OPERATION OF MUTATION ON POLAR QUIVERS

BY

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Abstract

In recent times, there has been a lot of interest in the study of quivers, both by mathematicians and theoretical physicists. We introduce a new concept of polar quivers and their mutation. The idea of polar quivers arises from the concept of anomaly free R-charges in theoretical physics. Mutation of polar quivers is build on mutation quivers with potential, which was defined by Derksen, Weyman and Zelevinsky. An R-charge assigns angles to the arrows of a quiver. In a polar quiver we assign angles and positive non-zero integers to vertices and impose conditions equivalent to the anomaly conditions for R-charges. We then establish that mutation of a polar quiver will give a polar quiver if and only if a simple additional condition is satisfied. We use families of quivers linked by mutation, from the work of Stern, as our source of examples. The results of this study have applications in geometry and theoretical physics.

Chapter 1

Introduction

Quiver algebras are an example of non-commutative algebras. A quiver Q consists of a set of vertices Q_0, a set of arrows Q_1 and two maps s and t, assigning to each arrow the starting and terminating vertices respectively. The path algebra $KQ = \mathbf{A}$ of the quiver Q over the field K has a basis the paths of the quiver, where a path is just a composition of arrows. Chapter 2 of this thesis provides the mathematical background for the study. Basic definitions and some important examples of quivers are given in this chapter.

Our study is built on quivers with potential. A quiver with potential (QP for short) is a quiver Q together with the potential $\mathcal{S}$, which is a sum of cycles in the path algebra of the quiver. Cyclic derivatives are partial derivatives of the potentials with respect to arrows of the quiver, which are in the cycles. For every potential $\mathcal{S}$ its Jacobian ideal $J(\mathcal{S})$ is the closure of the ideal in $\mathbf{A}$ generated by all elements of the cyclic derivative. These are relations of the quiver that define the quotient path algebra. In Chapter 3, we study quivers with potential and their mutation. Mutation of quivers with potential was first studied and defined by Derksen, Weyman and Zelevinsky [8]. In their work, they proved that there exist a non-degenerate potential which does not break as a result of mutation. They also stated and proved splitting theorem which is used in the reduction of the mutated quiver. We give some examples of quivers with potential and their mutations.

In chapter 4, we introduce a new idea of polar quivers which is inspired by the concept of R-charge from theoretical physics. An R-charge assigns angles to the arrows of a quiver, and this quiver has to satisfy the anomaly free conditions. In a polar quiver, which will be denoted as $(Q, \mathcal{S}, N, \theta)$, we assign angles and positive non-zero integers to vertices. These then induces angles for the arrows. A polar quiver has to satisfy an equivalent of the anomaly conditions we refer to as polar conditions. From the examples of polar quivers given in the last section of chapter 4, we can observe that polar co-ordinates are not unique.

The process of mutation of a polar quiver is defined in chapter 5. We give an illustration by mutating examples of polar quivers. In general we would expect mutation of a polar quiver to give rise to a polar quiver. As we will observe from example 5.3, this is not always the case.

In chapter 6, we state the main theorem of the study. The theorem gives the conditions a polar quiver must satisfy for its mutation to give a polar quiver. We follow up the examples from the previous chapter to illustrate the significance of the theorem.

Chapter 7 gives a summary of the study. We highlight some questions arising from this study and give recommendations on how this work can be extended.

Chapter 2

Mathematical background

2.1 Introduction

This chapter introduces the basic mathematics concepts that are fundamental to the understanding of the entire thesis. Such concepts as Vector spaces and algebras are covered in Section 2.2. Our main source for this section was the book by Connell [7], although many other sources were used.

Section 2.3 introduces the basic objects of our study which are quivers. Some important examples of quivers are given in this section. This section also introduces the path algebra and gives examples. The main sources of literature for this section include lecture notes by Crawley Boevey [6] and papers by Kearnes [15] and Savage [17].

Section 2.4 gives some of the sources used for the study. In section 2.5, we state the problem areas that our study has attempted to solve while section 2.6 gives the main objectives of the study. The main approach used in this study is stated in section 2.7. While the significance of the study is covered in section 2.8.

2.2 Vector space and Algebra

This is a section of basic algebraic definitions that might be helpful to the understanding of the path algebra which we introduce in the next section.

Definition 2.1. A **vector space** over the field K is a set V on which two operations are defined, called vector addition (+) and scalar multiplication (.). These operations must satisfy the following conditions;

i. *Closure*; For all $a \in K$ and all $u, v \in V$, $u + v$ and the scalar product $a \cdot v$ are uniquely defined and belong to V.

ii. *Associativity*: For all $a, b \in K$ and all $u, v, w \in V$, $u + (v + w) = (u + v) + w$ and $a \cdot (b \cdot v) = (a \cdot b) \cdot v$.

iii. *Commutativity of addition*: For all $u, v \in V$, $u + v = v + u$.

iv. *Distributive laws*: For all $a, b \in K$ and all $u, v \in V$, $a \cdot (u + v) = (a \cdot u) + (a \cdot v)$ and $(a + b) \cdot v = (a \cdot v) + (b \cdot v)$.

v. *Existence of an additive identity*: $\exists\, 0 \in V$ for which $v + 0 = v = 0 + v$ for all $v \in V$;

vi. *Existence of additive inverses*: For each $v \in V$ $\exists x \in V$ such that $v + x = 0 = x + v$, $x = -v$ is the additive inverse of v (the equations $x + v = 0$ and $v + x = 0$ have a solution $x \in V$ denoted by $-v$)

vii *Unitary law*: For all $v \in V$, $1 \cdot v = v$.

Definition 2.2. Given a vector space V over a field K, a subset W of V is called a **vector subspace** if W is a vector space over K under the operations already defined on V.

An algebra is just a vector space in which multiplication of vectors is defined. In an algebra, we can multiply elements of the algebra to get another element which belongs to the same algebra.

Definition 2.3. An **algebra** is a pair (K, A), where K is a field and A is a vector space over K, equipped with multiplication such that;

i. *Closure*: For all $x, y \in A$, $xy \in A$.

ii. *Distributivity*: $a(xy) = (ax)y = x(ay)$ and $a(x+y) = ax + ay$ for $a \in K$ and $x, y \in A$.

iii. A is *commutative* if $xy = yx$ $\forall x, y \in A$.

iv. A is *associative* if $(xy)z = x(yz)$ $\forall x, y, z \in A$.

v. A has a *multiplicative identity* 1_A such that $1_A x = x = x 1_A$ $\forall x \in A$.

vi. A is *finite dimensional* if the underlying vector space of A is finite dimensional.

Definition 2.4. A vector subspace I of A is called a **left ideal** if $xy \in I$ $\forall x \in A,\ y \in I$, and a **right ideal** if $yx \in I$ $\forall x \in A,\ y \in I$.

Definition 2.5. I is an **ideal** of A if it is both a right and left ideal.

2.3 Quivers and path algebras

In this section, we introduce quivers which are our objects of study. We also describe algebras related to some examples of quivers.

2.3.1 Quivers

Definition 2.6. A ***Quiver Q*** is an oriented graph. To be more precise, we allow multiple arrows between any two vertices. Formally, a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ where;

- Q_0 is the set of vertices $\{v\}$ which will be finite,
- Q_1 is the set of arrows which will be finite,
- s and t are two maps $s, t : Q_1 \longrightarrow Q_0$, assigning to each arrow the *starting vertex* and the *terminating vertex* respectively.

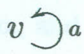
An arrow a starts at the vertex $s(a)$ and terminates at the vertex $t(a)$ indicated as $s(a) \xrightarrow{a} t(a)$.

Definition 2.7. The quiver is **finite** if both sets Q_0 and Q_1 are *finite*. For arrows $a \in Q_1$ with $s(a) = v_i$ and $t(a) = v_j$, we usually write. $a : v_i \longrightarrow v_j$ where $v_i, v_j \in Q_0$.

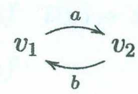
Example 2.8. The quiver Q with $Q_o = \{v_1, v_2, v_3, v_4\}$ and $Q_1 = \{a, b, d\}$ is represented by the diagram,

$$v_1 \xrightarrow{a} v_2 \xrightarrow{b} v_3 \xleftarrow{d} v_4$$

Example 2.9. A quiver Q consisting of one vertex and one arrow (also called the Jordan quiver or a loop).



Example 2.10. A quiver Q consisting of $Q_0 = \{v_1, v_2\}$ and $Q_1 = \{a, b\}$, this is called a 2-cycle.



Examples (2.9) and (2.10) are very important to our study, and we will refer to them later.

2.3.2 Path algebras

In this section we discuss an algebra **A** over the field K, which has a basis that consist of paths of a quiver Q.

Definition 2.11. A **non trivial path** x of length n in Q is a composition $a_1a_2a_3..........a_n$ of n arrows such that $t(a_{i+1}) = s(a_i)$ for $1 \leq i \leq n-1$. ie

$t(x) = t(a_1)$ and $s(x) = s(a_n)$ denotes the initial and final vertices of the path x. A path is **cyclic** if its starting and terminating vertices coincide.

For each vertex $v_i \in Q_0$ we will denote by e_{v_i} the trivial path which starts and terminates at the vertex v_i.

Definition 2.12. Let K be a fixed field. The **path algebra** KQ associated to a quiver Q is the K-algebra whose underlying vector space has basis the set of paths in Q, and with the product of paths given by *concatenation.* Thus, if $x = a_1...a_n$ and $y = b_1...b_m$ are two paths, then;

$$xy = \begin{cases} a_1...a_n b_1...b_m & if \quad t(y) = s(x), \ (i.e.\ t(b_1) = s(a_n)) \\ 0 & if \qquad\qquad otherwise \end{cases}$$

This multiplication is associative since concatenation of paths is associative. We also have for $x \in KQ$, $v_i, v_j \in Q_0$

$$e_{v_i} e_{v_j} = \begin{cases} e_{v_i} & if \quad v_i = v_j \\ 0 & if \quad v_i \neq v_j \end{cases}$$

$$e_{v_i} x = \begin{cases} x & if \quad t(x) = v_i \\ 0 & if \quad t(x) \neq v_i \end{cases}$$

$$x e_{v_j} = \begin{cases} x & if \quad s(x) = v_j \\ 0 & if \quad s(x) \neq v_j \end{cases}$$

Example 2.13. Let Q be the quiver $v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} v_3 \xleftarrow{a_3} v_4$, then KQ has a basis consisting of the paths e_{v_1}, e_{v_2}, e_{v_3}, e_{v_4}, a_1 a_2, a_3 and a_2a_1. The product $a_2 \cdot a_1$ of the paths a_1 and a_2 is the path a_2a_1. On the other hand the path $a_1a_2 = 0$. Some other products in the algebra are $a_2e_{v_2}a_1 = a_2a_1$, $a_2e_{v_2} = a_2$, $e_{v_2}a_1 = a_1$, $e_{v_1}a_2 = 0$, $a_2a_3 = 0$, etc.

Example 2.14. Let Q be the following quiver consisting of one vertex and one arrow (the Jordan quiver or loop), then $KQ \cong K[x]$ the algebra of polynomials in one variable.

$$x \circlearrowleft \bullet$$

Paths of this quiver include $e,\ x,\ x^2,\ x^3, \ldots, x^n$ where n is a nonnegative integer. An element of the path algebra will be of the form $\sum_{i=0}^{n} k_i x^i$ where $x^0 = e$ and $k_i \in K$ $i = 0, \ldots, n$. These are polynomials of degree n over the field K.

Example 2.15. Let Q be the following quiver,

$$v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{n-1} \longrightarrow v_n$$

then for every $v_1 \leq v_i \leq v_j \leq v_n$ there is a unique path from v_i to v_j. Let $f : KQ \longrightarrow M_n(K)$ be the linear map from the path algebra to the $n \times n$ matrices with entries in the field K that sends the unique path from v_i to v_{v_j} to the matrix $E_{v_j v_i}$ with (v_j, v_i) entry 1 and all other entries zero. Then f is an isomorphism onto the algebra of lower triangular matrices.

2.4 Literature review

A basic understanding of Ring theory, Field theory, Vector spaces and Algebras was important to our study. Although there are many sources of information in these areas, the papers [7, 15] were suitable for our study. A good understanding of category theory was necessary to this study. In my perspective, the book [12] was a rich source, and the paper [14] was quite illustrative with examples.

