly continuous on ϑ_1.

The linearity of Δ yields its uniform continuity relative to the weak-operator uniform structure on ϑ_1. From the Kaplansky density theorem [14], $\vartheta_{\bar{1}}$ is the unit ball in $B(H)^-$, and is compact in the weak-operator topology. Thus Δ has a unique weak-operator continuous extension to $\vartheta_{\bar{1}}$, and this extension has an obvious extension Δ from $\vartheta_{\bar{1}}$ to $B(H)^-$. It is easily checked that this extension is well defined and linear. For if $x \in H$, $(A, T) \longrightarrow ([\overline{\Delta}(AT) - \overline{\Delta}(A)T - A\overline{\Delta}(T)]x, x)$ is strongly continuous on $\vartheta_{\overline{1^*}} \times \vartheta_{\overline{1^*}}$, by strong continuity of operator multiplication on bounded sets, weak continuity of $\overline{\Delta}$ on $\vartheta_{\bar{1}}$ and boundedness of Δ (hence $\overline{\Delta}$). Since this mapping is 0 on $\vartheta_{1^*} \times \vartheta_{1^*}$, a strongly dense subset of $\vartheta_{\overline{1^*}} \times \vartheta_{\overline{1^*}}$; it is 0 on $\vartheta_{\overline{1^*}} \times \vartheta_{\overline{1^*}}$, for each x, so that $\overline{\Delta}$ is a derivation on $B(H)^-$ [7].

3.2.4: Lemma.

Every derivation Δ on a C^*-algebra is bounded.

Proof.

Since every derivation on a non-unital C*-algebra can be uniquely extended to its minimal unitization, the assertion follows from the fact that every generalised derivation on a unital C*-algebra is bounded.

3.3 Main results.

3.3.1: Lemma.

Every derivation Δ on a C^*-algebra $\mathcal{A}$ vanishes on the center $Z(\mathcal{A})$ of $\mathcal{A}$.

Proof.

Let $a \in Z(\mathcal{A})$. Then for all $x \in \mathcal{A}$, $x(\Delta a) = \Delta(xa) - (\Delta x)a = \Delta(ax) - a(\Delta x) = (\Delta a)x$ where $\Delta a \in Z(\mathcal{A})$. From $a(\Delta a) - (\Delta a)a = 0$, "the boundedness of a derivation and the general version of Kleinecke-Shirokov theorem" [7], we conclude that Δa is quasinilpotent but being central, this implies that $\Delta a = 0$.

3.3.2: Lemma.

If $\|T\| = \|x\| = 1$ and $\|Tx\|^2 \geq (1-\varepsilon)$, then $\|(T^*T - I)x\|^2 \leq 2\varepsilon$.

Proof.

$0 \leq \|(T^*T - I)x\|^2$

$$\begin{aligned}
\|(T^*T - I)x\|^2 &= \langle (T^*T - I)x, (T^*T - I)x \rangle \\
&= \langle T^*Tx - Ix, T^*Tx - Ix \rangle \\
&= \langle T^*Tx, T^*Tx \rangle - \langle T^*Tx, Ix \rangle - \langle Ix, T^*Tx \rangle + \langle Ix, Ix \rangle \\
&= \|T^*Tx\|^2 - 2\langle Tx, Tx \rangle + \|x\|^2 \\
&= \|T^*Tx\|^2 - 2\|Tx\|^2 + \|x\|^2 \\
&\leq (\|T^*\|\|T\|\|x)^2 - 2\|Tx\|^2 + \|x\|^2
\end{aligned}$$

$$= 1 - 2\|Tx\|^2 + 1$$
$$= 2(1 - \|Tx\|^2)$$
$$\leq 2(1 - (1 - \varepsilon)$$
$$= 2\varepsilon.$$

3.3.3: Lemma.

Let $\mu \in W(T)$. Then $\Delta_T \geq 2(\|T\|^2 - |\mu|^2)^{\frac{1}{2}}$.

Proof.

We note that $\|\Delta_T\| = sup\{\|TA - AT\| : A \in B(H), \|A\| = 1\}$. Since $\mu \in W(T)$, there exists $x_n \in H$ such that $\|x_n\| = 1$, $\|Tx_n\| \longrightarrow \|T\|$, and $(Tx_n, x_n) \longrightarrow \mu$. If we set $Tx_n = \alpha_n x_n + \beta_n y_n$, where $\langle x_n, y_n\rangle = 0$ and $\|y_n\| = 1$. Also, $V_n x_n = x_n$, $V_n y_n = -y_n$ and $V_n = 0$ on $\{x_n, y_n\}$. Then

$\|(TV_n - V_nT)x_n\|^2 = \|Tx_n - V_nTx_n\|^2$

$= \|\alpha_n x_n + \beta_n y_n - V_n(\alpha_n x_n + \beta_n y_n)\|^2$

$= \langle \alpha_n x_n + \beta_n y_n - V_n(\alpha_n x_n + \beta_n y_n), \alpha_n x_n + \beta_n y_n - V_n(\alpha_n x_n + \beta_n y_n)\rangle$

$= \langle \alpha_n x_n + \beta_n y_n, \alpha_n x_n + \beta_n y_n\rangle - \langle \alpha_n x_n + \beta_n y_n, V_n(\alpha_n x_n + \beta_n y_n)\rangle - \langle V_n(\alpha_n x_n + \beta_n y_n), \alpha_n x_n + \beta_n y_n\rangle + \langle V_n(\alpha_n x_n + \beta_n y_n), V_n(\alpha_n x_n + \beta_n y_n)\rangle$

$= [\langle \alpha_n x_n, \alpha_n x_n\rangle + \langle \alpha_n x_n, \beta_n y_n\rangle + \langle \beta_n y_n, \alpha_n x_n\rangle + \langle \beta_n y_n, \beta_n y_n\rangle] - [\langle \alpha_n x_n, V_n \alpha_n x_n\rangle + \langle \alpha_n x_n, \beta_n y_n\rangle + \langle \beta_n y_n, V_n \alpha_n x_n\rangle + \langle \beta_n y_n, V_n \beta_n y_n\rangle] - [\langle V_n \alpha_n x_n, \alpha_n x_n\rangle + \langle V_n \alpha_n x_n, \beta_n y_n\rangle + \langle V_n \beta_n y_n, \alpha_n x_n\rangle + \langle V_n \beta_n y_n, \beta_n y_n\rangle] + [\langle V_n \alpha_n x_n, V_n \alpha_n x_n\rangle + \langle V_n \alpha_n x_n, V_n \beta_n y_n\rangle + \langle V_n \beta_n y_n, V_n \alpha_n x_n\rangle + \langle V_n \beta_n y_n, V_n \beta_n y_n\rangle]$

$= [|\alpha|^2\|x_n\|^2 + \alpha_n\overline{\beta}_n\langle x_n, y_n\rangle + \beta_n\overline{\alpha}_n\langle y_n, x_n\rangle + |\beta|^2\|y_n\|^2] - [|\alpha_n|^2\|x_n\|^2 - \alpha_n\overline{\beta}_n\langle x_n, y_n\rangle + \beta_n\overline{\alpha}_n\langle y_n, x_n\rangle - |\beta_n|^2\|x_n\|^2] - [|\alpha_n|^2\|x_n\|^2 + \alpha_n\overline{\beta}_n\langle x_n, y_n\rangle - \beta_n\overline{\alpha}_n\langle y_n, x_n\rangle - |\beta_n|^2\|y_n\|^2] + [|\alpha_n|^2\|x_n\|^2 - \alpha_n\overline{\beta}_n\langle x_n, y_n\rangle - \beta_n\overline{\alpha}_n\langle y_n, x_n\rangle + |\beta_n|^2\|y_n\|^2]$

$= [|\alpha_n|^2\|x_n\|^2 + \beta_n\overline{\alpha}_n\langle y_n, x_n\rangle + |\beta_n|^2\|y_n\|^2] - [|\alpha_n|^2\|x_n\|^2 + \beta_n\overline{\alpha}_n\langle y_n, x_n\rangle - |\beta_n|^2\|y_n\|^2] - [|\alpha_n|^2\|x_n\|^2 - \beta_n\overline{\alpha}_n\langle y_n, x_n\rangle - |\beta_n|^2\|y_n\|^2] + [|\alpha_n|^2\|x_n\|^2 - \beta_n\overline{\alpha}_n\langle y_n, x_n\rangle +$

$|\beta_n|^2\|y_n\|^2]$

$= |\beta_n|^2\|y_n\|^2 + |\beta_n|^2\|y_n\|^2 = 2|\beta_n|^2$.