Quivers appear in many areas of mathematics and theoretical physics. A good understanding of quivers was fundamental to our study. It was important to have an understanding of path algebras and modules over path algebras which are part of the basic knowledge necessary to the study. Our study required concepts on the category of representation of quivers. For a reference in this area, I found the lecture notes in the paper [2] to be detailed and easy to study, other sources of literature on quivers that we studied include the papers [6, 11, 17].

Cluster algebra is a new and active area of study introduced and studied by Fomin and Zelevinsky in the papers [9] and [10]. Quivers have been studied in cluster algebras where mutation of quivers is defined as evidenced in [3]. Our study takes a slightly different direction in which we study mutation of quivers with potential and R-charge. Mutation of quivers with potential was first studied by Derksen, Weyman and Zelevinsky in the paper [8] in which they define the process of mutation of quivers with potential. Their work ignited a growth of interest as evidenced in papers such as [4, 19]. We also studied polar quivers. A source of literature in this area is the paper [1].

At the heart of this study are quivers with potential and R-charge. This study heavily relies on examples of quivers with potential and R-charge. The main source of these examples was the Phd thesis [18], Stern studied tilting mutation of geometric helixes. Another source of literature on R-charges was the paper [1]. We will provide additional information in the introduction of each chapter on the key references used.

2.5 Statement of the problem

Angles and positive non-zero integers can be assigned to the vertices of a quiver such that the quiver satisfies polar conditions. In general we expect mutation of polar quivers to give a polar quivers. This is not always the case. Mutation of these quivers can give mutant quivers which do not satisfy the polar conditions.

2.6 Objective of the study

The main aim of this study is to establish conditions under which mutation of a polar quiver gives a polar quiver.

2.7 Research methodology

The main method taken by our study is calculations involving examples that come from Stern's thesis [18]. We establish conditions a polar quiver must meet for its mutants to satisfy polar conditions. These are examples of del Pezzo quivers which are related to del Pezzo geometric surfaces.

2.8 Significance of the study

Given that polar quivers are based on anomaly free R-charges, we are hopeful that the results of our work will have an impact on the study of R-charges in theoretical physics.

Polar quivers are built on the concepts in mutation of quivers with potential. It is possible that the added information can be integrated in the study of non-degenerate quivers with potential.

Chapter 3

Mutation of quivers with potential

3.1 Introduction

Quivers with potential are the main objects of the study in this chapter. In section 3.2, we give a definition of potential as well as their algebraic importance. The key reference for this section is the paper by Derksen, Weyman and Zelevinsky [8]. In the same paper we find the formal definition of mutation of quivers with potential, which is covered in section 3.3. In section 3.4, we give worked examples of mutation of quivers with potential. The examples we used in this section are del Pezzo quivers which come from geometry, and were found in the Phd thesis of Stern [18].

3.2 Quivers with potential

Let $\mathbf{A} = KQ$ be the path algebra and $\mathbb{A} = KQ_1$ be the arrow span of the quiver Q, $\mathbb{A}^d = \underbrace{\mathbb{A} \otimes \mathbb{A} \otimes .... \otimes \mathbb{A}}_{d\ times}$ consists of paths of length d and $\mathbb{A}^0 = \sum_{v_i \in Q_0} K_{e_{v_i}}$ are paths of length zero. We can also define the path algebra $\mathbf{A}$ of Q as a graded algebra

$$\mathbf{A} = \bigoplus_{d=0}^{\infty} \mathbb{A}^d \tag{3.2.1}$$

and

$$\mathbb{A}^d_{cyc} = \bigoplus_{v_i \in Q_0} \mathbb{A}^d_{v_i v_i} \tag{3.2.2}$$

is the cyclic part of $\mathbb{A}^d$ which is the span of all paths $a_1....a_d$ with $s(a_d) = t(a_1)$ for $d \geq 1$.

Definition 3.1. We define the closed vector subspace $\mathbf{A}_{cyc} \subseteq \mathbf{A}$ by setting

$$\mathbf{A}_{cyc} = \prod_{d=1}^{\infty} \mathbb{A}^d_{cyc} \tag{3.2.3}$$

and call the elements of $\mathbf{A}_{cyc}$ **potentials**, denoted as $\mathcal{S}$ (the potential is non-zero if the path algebra $\mathbf{A}$ has oriented cycles, otherwise the potential is zero).

Definition 3.2. Suppose Q is a quiver with an arrow span $\mathbb{A}$, and $\mathcal{S} \in \mathbf{A}_{cyc}$ is a potential. The pair $(Q, \mathcal{S})$ (or $(\mathbb{A}, \mathcal{S})$) is a **quiver with potential** (QP for short) if it satisfies the following conditions:

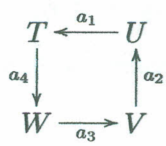
1. The quiver Q has no loops i.e. $\mathbb{A}_{v_i v_i} = 0$ for all $v_i \in Q_0$,

2. No two cyclically equivalent paths appear in the decomposition of $\mathcal{S}$.

Definition 3.3. For every $\xi \in \mathbb{A}^*$ we define the **cyclic derivative** δ_ξ as the continuous K-linear map $\mathbf{A}_{cyc} \longrightarrow \mathbf{A}$ acting on cyclic paths by

$$\delta_\xi(a_1....a_d) = \sum_{k=1}^{d} \xi(a_k)a_{k+1}....a_d a_1....a_{k-1} \tag{3.2.4}$$

Example 3.4. Consider the quiver



$\mathcal{S} = a_1a_2a_3a_4$ and let $\xi = a_3^* \in \mathbb{A}^*$, then

$$\begin{aligned}
\delta_\xi(\mathcal{S}) &= \sum_{k=1}^{4} \xi(a_k)a_{k+1}....a_4 a_1....a_{k-1} \\
\delta_{a_3^*}(a_1a_2a_3a_4) &= a_3^*(a_1)a_2a_3a_4 + a_3^*(a_2)a_3a_4a_1 + a_3^*(a_3)a_4a_1a_2 + a_3^*(a_4)a_1a_2a_3 \\
&= 0 + 0 + a_3^*(a_3)a_4a_1a_2 + 0 \\
&= a_4a_1a_2
\end{aligned}$$

Definition 3.5. For every potential $\mathcal{S}$ we define the **Jacobian ideal** $J(\mathcal{S})$ as the closure of the (two-sided) ideal in $\mathbf{A}$ generated by $\delta_\xi(\mathcal{S})$ for all $\xi \in \mathbb{A}^*$. Jacobian ideals are just relations on the quiver Q generated by cyclic derivatives on the potential $\delta_\xi(\mathcal{S})$.

Definition 3.6. We call the quotient $\mathbf{A}/J(\mathcal{S})$ the **Jacobian Algebra** of $\mathcal{S}$ and denote it by $\mathcal{P}(Q,\mathcal{S})$ or $\mathcal{P}(\mathbb{A},\mathcal{S})$.

Remark 3.7. The cyclic derivative $\delta_\xi : \mathbf{A}_{cyc} \longrightarrow \mathbf{A}$ does not depend on the choice of the path basis. Cyclic derivatives do not distinguish between the potentials that are equivalent as shown below.

Definition 3.8. Two potentials $\mathcal{S}$ and $\grave{\mathcal{S}}$ are **cyclically equivalent** if $\mathcal{S} - \grave{\mathcal{S}}$ lies in the closure of the span of all elements of the form $a_1....a_d - a_2....a_da_1$ where $a_1....a_d$ is a cyclic path.

Proposition 3.9. *If two potentials $\mathcal{S}$ and $\grave{\mathcal{S}}$ are cyclically equivalent, then $\delta_\xi(\mathcal{S}) = \delta_\xi(\grave{\mathcal{S}})$ for every $\xi \in \mathbb{A}^*$ hence $J(\mathcal{S}) = J(\grave{\mathcal{S}})$ and $\mathcal{P}(Q,\mathcal{S}) = \mathcal{P}(\mathbb{A},\grave{\mathcal{S}})$*

Proof. Let $\mathcal{S}$ and $\grave{\mathcal{S}}$ be two cyclically equivalent potentials, then $\mathcal{S} - \grave{\mathcal{S}}$ lies in the closure of the span of all elements of the form $a_1....a_d - a_k....a_da_1....a_{k-1}$ with $1 \leq k \leq d$, where $a_1....a_d$ is a cyclic path.
For any $\xi \in \mathbb{A}^*$,

$$\begin{aligned}
\delta_\xi(a_1....a_d - a_k....a_da_1....a_{k-1}) &= \delta_\xi(a_1....a_d) - \delta_\xi(a_k....a_da_1....a_{k-1}) \\
&= 0 \\
0 = \delta_\xi(\mathcal{S} - \grave{\mathcal{S}}) &= \delta_\xi(\mathcal{S}) - \delta_\xi(\grave{\mathcal{S}}) \\
\Rightarrow \delta_\xi(\mathcal{S}) &= \delta_\xi(\grave{\mathcal{S}})
\end{aligned}$$

Since $\delta_\xi(\mathcal{S}) = \delta_\xi(\grave{\mathcal{S}})$ for all $\xi \in \mathbb{A}^*$, the closure of the ideal in $\mathbf{A}$ generated by $\delta_\xi(\mathcal{S})$ and $\delta_\xi(\grave{\mathcal{S}})$ for all $\xi \in \mathbb{A}^*$ is the same. Hence $J(\mathcal{S}) = J(\grave{\mathcal{S}})$.
Finally $\mathbf{A}/J(\mathcal{S}) = \mathbf{A}/J(\grave{\mathcal{S}})$ which implies $\mathcal{P}(Q,\mathcal{S}) = \mathcal{P}(\mathbb{A},\grave{\mathcal{S}})$. □

Definition 3.10. For any arbitrary QP $(Q, \mathcal{S})$, we denote by $\mathcal{S}^{(2)} \in \mathbb{A}^2$ the degree 2 homogeneous component of $\mathcal{S}$. We call $(Q, \mathcal{S})$ *reduced* if $\mathcal{S}^{(2)} = 0$. We define the **trivial** and **reduced** arrow spans of $(Q, \mathcal{S})$ as the finite dimensional K-bimodule given respectively by;

$$\mathbb{A}_{triv} = \mathbb{A}_{triv}(\mathcal{S}) = \delta(\mathcal{S}^{(2)}), \quad \mathbb{A}_{red} = \mathbb{A}_{red}(\mathcal{S}) = \mathbf{A}/\delta(\mathcal{S}^{(2)})$$

Definition 3.11. Let $(Q, \mathcal{S})$ and $(\grave{Q}, \grave{\mathcal{S}})$ be two QPs on the same vertex in Q_0. By a **right equivalence** between $(Q, \mathcal{S})$ and $(\grave{Q}, \grave{\mathcal{S}})$ we mean an algebra isomorphism $\varphi : \mathbf{A} \longrightarrow \grave{\mathbf{A}}$ such that $\varphi|_K = id$ and $\varphi(\mathcal{S})$ is cyclically equivalent to $\grave{\mathcal{S}}$

Theorem 3.12. *(Splitting theorem) For every QP $(Q, \mathcal{S})$ with the trivial arrow span $\mathbb{A}_{triv}$ and the reduced arrow span $\mathbb{A}_{red}$, there exists a trivial QP $(\mathbb{A}_{triv}, \mathcal{S}_{triv})$ and a reduced QP $(\mathbb{A}_{red}, \mathcal{S}_{red})$ such that $(Q, \mathcal{S})$ is right-equivalent to the direct sum $(\mathbb{A}_{triv}, \mathcal{S}_{triv}) \oplus (\mathbb{A}_{red}, \mathcal{S}_{red})$. Furthermore, the right equivalence class of each of the QPs $(\mathbb{A}_{triv}, \mathcal{S}_{triv})$ and $(\mathbb{A}_{red}, \mathcal{S}_{red})$ is determined by the right-equivalence class of $(Q, \mathcal{S})$.*

Proof is provided in [8]. This theorem is essential especially in the reduction of the mutated quivers.

Condition 3.13. *Mutation of a QP $(Q, \mathcal{S})$ can be defined if at any vertex $v_k \in Q_0$ the quiver satisfies the following conditions:*

1. Q has no loops, i.e. $\mathbb{A}_{v_i v_i} = 0$ for each $v_i \in Q_0$,
2. Q has no oriented 2-cycles. For every vertex v_i, either $\mathbb{A}_{v_i v_k}$ or $\mathbb{A}_{v_k v_i}$ is zero.

3.3 Mutation of a quiver with potential

Let $(Q, \mathcal{S})$ have no loops or oriented 2-cycles, we associate to it a QP $\mu'_{v_k}(Q, \mathcal{S}) = (Q', \mathcal{S}')$ on the same set of vertices Q_0. $(Q', \mathcal{S}')$ is the unreduced mutated quiver. We define the homogeneous components $\mathbb{A}'_{v_i v_j}$ as follows:

$$\mathbb{A}'_{v_i v_j} = \begin{cases} (\mathbb{A}_{v_j v_i})^* & if \quad v_i = v_k \text{ or } v_j = v_k; \\ \mathbb{A}_{v_i v_j} \oplus \mathbb{A}_{v_i v_k}\mathbb{A}_{v_k v_j} & if \quad otherwise \end{cases} \tag{3.3.1}$$

Here the product $\mathbb{A}_{v_i v_k}\mathbb{A}_{v_k v_j}$ is understood as a subspace of $\mathbb{A}^2 \subseteq \mathbf{A}$. Thus, the K-bimodule $\mathbb{A}'$ is given by

$$\mathbb{A}' = \bar{e}_{v_k}\mathbb{A}\bar{e}_{v_k} \oplus \mathbb{A}\bar{e}_{v_k}\mathbb{A} \oplus (e_{v_k}\mathbb{A})^* \oplus (\mathbb{A}e_{v_k})^*, \tag{3.3.2}$$

where we use the notation

$$\bar{e}_{v_k} = 1 - e_{v_k} = \sum_{v_i \in Q_0 - \{v_k\}} e_{v_i}. \tag{3.3.3}$$

We associate to Q_1 the set of arrows Q'_1 in the following way:

i) The vertices of the quiver remain unchanged.

ii) Replace each arrow $a : v_i \longrightarrow v_k$ in Q by a new arrow $a^* : v_k \longrightarrow v_i$.

iii) Replace each arrow $b : v_k \longrightarrow v_j$ in Q by a new arrow $b^* : v_j \longrightarrow v_k$.

iv) All the arrows $c \in Q_1$ not incident to v_k remain unchanged.

v) Add a new arrow $[ba] : v_i \longrightarrow v_j$ for each pair of arrows $a : v_i \longrightarrow v_k$ and $b : v_k \longrightarrow v_j$ in Q.

We denote by $[ba] \in Q'_1 \cap \mathbb{A}_{v_i v_k} \mathbb{A}_{v_k v_j}$ the arrow in Q'_1 associated with the product ba (to avoid confusion since it is a single arrow). We associate to $\mathcal{S}$ the potential $\mu'_{v_k}(\mathcal{S}) = \mathcal{S}' \in \mathbf{A}$ given by

$$\mathcal{S}' = [\mathcal{S}] + \Delta_{v_k}, \tag{3.3.4}$$

where

$$\Delta_{v_k} = \Delta_{v_k}(\mathbb{A}) = \sum_{a,b \in Q_1 : s(b) = t(a) = v_k} [ba] a^* b^* \tag{3.3.5}$$

and $[\mathcal{S}]$ is obtained by substituting $[a_p a_{p+1}]$ for each factor $a_p a_{p+1}$ with $s(a_p) = t(a_{p+1}) = v_k$ in any cyclic path $a_1 .... a_d$ occurring in the expansion of $\mathcal{S}$. Note that none of the cyclic paths start nor terminate at v_k. Both $[\mathcal{S}]$ and Δ_{v_k} do not depend on the choice of a basis Q_1 of $\mathbb{A}$. The following proposition follows from the definitions.

Proposition 3.14. *Suppose that a QP $(Q, \mathcal{S})$ has no loops or oriented 2-cycles, and a QP $(\dot{Q}, \dot{\mathcal{S}})$ is such that $e_{v_k} \mathbb{A} = \mathbb{A} e_{v_k} = \{0\}$. Then we have by theorem (3.12),*

$$\mu'_{v_k}(Q \oplus \dot{Q}, \mathcal{S} + \dot{\mathcal{S}}) = \mu'_{v_k}(Q, \mathcal{S}) \oplus (\dot{Q}, \dot{\mathcal{S}}) \tag{3.3.6}$$

Theorem 3.15. *The equivalence class of the QP $(Q', \mathcal{S}') = \mu'(Q, \mathcal{S})$ is determined by the right-equivalence class of $(Q, \mathcal{S})$*

Proofs for (3.14) and (3.15) are provided in [8].

Note that even if a QP $(Q, \mathcal{S})$ is assumed to be reduced, the mutated quiver $(Q', \mathcal{S}') = \mu'(Q, \mathcal{S})$ is not necessarily reduced because the component $[\mathcal{S}]^{(2)} \in \mathbb{A}'^2$ may be non-zero. Combining Theorems (3.12) and (3.15) we obtain the following corollary.

Corollary 3.16. *Suppose a QP $(\mathbb{A}, \mathcal{S})$ has no loops or oriented 2-cycles, and let $(\mathbb{A}', \mathcal{S}') = \mu'(Q, \mathcal{S})$. Let $(\tilde{\mathbb{A}}, \tilde{\mathcal{S}})$ be a reduced QP such that*

$$(\mathbb{A}', \mathcal{S}') \cong (\mathbb{A}'_{triv}, \mathcal{S}'^{(2)}) \oplus (\tilde{\mathbb{A}}, \tilde{\mathcal{S}}) \tag{3.3.7}$$

Then the right-equivalence class of $(\tilde{\mathbb{A}}, \tilde{\mathcal{S}})$ is determined by the right-equivalence class of $(\mathbb{A}, \mathcal{S})$

Definition 3.17. In the situation of corollary (3.16) we use the notation

$$\mu_{v_k}(Q, \mathcal{S}) \doteq (\tilde{Q}, \tilde{\mathcal{S}}) \tag{3.3.8}$$

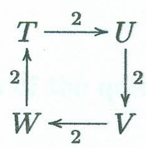
and call the correspondence

$$(Q, \mathcal{S}) \mapsto \mu_{v_k}(Q, \mathcal{S}) = (\tilde{Q}, \tilde{\mathcal{S}}) \tag{3.3.9}$$

the **mutation at vertex** v_k. See [8] for more details.

3.4 Examples on mutation of quivers with potential

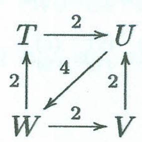
Example 3.18. Consider the quiver with potential Q given below,



the quiver has no loops or 2-cycles. The potential of the quiver is given by;

$$S = a^{(1)}_{TW}a^{(1)}_{WV}a^{(1)}_{VU}a^{(1)}_{UT} + a^{(1)}_{TW}a^{(2)}_{WV}a^{(1)}_{VU}a^{(2)}_{UT} + a^{(2)}_{TW}a^{(1)}_{WV}a^{(2)}_{VU}a^{(1)}_{UT} + a^{(2)}_{TW}a^{(2)}_{WV}a^{(2)}_{VU}a^{(2)}_{UT} \tag{3.4.1}$$

We will mutate the quiver at V. The mutated quiver Q' will be given by;



The four new arrows are $[a^{(1)}_{WV}a^{(1)}_{VU}]$, $[a^{(1)}_{WV}a^{(2)}_{VU}]$, $[a^{(2)}_{WV}a^{(1)}_{VU}]$ and $[a^{(2)}_{WV}a^{(2)}_{VU}]$. While the unreduced potential of the mutated quiver will be given by;

$$
\begin{aligned}
S' = {} & a_{TW}^{(1)}[a_{WV}^{(1)}a_{VU}^{(1)}]a_{UT}^{(1)} + a_{TW}^{(1)}[a_{WV}^{(2)}a_{VU}^{(1)}]a_{UT}^{(2)} + a_{TW}^{(2)}[a_{WV}^{(1)}a_{VU}^{(2)}]a_{UT}^{(1)} + \\
& a_{TW}^{(2)}[a_{WV}^{(2)}a_{VU}^{(2)}]a_{UT}^{(2)} + [a_{WV}^{(1)}a_{VU}^{(1)}]a_{VU}^{(1)*}a_{WV}^{(1)*} + [a_{WV}^{(1)}a_{VU}^{(2)}]a_{VU}^{(2)*}a_{WV}^{(1)*} + \\
& [a_{WV}^{(2)}a_{VU}^{(1)}]a_{VU}^{(1)*}a_{WV}^{(2)*} + [a_{WV}^{(2)}a_{VU}^{(2)}]a_{VU}^{(2)*}a_{WV}^{(2)*}
\end{aligned}
\tag{3.4.2}
$$

We can rename the arrows of the quiver. First the four new arrows as,

$$
[a_{WV}^{(1)}a_{VU}^{(1)}] = a_{WU}^{(1)} \qquad [a_{WV}^{(2)}a_{VU}^{(1)}] = a_{WU}^{(3)}
$$

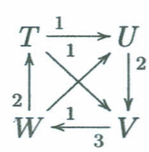
$$
[a_{WV}^{(1)}a_{VU}^{(2)}] = a_{WU}^{(2)} \qquad [a_{WV}^{(2)}a_{VU}^{(2)}] = a_{WU}^{(4)}
$$

On reduction, the potential then takes the form;

$$
\begin{aligned}
\tilde{S} = {} & a_{TW}^{(1)}a_{WU}^{(1)}a_{UT}^{(1)} + a_{TW}^{(1)}a_{WU}^{(3)}a_{UT}^{(2)} + a_{TW}^{(2)}a_{WU}^{(2)}a_{UT}^{(1)} + \\
& a_{TW}^{(2)}a_{WU}^{(4)}a_{UT}^{(2)} + a_{WU}^{(1)}a_{UV}^{(1)}a_{VW}^{(1)} + a_{WU}^{(2)}a_{UV}^{(2)}a_{VW}^{(1)} + \\
& a_{WU}^{(3)}a_{UV}^{(1)}a_{VW}^{(2)} + a_{WU}^{(4)}a_{UV}^{(2)}a_{VW}^{(2)}
\end{aligned}
\tag{3.4.3}
$$

From the potential, we can see that the mutated quiver has no loops or 2-cycles.