$\Longrightarrow \|(TV_n - V_nT)x_n\| = 2|\beta_n| \geq 2(\|T\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \epsilon_n$, where $\epsilon_n \longrightarrow 0$ and since $\alpha_n \longrightarrow \mu$ the proof is complete. see [19].

3.3.4: Theorem.

$\|\Delta_T\| = 2\|T\|$ *if and only if* $0 \in W(T)$.

Proof.

From lemma 3.3.3, we have that $\|\Delta_T\| \geq 2\|T\|$ if $0 \in W(T)$. Since $\|\Delta_T\| \leq 2\|T\|$ for any T, sufficiency is proved. We assume that the $\|\Delta_T\| = 2\|T\|$, and hence there exists x_n and A_n such that $\|x_n\| = \|A_n\| = 1$ and $\|(TA_n - A_nT)x_n\| \to 2\|T\|$. Clearly, $\|A_nx_n\| \to 1$, $\|Tx_n\| \to \|T\|$ and $\|TA_nx_n\| \to \|T\|$. Moreover, since $\|(TA_n - A_nT)x_n\| \to 2\|T\|$, $TA_nx_n = -A_nTx_n + \overrightarrow{\epsilon}_n$ where $\|\overrightarrow{\epsilon}_n\| \longrightarrow 0$. Let $(Tx_n, x_n) \to \mu$ by choosing subsequence if necessary, i.e. $\mu \in W(T)$. We observe that $(TA_nx_n, A_nx_n) = -(A_nTx_n, A_nx_n) + \epsilon_n$

$= -(Tx_n, A_n^*A_nx_n)$

$= -(Tx_n, x_n) + \epsilon_n'$ where the last step follows from lemma 3.3.2. Thus , $lim_{n\longrightarrow\infty}(TA_nx_n, A_nx_n) = -\mu$. Since $\mu, -\mu \in W(T)$, it follows that $0 \in W(T)$.

3.3.5: Theorem

If $0 \in W(T)$, *then* $\|T\|^2 + |\lambda|^2 \leq \|T + \lambda\|^2$ *for all* $\lambda \in \mathbb{C}$. *Conversely, if* $\|T\| \leq \|T + \lambda\|$ *for all* $\lambda \in \mathbb{C}$, *then* $0 \in W(T)$.

Proof.

If $0 \in W(T)$, then there exists $x_n \in H$, $\|x_n\| = 1$ such that

$\|(T+\lambda)x_n\|^2 = \|Tx_n\|^2 + Re\overline{\lambda}(Tx_n, x_n) + |\lambda|^2 \rightarrow \|T\|^2 + |\lambda|^2$.

Conversely, let $\|T\| \leq \|T+\lambda\| \; \forall\lambda \in \mathbb{C}$. We assume that $0 \notin W(T)$. By rotating T, we may assume that $ReW(T) \geq \tau > 0$. Let

$\zeta = \{x \in H : \|x\| = 1 and Re(Tx, x) \leq \tau/2\}$, $\eta = sup\{\|Tx\| : x \in \zeta\}$. Then $\eta < \|T\|$. Let $\mu = min\{\tau/2, (\|T\| - \eta)/2\}$ and consider $(T - \mu)$. If $x \in \zeta$, then $\|(T-\mu)x\| \leq \|Tx\| + \mu \leq \eta + \mu < \|T\|$.

Let $Tx = (a+ib)x + y$ where $x \notin \zeta$, $\|x\| = 1$ and $(x, y) = 0$. Then

$$\begin{aligned}\|(T-\mu)x\|^2 &= (a-\mu)^2 + b^2 + \|y\|^2 \\ &= \|Tx\|^2 + (\mu^2 - 2a\mu) \\ &< \|T\|^2 \text{ since } a > \mu > 0\end{aligned}$$

i.e. $\|T - \mu\| < \|T\|$, contrary to the hypothesis

3.3.6:Corollary. (Pythagorean relation for operator.)