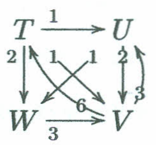
Example 3.19. Consider the quiver with potential in the diagram below,



The quiver has no loops or 2-cycles and its potential is given by,

$$\begin{aligned} \mathcal{S} &= a_{UT}a^{(1)}_{TW}a^{(3)}_{WV}a^{(2)}_{VU} + a_{UT}a^{(2)}_{TW}a^{(3)}_{WV}a^{(1)}_{VU} + a_{UW}a^{(1)}_{WV}a^{(1)}_{VU} + \\ &\quad a_{UW}a^{(2)}_{WV}a^{(2)}_{VU} + a_{VT}a^{(1)}_{TW}a^{(2)}_{WV} + a_{VT}a^{(2)}_{TW}a^{(1)}_{WV} \end{aligned}$$

We mutate the quiver at W to get an unreduced quiver Q' given by the diagram below,



The new arrows created by the process of mutation are;

From V to T,

$[a^{(1)}_{TW}a^{(1)}_{WV}]$

$[a^{(1)}_{TW}a^{(2)}_{WV}]$

$[a^{(1)}_{TW}a^{(3)}_{WV}]$

$[a^{(2)}_{TW}a^{(1)}_{WV}]$

$[a^{(2)}_{TW}a^{(2)}_{WV}]$

$[a^{(2)}_{TW}a^{(3)}_{WV}]$

From V to U,

$[a_{UW}a^{(1)}_{WV}]$

$[a_{UW}a^{(2)}_{WV}]$

$[a_{UW}a^{(3)}_{WV}]$

and the potential for the unreduced quiver is given by;

$$\begin{aligned}
S' &= a_{UT}[a^{(1)}_{TW}a^{(3)}_{WV}]a^{(2)}_{VU} + a_{UT}[a^{(2)}_{TW}a^{(3)}_{WV}]a^{(1)}_{VU} + [a_{UW}a^{(1)}_{WV}]a^{(1)}_{VU} + \\
&\quad [a_{UW}a^{(2)}_{WV}]a^{(2)}_{VU} + a_{VT}[a^{(1)}_{TW}a^{(2)}_{WV}] + a_{VT}[a^{(2)}_{TW}a^{(1)}_{WV}] + \\
&\quad [a^{(1)}_{TW}a^{(1)}_{WV}]a^{(1)*}_{WV}a^{(1)*}_{TW} + [a^{(1)}_{TW}a^{(2)}_{WV}]a^{(2)*}_{WV}a^{(1)*}_{TW} + [a^{(1)}_{TW}a^{(3)}_{WV}]a^{(3)*}_{WV}a^{(1)*}_{TW} + \\
&\quad [a^{(2)}_{TW}a^{(1)}_{WV}]a^{(1)*}_{WV}a^{(2)*}_{TW} + [a^{(2)}_{TW}a^{(2)}_{WV}]a^{(2)*}_{WV}a^{(2)*}_{TW} + [a^{(2)}_{TW}a^{(3)}_{WV}]a^{(3)*}_{WV}a^{(2)*}_{TW} + \\
&\quad [a_{UW}a^{(1)}_{WV}]a^{(1)*}_{WV}a^{*}_{UW} + [a_{UW}a^{(2)}_{WV}]a^{(2)*}_{WV}a^{*}_{UW} + [a_{UW}a^{(3)}_{WV}]a^{(3)*}_{WV}a^{*}_{UW}
\end{aligned}$$

There are two cycles in the quiver, and this is evident in the unreduced potential S'. We must undertake the reduction process to identify and eliminate equivalent paths that are creating the 2-cycles. This is done by taking cyclic derivatives with respect to the arrows in the 2-cycles. Taking cyclic derivative with respect to a_{VT} gives, $[a^{(1)}_{TW}a^{(2)}_{WV}] = [a^{(2)}_{TW}a^{(1)}_{WV}]$ and, taking cyclic derivative with respect to $[a_{UW}a^{(1)}_{WV}]$ and $[a_{UW}a^{(2)}_{WV}]$ gives $a^{(1)}_{VU} = a^{(1)*}_{WV}a^{*}_{UW}$ and $a^{(2)}_{VU} = a^{(2)*}_{WV}a^{*}_{UW}$ respectively. Substituting this result into the potential S', we get pairs of equivalent 2-cycles which are eliminated in the reduction process. The number of new arrows are reduced through this process. We rename the remaining arrows as, From V to T,

$$[a_{TW}^{(1)}a_{WV}^{(1)}] \in Q' = a_{TV}^{(1)} \in \tilde{Q}$$

$$[a_{TW}^{(1)}a_{WV}^{(2)}] \in Q' = a_{TV}^{(2)} \in \tilde{Q}$$

$$[a_{TW}^{(1)}a_{WV}^{(3)}] \in Q' = a_{TV}^{(3)} \in \tilde{Q}$$

$$[a_{TW}^{(2)}a_{WV}^{(1)}] \in Q' = a_{TV}^{(2)} \in \tilde{Q}$$

$$[a_{TW}^{(2)}a_{WV}^{(2)}] \in Q' = a_{TV}^{(4)} \in \tilde{Q}$$

$$[a_{TW}^{(2)}a_{WV}^{(3)}] \in Q' = a_{TV}^{(5)} \in \tilde{Q}$$

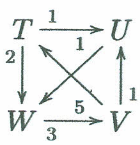
From V to U,

$$[a_{UW}a_{WV}^{(1)}] \in Q' = 0 \in \tilde{Q}$$

$$[a_{UW}a_{WV}^{(2)}] \in Q' = 0 \in \tilde{Q}$$

$$[a_{UW}a_{WV}^{(3)}] \in Q' = a_{UV} \in \tilde{Q}$$

This is shown in the reduced quiver below,



with a potential given by;

$$\begin{aligned}\tilde{S} \;=\; & a_{UT}a_{TV}^{(3)}a_{VW}^{(2)}a_{WU} + a_{UT}a_{TV}^{(5)}a_{VW}^{(1)}a_{WU} + a_{TV}^{(1)}a_{VW}^{(1)}a_{WT}^{(1)} + \\ & a_{TV}^{(2)}a_{VW}^{(2)}a_{WT}^{(1)} + a_{TV}^{(3)}a_{VW}^{(3)}a_{WT}^{(1)} + a_{TV}^{(2)}a_{VW}^{(1)}a_{WT}^{(2)} + \\ & a_{TV}^{(4)}a_{VW}^{(2)}a_{WT}^{(2)} + a_{TV}^{(5)}a_{VW}^{(3)}a_{WT}^{(2)} + a_{UV}a_{VW}^{(3)}a_{WU}\end{aligned}$$

Mutation of quivers, and of quivers with potential is a reversible process. We can get back to our original quiver from the mutated quiver by carrying out mutation at the very vertex that was mutated. The theorem below gives a detailed account for this.

Theorem 3.20. *(Every mutation is an involution) The correspondence*

$\mu_k : (Q, \mathcal{S}) \longrightarrow (\tilde{\mathbb{A}}, \tilde{\mathcal{S}})$ acts as an involution on the right-equivalence classes of the reduced QPs that are without loops or oriented 2-cycles, that is $\mu_k^2(Q, \mathcal{S})$ is right equivalent to $(Q, \mathcal{S})$

Proof of the theorem is provided in [8].

Chapter 4

Polar quivers with potential

4.1 Introduction

In this chapter[1] we introduce polar quivers. Angles and positive non-zero integers can be assigned to the vertices of a quiver with potential in such a way that the quiver satisfy the polar conditions. In section 4.2, we state the three polar conditions. In section 4.3, we give examples of polar quivers with potential. This section gives an illustration of the polar conditions.

4.2 Polar quivers

Definition 4.1. A quadruple $(Q, \mathcal{S}, N, \theta)$, where Q is a finite quiver with potential $\mathcal{S}$, $N : Q_0 \to \mathbb{Z}^{>0}$, $\theta : Q_0 \to [0, 2\pi)$ where θ can be extended to

[1] Unless stated otherwise, the work covered from this chapter onwards is original and therefore has no references.

a function $Q_1 \to [0, 2\pi)$ by $\theta(a) := (\theta(t(a)) - \theta(s(a)))_{\bmod 2\pi}$, is called a **polar quiver** if all the following conditions hold:

1. For any $v \in Q_0$ a total $N^T(v)$ can be defined such that

$$N^T(v) := \sum_{a \,|\, t(a)=v} N(s(a)) = \sum_{a \,|\, s(a)=v} N(t(a)) \tag{4.2.1}$$

2. For any $v \in Q_0$ totals

$$\begin{aligned} \theta^L(v) &:= \sum_{a \,|\, t(a)=v} \theta(a) N(s(a)), \\ \theta^R(v) &:= \sum_{a \,|\, s(a)=v} \theta(a) N(t(a)), \end{aligned}$$

can be defined such that

$$\theta^L(v) + \theta^R(v) = 2\pi(N^T(v) - N(v)). \tag{4.2.2}$$

where

$$N^T(v) > N(v) \tag{4.2.3}$$

3. For any cyclic element $a_1 \dots a_n \in \mathcal{S}$, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi \tag{4.2.4}$$

The next section gives examples of polar quivers which offer illustrations for each polar conditions;

4.3 Examples of polar quivers

In this section, we give some examples of polar quivers. Through calculations, we verify that they satisfy all polar conditions. The last two examples are of particular interest for they concern the same quiver with potential, but with different polar co-ordinates. From these examples we conclude that polar co-ordinates for a particular quiver with potential are not unique.

Example 4.2. Consider the diagram of $P^1 \times P^1$ quiver with potential below. We show that it is a polar quiver,

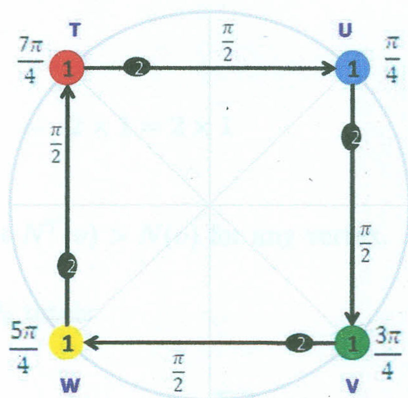


Figure 4.3.1: Diagram of $P^1 \times P^1$. Vertices are given names T, U, V and W, while the number in the vertices indicates the number of grouped vertices.

The potential for the quiver is given by;

$$\mathcal{S} = a^{(1)}_{TW}a^{(1)}_{WV}a^{(1)}_{VU}a^{(1)}_{UT}+a^{(1)}_{TW}a^{(2)}_{WV}a^{(1)}_{VU}a^{(2)}_{UT}+a^{(2)}_{TW}a^{(1)}_{WV}a^{(2)}_{VU}a^{(1)}_{UT}+a^{(2)}_{TW}a^{(2)}_{WV}a^{(2)}_{VU}a^{(2)}_{UT}$$

We show that this is a polar quiver,

1.

$$N^T(v) := \sum_{a\,|\,t(a)=v} N(s(a)) = \sum_{a\,|\,s(a)=v} N(t(a))$$

$$N^T(T) \;=\; 2 \times 1 = 2 \times 1 \qquad\qquad N(T) = 1$$

$$N^T(U) \;=\; 2 \times 1 = 2 \times 1 \qquad\qquad N(U) = 1$$

$$N^T(V) \;=\; 2 \times 1 = 2 \times 1 \qquad\qquad N(V) = 1$$

$$N^T(W) \;=\; 2 \times 1 = 2 \times 1 \qquad\qquad N(W) = 1$$

It is clear that $N^T(v) > N(v)$ for any vertex.

2. For any $v \in Q_0$ totals

$$\begin{aligned} \theta^L(v) &:= \sum_{a\,|\,t(a)=v} \theta(a)N(s(a)), \\ \theta^R(v) &:= \sum_{a\,|\,s(a)=v} \theta(a)N(t(a)), \end{aligned}$$

can be defined such that

$$\theta^L(v) + \theta^R(v) = 2\pi(N^T(v) - N(v)).$$

where

$$N^T(v) > N(v)$$

$$\begin{aligned}\theta^L(T) &= 2 \times \frac{\pi}{2} \times 1 \\ &= \pi\end{aligned} \qquad \begin{aligned}\theta^R(T) &= 2 \times \frac{\pi}{2} \times 1 \\ &= \pi\end{aligned}$$

$$\begin{aligned}\theta^L(U) &= 2 \times \frac{\pi}{2} \times 1 \\ &= \pi\end{aligned} \qquad \begin{aligned}\theta^R(U) &= 2 \times \frac{\pi}{2} \times 1 \\ &= \pi\end{aligned}$$

$$\begin{aligned}\theta^L(V) &= 2 \times \frac{\pi}{2} \times 1 \\ &= \pi\end{aligned} \qquad \begin{aligned}\theta^R(V) &= 2 \times \frac{\pi}{2} \times 1 \\ &= \pi\end{aligned}$$

$$\begin{aligned}\theta^L(W) &= 2 \times \frac{\pi}{2} \times 1 \\ &= \pi\end{aligned} \qquad \begin{aligned}\theta^R(W) &= 2 \times \frac{\pi}{2} \times 1 \\ &= \pi\end{aligned}$$

and their sums are;

$$\begin{aligned}\theta^L(T) + \theta^R(T) &= \pi + \pi \\ &= 2\pi \\ &= 2\pi(2-1) \\ &= 2\pi(N^T(T) - N(T))\end{aligned}$$

$$\begin{aligned}\theta^L(T)+\theta^R(T) &= \pi+\pi \\ &= 2\pi \\ &= 2\pi(2-1) \\ &= 2\pi(N^T(U)-N(U))\end{aligned}$$

$$\begin{aligned}\theta^L(T)+\theta^R(T) &= \pi+\pi \\ &= 2\pi \\ &= 2\pi(2-1) \\ &= 2\pi(N^T(V)-N(V))\end{aligned}$$

$$\begin{aligned}\theta^L(T)+\theta^R(T) &= \pi+\pi \\ &= 2\pi \\ &= 2\pi(2-1) \\ &= 2\pi(N^T(W)-N(W))\end{aligned}$$

3. For any cyclic element $a_1 \dots a_n \in \mathcal{S}$, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi$$

the quiver has a potential given by

$$\mathcal{S} = a_{TW}^{(1)}a_{WV}^{(1)}a_{VU}^{(1)}a_{UT}^{(1)}+a_{TW}^{(1)}a_{WV}^{(2)}a_{VU}^{(1)}a_{UT}^{(2)}+a_{TW}^{(2)}a_{WV}^{(1)}a_{VU}^{(2)}a_{UT}^{(1)}+a_{TW}^{(2)}a_{WV}^{(2)}a_{VU}^{(2)}a_{UT}^{(2)}$$

In any cyclic component $a_1 \dots a_n$ of the potential, $n > 2$ and

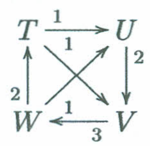
$$\sum_{i=1}^{4} \theta(a_i) = 2\pi$$

as shown below

$$\begin{aligned} \theta(a^{(1,2)}_{TW} a^{(1,2)}_{WV} a^{(1,2)}_{VU} a^{(1,2)}_{UT}) &= \theta(a^{(1,2)}_{TW}) + \theta(a^{(1,2)}_{WV}) + \theta(a^{(1,2)}_{VU}) + \theta(a^{(1,2)}_{UT}) \\ &= \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} \\ &= 2\pi, \end{aligned}$$

where $a^{(1,2)}_{TW}$ implies $a^{(1)}_{TW}$ or $a^{(2)}_{TW}$ and so on.

The next two examples of the same quiver with potential but with different polar co-ordinates. These examples are a good illustration that for a given quiver with potential, the polar co-ordinates are not unique.



$$\begin{aligned} \mathcal{S} = \; & a_{UT} a^{(1)}_{TW} a^{(3)}_{WV} a^{(2)}_{VU} + a_{UT} a^{(2)}_{TW} a^{(3)}_{WV} a^{(1)}_{VU} + \\ & a_{UW} a^{(1)}_{WV} a^{(1)}_{VU} + a_{UW} a^{(2)}_{WV} a^{(2)}_{VU} + \\ & a_{VT} a^{(1)}_{TW} a^{(2)}_{WV} + a_{VT} a^{(2)}_{TW} a^{(1)}_{WV} \end{aligned}$$

Example 4.3. In this example, we assign to the quiver with potential a set of polar co-ordinates that makes the quiver polar. This will be verified in the calculations that follow.

We show that it is a polar quiver.

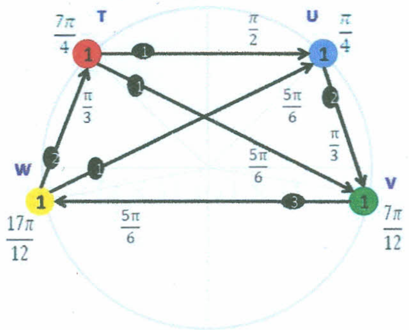


Figure 4.3.2: Diagram of P^2 quiver blown up at one point. Vertices are given names T, U, V and W, while the number in the vertices indicates number of grouped vertices. Angles are assigned such that the quiver satisfies all polar conditions but none of its mutants does.

1.

$$N^T(v) := \sum_{a \,|\, t(a)=v} N(s(a)) = \sum_{a \,|\, s(a)=v} N(t(a))$$

$$\begin{aligned} N^T(T) &= 2 \times 1 = (1 \times 1) + (1 \times 1) && N(T) = 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} N^T(U) &= (1 \times 1) + (1 \times 1) = 2 \times 1 && N(U) = 1 \\ &= 2 \end{aligned}$$

$$
\begin{aligned}
N^T(V) &= (2\times 1)+(1\times 1) = 3\times 1 \qquad\qquad N(V)=1\\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
N^T(W) &= 3\times 1 = (2\times 1)+(1\times 1) \qquad\qquad N(W)=1\\
&= 3
\end{aligned}
$$

It is clear that $N^T(v) > N(v)$ for any vertex.

2. For any $v \in Q_0$ totals

$$
\begin{aligned}
\theta^L(v) &:= \sum_{a\,|\,t(a)=v} \theta(a)N(s(a)),\\
\theta^R(v) &:= \sum_{a\,|\,s(a)=v} \theta(a)N(t(a)),
\end{aligned}
$$

can be defined such that

$$
\theta^L(v)+\theta^R(v) = 2\pi(N^T(v)-N(v)).
$$

where

$$
N^T(v) > N(v)
$$

$$
\begin{aligned}
\theta^L(T) &= 2\times\frac{\pi}{3}\times 1 & \theta^R(T) &= \left(1\times\frac{\pi}{2}\times 1\right)+\left(1\times\frac{5\pi}{6}\times 1\right)\\
&= \frac{2\pi}{3} & &= \frac{4\pi}{3}
\end{aligned}
$$

$$\begin{aligned}\theta^L(U) &= \left(1\times\frac{\pi}{2}\times 1\right)+\left(1\times\frac{5\pi}{6}\times 1\right) & \theta^R(U) &= 2\times\frac{\pi}{3}\times 1\\ &= \frac{4\pi}{3} & &= \frac{2\pi}{3}\end{aligned}$$

$$\begin{aligned}\theta^L(V) &= \left(2\times\frac{\pi}{3}\times 1\right)+\left(1\times\frac{5\pi}{6}\times 1\right) & \theta^R(V) &= 3\times\frac{5\pi}{6}\times 1\\ &= \frac{3\pi}{2} & &= \frac{5\pi}{2}\end{aligned}$$

$$\begin{aligned}\theta^L(W) &= 3\times\frac{5\pi}{6}\times 1 & \theta^R(W) &= \left(2\times\frac{\pi}{3}\times 1\right)+\left(1\times\frac{5\pi}{6}\times 1\right)\\ &= \frac{5\pi}{2} & &= \frac{3\pi}{2}\end{aligned}$$

and their sums are;

$$\begin{aligned}\theta^L(T)+\theta^R(T) &= \frac{2\pi}{3}+\frac{4\pi}{3}\\ &= 2\pi\\ &= 2\pi(2-1)\\ &= 2\pi(N^T(T)-N(T))\end{aligned}$$

$$\begin{aligned}\theta^L(U)+\theta^R(U) &= \frac{4\pi}{3}+\frac{2\pi}{3}\\ &= 2\pi\\ &= 2\pi(2-1)\\ &= 2\pi(N^T(U)-N(U))\end{aligned}$$

$$\begin{aligned}\theta^L(V)+\theta^R(V) &= \frac{3\pi}{2}+\frac{5\pi}{2}\\ &= 4\pi\\ &= 2\pi(3-1)\\ &= 2\pi(N^T(V)-N(V))\end{aligned}$$

$$\begin{aligned}\theta^L(W)+\theta^R(W) &= \frac{5\pi}{2}+\frac{3\pi}{2}\\ &= 4\pi\\ &= 2\pi(3-1)\\ &= 2\pi(N^T(W)-N(W))\end{aligned}$$

3. For any cyclic element $a_1 \dots a_n \in \mathcal{S}$, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi$$

The quiver has a potential given by;

$$\begin{aligned}\mathcal{S} \;=\; & a_{UT}a^{(1)}_{TW}a^{(3)}_{WV}a^{(2)}_{VU} + a_{UT}a^{(2)}_{TW}a^{(3)}_{WV}a^{(1)}_{VU} + a_{UW}a^{(1)}_{WV}a^{(1)}_{VU} + \\ & a_{UW}a^{(2)}_{WV}a^{(2)}_{VU} + a_{VT}a^{(1)}_{TW}a^{(2)}_{WV} + a_{VT}a^{(2)}_{TW}a^{(1)}_{WV}\end{aligned}$$

In any cyclic component $a_1 \dots a_n$ of the potential, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi$$

as shown below

$$\begin{aligned}
\theta(a_{UT}a^{(1)}_{TW}a^{(3)}_{WV}a^{(2)}_{VU}) &= \theta(a_{UT}a^{(2)}_{TW}a^{(3)}_{WV}a^{(1)}_{VU}) \\
&= \theta(a_{UT}) + \theta(a^{(2)}_{TW}) + \theta(a^{(3)}_{WV}) + \theta(a^{(1)}_{VU}) \\
&= \frac{\pi}{2} + \frac{\pi}{3} + \frac{5\pi}{6} + \frac{\pi}{3} \\
&= 2\pi
\end{aligned}$$

$$\begin{aligned}
\theta(a_{UW}a^{(1)}_{WV}a^{(1)}_{VU}) &= \theta(a_{UW}a^{(2)}_{WV}a^{(2)}_{VU}) \\
&= \theta(a_{UW}) + \theta(a^{(2)}_{WV}) + \theta(a^{(2)}_{VU}) \\
&= \frac{5\pi}{6} + \frac{5\pi}{6} + \frac{\pi}{3} \\
&= 2\pi
\end{aligned}$$

$$\begin{aligned}
\theta(a_{VT}a^{(1)}_{TW}a^{(2)}_{WV}) &= \theta(a_{VT}a^{(2)}_{TW}a^{(1)}_{WV}) \\
&= \theta(a_{VT}) + \theta(a^{(2)}_{TW}) + \theta(a^{(1)}_{WV}) \\
&= \frac{5\pi}{6} + \frac{\pi}{3} + \frac{5\pi}{6} \\
&= 2\pi
\end{aligned}$$

All conditions are satisfied, hence a polar quiver.

Example 4.4. In this example, we have the same quiver as in the previous example but with a different set of polar co-ordinates. Just like in example

4.3, we will show that this is a polar quiver.

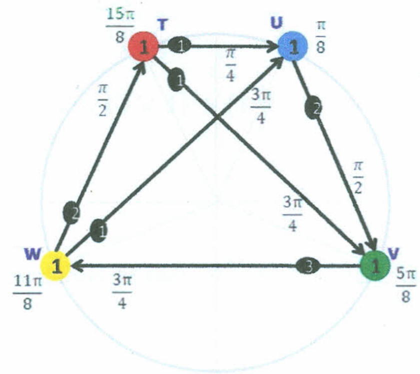


Figure 4.3.3: Diagram of P^2 quiver blown up at one point. Vertices are given names T, U, V and W, while the number in the vertices indicates the number of grouped vertices. Angles are assigned such that the quiver and all its mutants satisfy polar conditions.

We show that this is indeed a polar quiver,

1.

$$N^T(v) := \sum_{a\,|\,t(a)=v} N(s(a)) = \sum_{a\,|\,s(a)=v} N(t(a))$$

$$\begin{aligned} N^T(T) &= 2\times 1 = (1\times 1)+(1\times 1) & N(T)=1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} N^T(U) &= (1\times 1)+(1\times 1) = 2\times 1 & N(U)=1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} N^T(V) &= (2\times 1)+(1\times 1) = 3\times 1 \qquad\qquad N(V)=1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} N^T(W) &= 3\times 1 = (2\times 1)+(1\times 1) \qquad\qquad N(W)=1 \\ &= 3 \end{aligned}$$

From the calculations above, $N^T(v) > N(v)$ for any vertex.