Let T be a bounded linear operator. Then there exists a unique $z_o \in \mathbb{C}$, such that $\|T - z_o\|^2 + |\lambda|^2 \leq \|(T - z_o) + \lambda\|^2 \; \forall\lambda \in \mathbb{C}$. Moreover, $0 \in W(T - \lambda)$ if and only if $\lambda = z_o$.

Proof.

Now, there exists a $z_o \in \mathbb{C}$ such that $\|T - z_o\| \leq \|(T - z_o) + \lambda\| \; \forall\lambda \in \mathbb{C}$. The rest of the proof easily follow from theorem 3.3.5.

3.3.7: Theorem.

Let Δ_T be a derivation on $B(H)$. Then $\|\Delta_{T/B(H)}\| = sup\{\|TA - AT\| : A \in B(H), \|A\| = 1\} = inf_{\lambda\in\mathbb{C}}\{2\|T - \lambda\|\}$.

Proof.

Since $\|TA - AT\| = \|(T-\lambda)A - A(T-\lambda)\| \leq 2\|T-\lambda\|\|A\|$. It follows therefore that $\|\Delta_T\| \leq inf_{\lambda\in\mathbb{C}}\{2\|T-\lambda\|\}$.

On the other hand, $\|T-\lambda\|$ is larger for λ large. So $inf\|T-\lambda\|$ must be taken on at some point, say Z_o. But $\|T-Z_o\| \leq \|(T-Z_o+\lambda\| \; \forall\lambda \in \mathbb{C}$ implies that $0 \in W(T-Z_o)$. Hence $\|\Delta_T\| = \|\Delta_{T-Z_o}\| = 2\|T-Z_o\|$. □

3.3.8: Definition.

A C*-algebra $\mathcal{A}$ is irreducible if the commutant of $\mathcal{A}$ contains only the scalars.

3.3.9: Theorem.

Let $B(H)$ be an irreducible C-algebra on H. Let $T \in B(H)$. Then* $\|\Delta_{T/B(H)}\| = sup\{\|TA-AT\| : A \in B(H), \|A\| = 1\} = inf_{\lambda\in\mathbb{C}}\{2\|T-\lambda\|\}$.

See[19] for proof.

3.3.10: Theorem.

Let $A, B \in B(H)$. Then $\|T_{A,B}\| = sup\{\|AX-XB\| : X \in B(H), \|X\| = 1\} = inf_{\lambda\in\mathbb{C}}\{\|A-\lambda\| + \|B-\lambda\|\}$.

Proof.

$\|T_{A,B}\| \leq inf\{\|A-\lambda\| + \|B-\lambda\|\}$ follows from theorem 3.3.7. If we let $inf_{\lambda\in C}\{\|A-\lambda\| + \|B-\lambda\|\} = \|A-\lambda_o\| + \|B-\lambda_o\|$. Then it follows from [19] lemma 6 and theorem 7 that $\|T_{A,B}\| = \|T_{(A-\lambda_o, B-\lambda_o)}\| = \|A-\lambda_o\| + \|B-\lambda_o\|$. If $A = B$, then the norm of $T_{A,B}$ is an inner derivation induced by A or B respectively i.e. $\|T_{A,A}\| = inf\{\|A-\lambda\| + \|A-\lambda\| : \lambda \in \mathcal{C}\}$

$= 2inf\{\|A-\lambda\| : \lambda \in \mathcal{C}\}$

$= 2R_A$ where R_A is the radius of the spectrum of A.

If $B(H)$ is irreducible then $\|T_{A,A}\| = 2inf\{\|A-\lambda\| : \lambda \in \mathcal{C}\}$ implies that λ is the center of $B(H)$. Further if X is close to λ, then the norm is small hence X almost commute with the elements of the unit ball of $B(H)$.

3.4 Conclusion

In this thesis, the problem stated in 1.4 has been solved. We have shown in section 2.4 that the constant $c = 2$ i.e. $\|T_{a,b}\| \geq 2\|a\|\|b\|$ by taking $T_{a,b} = a \otimes b + b \otimes a$. We have also shown that $\|T_{A,B} = inf_{\lambda \in C}\{\|A - \lambda\| + \|B - \lambda\|\}$ which in turn is an inner derivation when A coincides with B.

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