2. For any $v \in Q_0$ totals

$$\begin{aligned} \theta^L(v) &:= \sum_{a\,|\,t(a)=v} \theta(a)N(s(a)), \\ \theta^R(v) &:= \sum_{a\,|\,s(a)=v} \theta(a)N(t(a)), \end{aligned}$$

can be defined such that

$$\theta^L(v)+\theta^R(v) = 2\pi(N^T(v)-N(v)).$$

where

$$N^T(v) > N(v)$$

$$\begin{aligned} \theta^L(T) &= 2\times\frac{\pi}{2}\times 1 \qquad \theta^R(T) = \left(1\times\frac{\pi}{4}\times 1\right)+\left(1\times\frac{3\pi}{4}\times 1\right) \\ &= \pi \qquad\qquad\qquad\qquad\quad = \pi \end{aligned}$$

$$\begin{aligned} \theta^L(U) &= \left(1\times\frac{\pi}{4}\times 1\right)+\left(1\times\frac{3\pi}{4}\times 1\right) \qquad \theta^R(U) = 2\times\frac{\pi}{2}\times 1 \\ &= \pi \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \pi \end{aligned}$$

$$\begin{aligned}\theta^L(V) &= \left(2\times\frac{\pi}{2}\times 1\right)+\left(1\times\frac{3\pi}{4}\times 1\right) \\ &= \frac{7\pi}{4}\end{aligned}\qquad \begin{aligned}\theta^R(V) &= 3\times\frac{3\pi}{4}\times 1 \\ &= \frac{9\pi}{4}\end{aligned}$$

$$\begin{aligned}\theta^L(W) &= 3\times\frac{3\pi}{4}\times 1 \\ &= \frac{9\pi}{4}\end{aligned}\qquad \begin{aligned}\theta^R(W) &= \left(2\times\frac{\pi}{2}\times 1\right)+\left(1\times\frac{3\pi}{4}\times 1\right) \\ &= \frac{7\pi}{4}\end{aligned}$$

and their sums are;

$$\begin{aligned}\theta^L(T)+\theta^R(T) &= \pi+\pi \\ &= 2\pi \\ &= 2\pi(2-1) \\ &= 2\pi(N^T(T)-N(T))\end{aligned}$$

$$\begin{aligned}\theta^L(U)+\theta^R(U) &= \pi+\pi \\ &= 2\pi \\ &= 2\pi(2-1) \\ &= 2\pi(N^T(U)-N(U))\end{aligned}$$

$$
\begin{aligned}
\theta^L(V) + \theta^R(V) &= \frac{7\pi}{4} + \frac{9\pi}{4} \\
&= 4\pi \\
&= 2\pi(3-1) \\
&= 2\pi(N^T(V) - N(V))
\end{aligned}
$$

$$
\begin{aligned}
\theta^L(W) + \theta^R(W) &= \frac{9\pi}{4} + \frac{7\pi}{4} \\
&= 4\pi \\
&= 2\pi(3-1) \\
&= 2\pi(N^T(W) - N(W))
\end{aligned}
$$

3. For any cyclic element $a_1 \dots a_n \in \mathcal{S}$, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi$$

The quiver has a potential given by;

$$
\begin{aligned}
\mathcal{S} = \; & a_{UT}a_{TW}^{(1)}a_{WV}^{(3)}a_{VU}^{(2)} + a_{UT}a_{TW}^{(2)}a_{WV}^{(3)}a_{VU}^{(1)} + a_{UW}a_{WV}^{(1)}a_{VU}^{(1)} + \\
& a_{UW}a_{WV}^{(2)}a_{VU}^{(2)} + a_{VT}a_{TW}^{(1)}a_{WV}^{(2)} + a_{VT}a_{TW}^{(2)}a_{WV}^{(1)}
\end{aligned}
$$

In any cyclic component $a_1 \dots a_n$ of the potential, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi$$

as shown below

$$\begin{aligned}
\theta(a_{UT}a_{TW}^{(1)}a_{WV}^{(3)}a_{VU}^{(2)}) &= \theta(a_{UT}a_{TW}^{(2)}a_{WV}^{(3)}a_{VU}^{(1)}) \\
&= \theta(a_{UT}) + \theta(a_{TW}^{(2)}) + \theta(a_{WV}^{(3)}) + \theta(a_{VU}^{(1)}) \\
&= \frac{\pi}{4} + \frac{\pi}{2} + \frac{3\pi}{4} + \frac{\pi}{2} \\
&= 2\pi
\end{aligned}$$

$$\begin{aligned}
\theta(a_{UW}a_{WV}^{(1)}a_{VU}^{(1)}) &= \theta(a_{UW}a_{WV}^{(2)}a_{VU}^{(2)}) \\
&= \theta(a_{UW}) + \theta(a_{WV}^{(2)}) + \theta(a_{VU}^{(2)}) \\
&= \frac{3\pi}{4} + \frac{3\pi}{4} + \frac{\pi}{2} \\
&= 2\pi
\end{aligned}$$

$$\begin{aligned}
\theta(a_{VT}a_{TW}^{(1)}a_{WV}^{(2)}) &= \theta(a_{VT}a_{TW}^{(2)}a_{WV}^{(1)}) \\
&= \theta(a_{VT}) + \theta(a_{TW}^{(2)}) + \theta(a_{WV}^{(1)}) \\
&= \frac{3\pi}{4} + \frac{\pi}{2} + \frac{3\pi}{4} \\
&= 2\pi
\end{aligned}$$

All conditions are satisfied, hence a polar quiver.

Lemma 4.5. *For any two distinct paths x and y of the quiver with a common starting and target vertices, we have $\theta(x) \cong_{mod\, 2\pi} \theta(y)$.*

Proof. Let $x = a_1 \dots a_n$ be a path of the quiver Q,

$$\begin{aligned}
(\theta(x))_{mod\ 2\pi} &= \sum_{i=1}^{n} \theta(a_i) \\
&= (\theta(t(a_n)) - \theta(s(a_n)))_{mod\ 2\pi} + \cdots + (\theta(t(a_1)) - \theta(s(a_1)))_{mod\ 2\pi} \\
&\cong_{mod\ 2\pi} ((\theta(t(a_n)) - \theta(s(a_n))) + \cdots + (\theta(t(a_1)) - \theta(s(a_1))))_{mod\ 2\pi} \\
&\cong_{mod\ 2\pi} (\theta(t(a_1)) - \theta(s(a_n)))_{mod\ 2\pi} \\
&\cong_{mod\ 2\pi} (\theta(t(x)) - \theta(s(x)))_{mod\ 2\pi}
\end{aligned}$$

Let y be any another path with $s(x) = s(y)$ and $t(x) = t(y)$, then

$$\begin{aligned}
\theta(x) &\cong_{mod\ 2\pi} (\theta(t(x)) - \theta(s(x)))_{mod\ 2\pi} \\
&\cong_{mod\ 2\pi} (\theta(t(y)) - \theta(s(y)))_{mod\ 2\pi} \\
&\cong_{mod\ 2\pi} \theta(y)
\end{aligned}$$

□

This lemma will become particularly important in the proof of the major theorem in Chapter 6.

Chapter 5

Mutation of Polar quivers

5.1 Introduction

This chapter introduces mutation of Polar quivers. Mutation of quivers with potential was covered in Chapter 2, chapter 4 introduced polar quivers, with some interesting examples. Section 5.2 of this chapter gives a continuation with mutation of polar quivers. It is in this section that we state the process of mutation for polar quivers. Worked examples that illustrate the process of mutation of polar quivers are given in section 5.3.

5.2 Mutation of polar quivers

Definition 5.1. Let $(Q, \mathcal{S}, N, \theta)$ be a polar quiver, at any vertex $v \in Q_0$ define the **mutation** of $(Q, \mathcal{S}, N, \theta)$ at v to be $(\tilde{Q}, \tilde{\mathcal{S}}, \tilde{N}, \tilde{\theta})$ where;

- $\tilde{Q}$ and $\tilde{\mathcal{S}}$ are obtained by mutation of $(Q, \mathcal{S})$ at v.
- For a vertex $u \neq v$, $\tilde{N}(u) = N(u)$ and $\tilde{\theta}(u) = \theta(u)$.

- $$\tilde{N}(v) = N^T(v) - N(v) \geq 1 \tag{5.2.1}$$

- The position of v after mutation is given by,

$$\tilde{\theta}(v) = \left(\theta(v) - \frac{\theta^L(v)}{\tilde{N}(v)}\right)_{\text{mod } 2\pi} = \left(\theta(v) + \frac{\theta^R(v)}{\tilde{N}(v)}\right)_{\text{mod } 2\pi} . \tag{5.2.2}$$

Mutation of the quiver at any vertex say v alters the position of the vertex, and hence the orientation of vertices and arrows in the quiver. The angles $\frac{\theta^L(v)}{\tilde{N}(v)}$ and $\frac{\theta^R(v)}{\tilde{N}(v)}$ determines the new position of v. If the sum of $\frac{\theta^L(v)}{\tilde{N}(v)}$ and $\frac{\theta^R(v)}{\tilde{N}(v)}$ is 2π, we have $\tilde{\theta}(v) = \left(\theta(v) - \frac{\theta^L(v)}{\tilde{N}(v)}\right)_{\text{mod } 2\pi} = \left(\theta(v) + \frac{\theta^R(v)}{\tilde{N}(v)}\right)_{\text{mod } 2\pi}$, i.e. the new position for v is unique. If the sum is not 2π, mutation at v will give rise to two different positions for v, hence the second condition ensures that the new position for the mutated vertex is unique.

We can visualize the third condition as implying that elements of the potential define a convex polygon. The polar quiver has a cyclic order to the vertices imposed by the angles. The third condition ensures that this ordering is consistent with each element of the potential. From the examples below, we will observe that mutation preserves this condition although there are some cases in which it fails. In chapter 6, we will state and prove the conditions that if a polar quiver satisfies, this problem will be resolved.

5.3 Examples on mutation of polar quivers

In this section, we illustrate the process of mutating polar quivers by examples. The polar co-ordinates for the mutated vertex are changed. In Example 5.2, the mutated vertex W has the same polar co-ordinate as U which was unaffected by mutation. Example 5.3 highlights another effect of mutation on polar quivers. Although we started with a polar quiver, the resulting quiver after mutation breaks some of the polar conditions. There are cases where everything goes right, as in eExample 5.4, where mutation of a polar quiver gave a polar quiver.

Example 5.2. In this example, we mutate the polar quiver in Example 4.2 shown below,

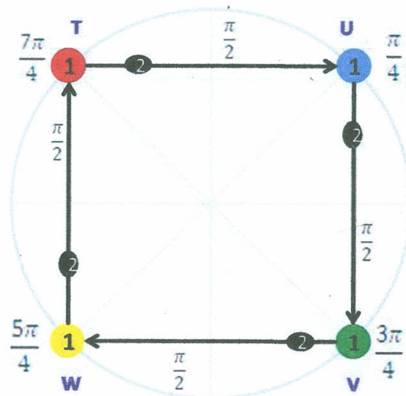


Figure 5.3.1: Diagram of $P^1 \times P^1$. Vertices are given names T, U, V and W, while the number in the vertices indicates the number of grouped vertices.

We mutate the quiver at W (the yellow vertex). By the definition of mutation for a polar quivers, any other vertices not same as W will keep their position after mutation. The new position for W is given by;

$$\tilde{\theta}(W) = \left(\theta(W) - \frac{\theta^L(W)}{\tilde{N}(W)}\right)_{\text{mod } 2\pi} = \left(\theta(W) + \frac{\theta^R(W)}{\tilde{N}(W)}\right)_{\text{mod } 2\pi}$$

$$\begin{aligned}\tilde{\theta}(W) &= \left(\theta(W) - \frac{\theta^L(W)}{\tilde{N}(W)}\right)_{\text{mod } 2\pi} \\ &= \left(\frac{5\pi}{4} - \frac{\pi}{1}\right)_{\text{mod } 2\pi} \\ &= \frac{\pi}{4}\end{aligned}$$

This is the same position occupied by U the blue vertex.

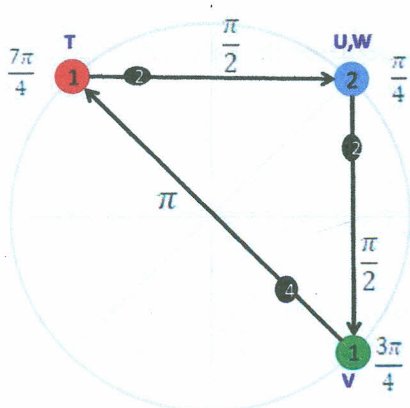


Figure 5.3.2: $P^1 \times P^1$ quiver mutated at W. The number 2 at the blue vertex implies that two vertices U and W are sharing the same position.

The next task is to check if this quiver satisfies all polar conditions.

1.

$$N^T(v) := \sum_{a \mid t(a)=v} N(s(a)) = \sum_{a \mid s(a)=v} N(t(a))$$

$$\begin{aligned} N^T(T) &= 4\times 1 = 2\times 2 \\ &= 4 \end{aligned} \qquad N(T) = 1$$

$$\begin{aligned} N^T(U,W) &= 2\times 1 = 2\times 1 \\ &= 2 \end{aligned} \qquad N(U,W) = 1$$

$$\begin{aligned} N^T(V) &= 2\times 2 = 4\times 1 \\ &= 4 \end{aligned} \qquad N(V) = 1$$

It is clear that $N^T(v) > N(v)$ for all vertices.

2. For any $v \in Q_0$ totals

$$\begin{aligned} \theta^L(v) &:= \sum_{a \mid t(a)=v} \theta(a)N(s(a)), \\ \theta^R(v) &:= \sum_{a \mid s(a)=v} \theta(a)N(t(a)), \end{aligned}$$

can be defined such that

$$\theta^L(v) + \theta^R(v) = 2\pi(N^T(v) - N(v)).$$

where

$$N^T(v) > N(v)$$

$$\theta^L(T) = 4 \times \pi \times 1 = 4\pi \qquad \theta^R(T) = 2 \times \frac{\pi}{2} \times 2 = 2\pi$$

$$\theta^L(U,W) = 2 \times \frac{\pi}{2} \times 1 = \pi \qquad \theta^R(U,W) = 2 \times \frac{\pi}{2} \times 1 = \pi$$

$$\theta^L(V) = 2 \times \frac{\pi}{2} \times 2 = 2\pi \qquad \theta^R(V) = 4 \times \pi \times 1 = 4\pi$$

and their sums are;

$$\begin{aligned}\theta^L(T) + \theta^R(T) &= 4\pi + 2\pi \\ &= 6\pi \\ &= 2\pi(4-1) \\ &= 2\pi(N^T(T) - N(T))\end{aligned}$$

$$\begin{aligned}\theta^L(U,W) + \theta^R(U,W) &= \pi + \pi \\ &= 2\pi \\ &= 2\pi(2-1) \\ &= 2\pi(N^T(U,W) - N(U,W))\end{aligned}$$

$$\begin{aligned}\theta^L(V) + \theta^R(V) &= 2\pi + 4\pi \\ &= 6\pi \\ &= 2\pi(4-1) \\ &= 2\pi(N^T(V) - N(V))\end{aligned}$$

3. For any cyclic element $a_1 \ldots a_n \in \mathcal{S}$, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi$$

The quiver has a potential given by

$$\begin{aligned}\tilde{\mathcal{S}} &= a^{(1)}_{WT}a^{(1)}_{TV}a^{(1)}_{VW} + a^{(1)}_{WT}a^{(3)}_{TV}a^{(2)}_{VW} + a^{(2)}_{WT}a^{(2)}_{TV}a^{(1)}_{VW} + a^{(2)}_{WT}a^{(4)}_{TV}a^{(2)}_{VU} + \\ &\quad a^{(1)}_{UT}a^{(1)}_{TV}a^{(1)}_{VU} + a^{(1)}_{UT}a^{(3)}_{TV}a^{(2)}_{VU} + a^{(2)}_{UT}a^{(2)}_{TV}a^{(1)}_{VU} + a^{(2)}_{UT}a^{(4)}_{TV}a^{(2)}_{VU}\end{aligned}$$

In any cyclic component $a_1 \ldots a_n$ of the potential, $n > 2$ and

$$\sum_{i=1}^{3} \theta(a_i) = 2\pi$$

as shown below

$$\begin{aligned}\theta(a^{(1,2)}_{WT}a^{(1,2,3,4)}_{TV}a^{(1,2)}_{VW}) &= \theta(a^{(1,2)}_{UT}a^{(1,2,3,4)}_{TV}a^{(1,2)}_{VU}) \\ &= \theta(a^{(1,2)}_{UT}) + \theta(a^{(1,2,3,4)}_{TV}) + \theta(a^{(1,2)}_{VU}) \\ &= \frac{\pi}{2} + \pi + \frac{\pi}{2} \\ &= 2\pi\end{aligned}$$

Where $a_{WT}^{(1,2)}$ implies $a_{WT}^{(1)}$ or $a_{WT}^{(2)}$ and so on.

All Properties are satisfied, hence it is a polar quiver.

In the next two examples, we have the same quiver with potential but with different polar co-ordinates. We mutate these quivers at the same vertex to illustrate the process of mutation, and to check if the resulting quivers satisfy the polar conditions.

Example 5.3. In this example, we have a polar quiver in Example 4.3 shown below.

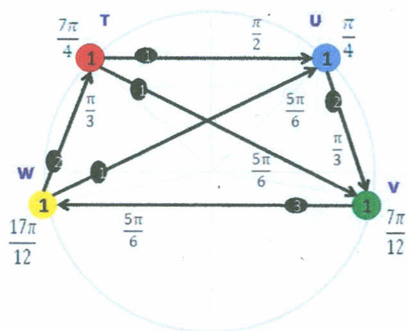


Figure 5.3.3: Diagram of P^2 quiver blown up at one point. It is a polar quiver, but not any of its mutants.

We mutate this quiver at W. In the mutated quiver, any vertex not W will have the same position as in the old quiver. W will have a new position in the mutated quiver given by,

$$\tilde{\theta}(W) = \left(\theta(W) - \frac{\theta^L(W)}{\tilde{N}(W)}\right)_{\text{mod } 2\pi} = \left(\theta(W) + \frac{\theta^R(W)}{\tilde{N}(W)}\right)_{\text{mod } 2\pi}$$

$$
\begin{aligned}
\tilde{\theta}(W) &= \left(\theta(W) - \frac{\theta^L(W)}{\tilde{N}(W)}\right)_{\bmod 2\pi} \\
&= \left(\frac{17\pi}{12} - \frac{\frac{5\pi}{2}}{2}\right)_{\bmod 2\pi} \\
&= \frac{17\pi}{12} - \frac{5\pi}{4} \\
&= \frac{\pi}{6}
\end{aligned}
$$

The diagram below best illustrates this.

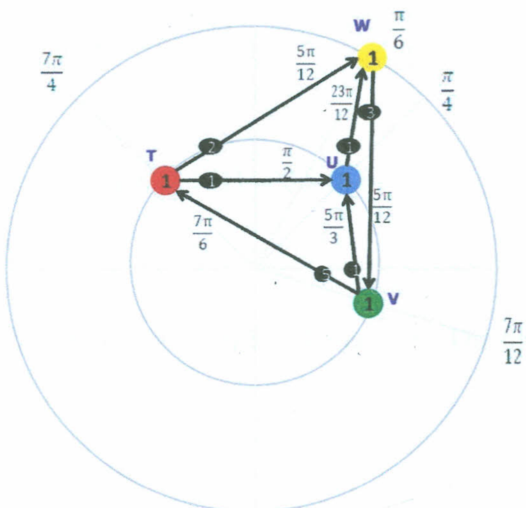


Figure 5.3.4: Diagram of a mutation of P^2 quiver blown up at one point. This quiver does not satisfy the second and third polar conditions.

We check if this quiver satisfies all polar conditions.

1.

$$
N^T(v) := \sum_{a \,|\, t(a)=v} N(s(a)) = \sum_{a \,|\, s(a)=v} N(t(a))
$$

$$\begin{aligned} N^T(T) &= 5\times 1 = (2\times 2)+(1\times 1) \\ &= 5 \end{aligned} \qquad N(T)=1$$

$$\begin{aligned} N^T(U) &= (1\times 1)+(1\times 1) = 1\times 2 \\ &= 2 \end{aligned} \qquad N(U)=1$$

$$\begin{aligned} N^T(V) &= (3\times 2) = (5\times 1)+(1\times 1) \\ &= 6 \end{aligned} \qquad N(V)=1$$

$$\begin{aligned} N^T(W) &= (2\times 1)+(1\times 1) = 3\times 1 \\ &= 3 \end{aligned} \qquad N(W)=2$$

It is clear that $N^T(v) > N(v)$ for all vertices. The first condition is satisfied.

2. For any $v \in Q_0$ totals

$$\begin{aligned} \theta^L(v) &:= \sum_{a\,|\,t(a)=v} \theta(a)N(s(a)), \\ \theta^R(v) &:= \sum_{a\,|\,s(a)=v} \theta(a)N(t(a)), \end{aligned}$$

can be defined such that

$$\theta^L(v)+\theta^R(v) = 2\pi(N^T(v)-N(v)).$$

where

$$N^T(v) > N(v)$$

$$\begin{aligned} \theta^L(T) &= 5 \times \frac{7\pi}{6} \times 1 \\ &= \frac{35\pi}{6} \end{aligned} \qquad \begin{aligned} \theta^R(T) &= \left(1 \times \frac{\pi}{2} \times 1\right) + \left(2 \times \frac{5\pi}{12} \times 2\right) \\ &= \frac{13\pi}{6} \end{aligned}$$

$$\begin{aligned} \theta^L(U) &= \left(1 \times \frac{\pi}{2} \times 1\right) + \left(1 \times \frac{5\pi}{3} \times 1\right) \\ &= \frac{13\pi}{6} \end{aligned} \qquad \begin{aligned} \theta^R(U) &= 1 \times \frac{23\pi}{12} \times 2 \\ &= \frac{23\pi}{6} \end{aligned}$$

$$\begin{aligned} \theta^L(V) &= 3 \times \frac{5\pi}{12} \times 2 \\ &= \frac{5\pi}{2} \end{aligned} \qquad \begin{aligned} \theta^R(V) &= \left(5 \times \frac{7\pi}{6} \times 1\right) + \left(1 \times \frac{5\pi}{3} \times 1\right) \\ &= \frac{15\pi}{2} \end{aligned}$$

$$\begin{aligned} \theta^L(W) &= \left(2 \times \frac{5\pi}{12} \times 1\right) + \left(1 \times \frac{23\pi}{12} \times 1\right) \\ &= \frac{11\pi}{4} \end{aligned} \qquad \begin{aligned} \theta^R(W) &= 3 \times \frac{5\pi}{12} \times 1 \\ &= \frac{5\pi}{4} \end{aligned}$$

and their sums are;

$$\begin{aligned} \theta^L(T) + \theta^R(T) &= \frac{35\pi}{6} + \frac{13\pi}{6} \\ &= 8\pi \\ &= 2\pi(5 - 1) \\ &= 2\pi(N^T(T) - N(T)) \end{aligned}$$

$$\begin{aligned}
\theta^L(U) + \theta^R(U) &= \frac{13\pi}{6} + \frac{23\pi}{6} \\
&= 6\pi \\
&\neq 2\pi(2-1) \\
&= 2\pi(N^T(U) - N(U))
\end{aligned}$$

$$\begin{aligned}
\theta^L(V) + \theta^R(V) &= \frac{5\pi}{2} + \frac{15\pi}{2} \\
&= 10\pi \\
&= 2\pi(6-1) \\
&= 2\pi(N^T(V) - N(V))
\end{aligned}$$

$$\begin{aligned}
\theta^L(W) + \theta^R(W) &= \frac{11\pi}{4} + \frac{5\pi}{4} \\
&= 4\pi \\
&\neq 2\pi(3-2) \\
&= 2\pi(N^T(W) - N(W))
\end{aligned}$$

This condition is broken at U and W.

3. For any cyclic element $a_1 \dots a_n \in \mathcal{S}$, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi$$

The quiver has a potential given by;

$$
\begin{aligned}
\tilde{S} \;=\; & a_{UT}a_{TV}^{(3)}a_{VW}^{(2)}a_{WU} + a_{UT}a_{TV}^{(5)}a_{VW}^{(1)}a_{WU} + a_{TV}^{(1)}a_{VW}^{(1)}a_{WT}^{(1)} + \\
& a_{TV}^{(2)}a_{VW}^{(2)}a_{WT}^{(1)} + a_{TV}^{(3)}a_{VW}^{(3)}a_{WT}^{(1)} + a_{TV}^{(2)}a_{VW}^{(1)}a_{WT}^{(2)} + \\
& a_{TV}^{(4)}a_{VW}^{(2)}a_{WT}^{(2)} + a_{TV}^{(5)}a_{VW}^{(3)}a_{WT}^{(2)} + a_{UV}a_{VW}^{(3)}a_{WU}
\end{aligned}
$$

In any cyclic component $a_1 \ldots a_n$ of the potential, $n > 2$. The arrow a_{WU} is not compatible with the polar ordering for the quiver. Any cyclic component $a_1 \ldots a_n$ of the potential with this arrow in the composition will have

$$\sum_i \theta(a_i) = 4\pi$$

as shown below

$$
\begin{aligned}
\theta(a_{UT}a_{TV}^{(3)}a_{VW}^{(2)}a_{WU}) &= \theta(a_{UT}a_{TV}^{(5)}a_{VW}^{(1)}a_{WU}) \\
&= \theta(a_{UT}) + \theta(a_{TV}^{(5)}) + \theta(a_{VW}^{(1)}) + \theta(a_{WU}) \\
&= \frac{\pi}{2} + \frac{7\pi}{6} + \frac{5\pi}{12} + \frac{23\pi}{12} \\
&= 4\pi
\end{aligned}
$$

$$
\begin{aligned}
\theta(a_{TV}^{(1,2,3,4,5)}a_{VW}^{(1,2,3)}a_{WT}^{(1,2)}) &= \theta(a_{TV}^{(1,2,3,4,5)}) + \theta(a_{VW}^{(1,2,3)}) + \theta(a_{WT}^{(1,2)}) \\
&= \frac{7\pi}{6} + \frac{5\pi}{12} + \frac{5\pi}{12} \\
&= 2\pi
\end{aligned}
$$

$$
\begin{aligned}
\theta(a_{UV}a^{(3)}_{VW}a_{WU}) &= \theta(a_{UV}) + \theta(a^{(3)}_{VW}) + \theta(a_{WU}) \\
&= \frac{5\pi}{3} + \frac{5\pi}{12} + \frac{23\pi}{12} \\
&= 4\pi
\end{aligned}
$$

The second and third polar condition are broken, hence this is not a polar quiver.

Example 5.4. In this example, we have the same polar quiver as in Example 4.4 shown below;

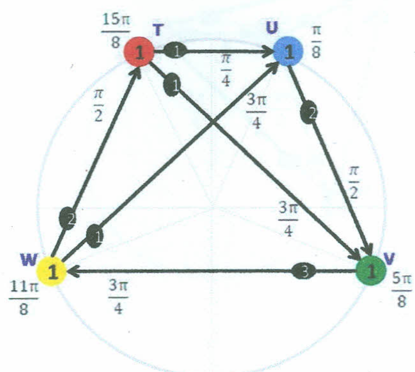


Figure 5.3.5: P^2 quiver blown up at one point. This is a polar quiver, so are its mutants.

We mutate this quiver at W. In the new quiver, W will have a new position given by;

$$\begin{aligned}\tilde{\theta}(W) &= \left(\theta(W) - \frac{\theta^L(W)}{\tilde{N}(W)}\right)_{\text{mod } 2\pi} \\ &= \left(\frac{11\pi}{8} - \frac{\frac{9\pi}{4}}{2}\right)_{\text{mod } 2\pi} \\ &= \frac{11\pi}{8} - \frac{9\pi}{8} \\ &= \frac{\pi}{4}\end{aligned}$$

The position of other vertices is not changed in the mutated quiver.

1.

$$N^T(v) := \sum_{a\,|\,t(a)=v} N(s(a)) = \sum_{a\,|\,s(a)=v} N(t(a))$$

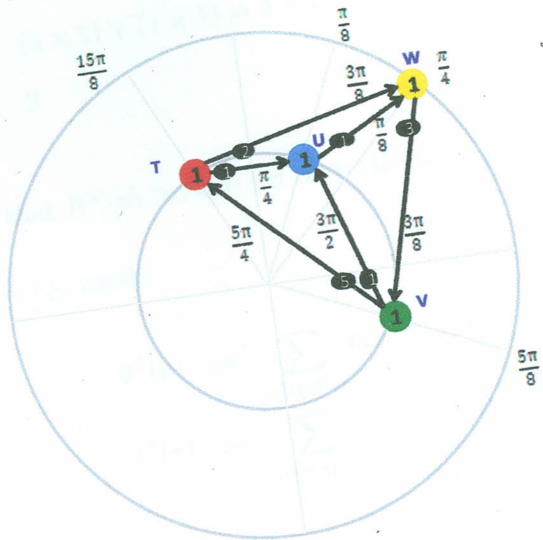


Figure 5.3.6: Diagram of a mutation of P^2 quiver blown up at one point.

This quiver satisfies all polar conditions.

$$N(T) = 1$$

$$\begin{aligned} N^T(T) &= 5 \times 1 = (2 \times 2) + (1 \times 1) \\ &= 5 \end{aligned}$$

$$N(U) = 1$$

$$\begin{aligned} N^T(U) &= (1 \times 1) + (1 \times 1) = 1 \times 2 \\ &= 2 \end{aligned}$$

$$N(V) = 1$$

$$\begin{aligned} N^T(V) &= (3 \times 2) = (5 \times 1) + (1 \times 1) \\ &= 6 \end{aligned}$$

$$N^T(W) = (2 \times 1) + (1 \times 1) = 3 \times 1 = 3 \qquad N(W) = 2$$

It is clear that $N^T(v) > N(v)$ for all vertices.

2. For any $v \in Q_0$ totals

$$\theta^L(v) := \sum_{a \mid t(a)=v} \theta(a) N(s(a)),$$
$$\theta^R(v) := \sum_{a \mid s(a)=v} \theta(a) N(t(a)),$$

can be defined such that

$$\theta^L(v) + \theta^R(v) = 2\pi(N^T(v) - N(v)).$$

where

$$N^T(v) > N(v)$$

$$\theta^L(T) = 5 \times \frac{5\pi}{4} \times 1 = \frac{25\pi}{4} \qquad \theta^R(T) = \left(2 \times \frac{3\pi}{8} \times 2\right) + \left(1 \times \frac{\pi}{4} \times 1\right) = \frac{7\pi}{4}$$

$$\theta^L(U) = \left(1 \times \frac{\pi}{4} \times 1\right) + \left(1 \times \frac{3\pi}{2} \times 1\right) = \frac{7\pi}{4} \qquad \theta^R(U) = 1 \times \frac{\pi}{8} \times 2 = \frac{\pi}{4}$$

$$\theta^L(V) = 3 \times \frac{3\pi}{8} \times 2 = \frac{9\pi}{4} \qquad \theta^R(V) = \left(5 \times \frac{5\pi}{4} \times 1\right) + \left(1 \times \frac{3\pi}{2} \times 1\right) = \frac{31\pi}{4}$$

$$\theta^L(W) = \left(2 \times \frac{3\pi}{8} \times 1\right) + \left(1 \times \frac{\pi}{8} \times 1\right) = \frac{7\pi}{8} \qquad \theta^R(W) = 3 \times \frac{3\pi}{8} \times 1 = \frac{9\pi}{8}$$

and their sums are;

$$\begin{aligned}\theta^L(T) + \theta^R(T) &= \frac{25\pi}{4} + \frac{7\pi}{4} \\ &= 8\pi \\ &= 2\pi(5-1) \\ &= 2\pi(N^T(T) - N(T))\end{aligned}$$

$$\begin{aligned}\theta^L(U) + \theta^R(U) &= \frac{7\pi}{4} + \frac{\pi}{4} \\ &= 2\pi \\ &= 2\pi(2-1) \\ &= 2\pi(N^T(U) - N(U))\end{aligned}$$

$$\begin{aligned}\theta^L(V) + \theta^R(V) &= \frac{9\pi}{4} + \frac{31\pi}{4} \\ &= 10\pi \\ &= 2\pi(6-1) \\ &= 2\pi(N^T(V) - N(V))\end{aligned}$$

$$
\begin{aligned}
\theta^L(W) + \theta^R(W) &= \frac{7\pi}{8} + \frac{9\pi}{8} \\
&= 2\pi \\
&= 2\pi(3-2) \\
&= 2\pi(N^T(W) - N(W))
\end{aligned}
$$

The second polar condition is fulfilled.

3. For any cyclic element $a_1 \dots a_n \in \mathcal{S}$, $n > 2$ and

$$\sum_i \theta(a_i) = 2\pi$$

The quiver has a potential given by;

$$
\begin{aligned}
\tilde{\mathcal{S}} = \; & a_{UT}a^{(3)}_{TV}a^{(2)}_{VW}a_{WU} + a_{UT}a^{(5)}_{TV}a^{(1)}_{VW}a_{WU} + a^{(1)}_{TV}a^{(1)}_{VW}a^{(1)}_{WT} + \\
& a^{(2)}_{TV}a^{(2)}_{VW}a^{(1)}_{WT} + a^{(3)}_{TV}a^{(3)}_{VW}a^{(1)}_{WT} + a^{(2)}_{TV}a^{(1)}_{VW}a^{(2)}_{WT} + \\
& a^{(4)}_{TV}a^{(2)}_{VW}a^{(2)}_{WT} + a^{(5)}_{TV}a^{(3)}_{VW}a^{(2)}_{WT} + a_{UV}a^{(3)}_{VW}a_{WU}
\end{aligned}
$$

Any cyclic element of the potential is of length > 2 and the sum of angles is calculated below;

$$
\begin{aligned}
\theta(a_{UT}a^{(3)}_{TV}a^{(2)}_{VW}a_{WU}) &= \theta(a_{UT}a^{(5)}_{TV}a^{(1)}_{VW}a_{WU}) \\
&= \theta(a_{UT}) + \theta(a^{(5)}_{TV}) + \theta(a^{(1)}_{VW}) + \theta(a_{WU}) \\
&= \frac{\pi}{4} + \frac{5\pi}{4} + \frac{3\pi}{8} + \frac{\pi}{8} \\
&= 2\pi
\end{aligned}
$$

$$
\begin{aligned}
\theta(a_{TV}^{(1,2,3,4,5)} a_{VW}^{(1,2,3)} a_{WT}^{(1,2)}) &= \theta(a_{TV}^{(1,2,3,4,5)}) + \theta(a_{VW}^{(1,2,3)}) + \theta(a_{WT}^{(1,2)}) \\
&= \frac{5\pi}{4} + \frac{3\pi}{8} + \frac{3\pi}{8} \\
&= 2\pi
\end{aligned}
$$

$$
\begin{aligned}
\theta(a_{UV} a_{VW}^{(3)} a_{WU}) &= \theta(a_{UV}) + \theta(a_{VW}^{(3)}) + \theta(a_{WU}) \\
&= \frac{3\pi}{2} + \frac{3\pi}{8} + \frac{\pi}{8} \\
&= 2\pi
\end{aligned}
$$

This quiver meets the third polar condition, which implies that all arrows are compatible with the polar ordering and hence consistent with the potential.

From the last two examples, we mutated a quiver with potential but having two different sets of polar co-ordinates. In Example 5.3 the mutated quiver failed to satisfy the second and third polar conditions, while in Example 5.4, the mutated quiver satisfied all the polar conditions. The polar quiver in Example 5.4 must be having some special properties that enables its mutants to meet all polar conditions. These are the conditions that will be investigated in the next chapter.

Chapter 6

Main theorem on mutation of polar quivers

6.1 Introduction

From the previous chapter, we had two cases, Examples 5.3 and 5.4, of a quiver with potential but with two different R-charges. On mutation, the quiver in Example 5.3 fails to satisfy the second and third polar conditions, while the quiver in Example 5.4 does. In this chapter we state and prove the preconditions a polar quiver must satisfy for its mutants to satisfy all polar conditions. In the last section, we highlight the importance of the theorem by revisiting these two examples.

6.2 Main theorem

Given a polar quiver, its mutation can either give a polar quiver or a quiver that breaks some of the polar conditions. There are certain conditions that if a polar quiver satisfies, the resulting quiver after mutation will be a polar quiver. These conditions are stated in the following theorem.

Theorem 6.1. *Let $(Q, \mathcal{S}, N, \theta)$ be a polar quiver. The mutated quadruple $(\tilde{Q}, \tilde{\mathcal{S}}, \tilde{N}, \tilde{\theta})$ is a polar quiver if and only if for any vertex $v \in Q$, $\frac{\theta^L(v)}{\tilde{N}(v)} > \theta(a)$ for all $a \,|\, t(a) = v$ and $\frac{\theta^R(v)}{\tilde{N}(v)} > \theta(b)$ for all $b \,|\, s(b) = v$.*

Proof. We prove the theorem for each of the three polar conditions.

Let $(Q, \mathcal{S}, N, \theta)$ be a polar quiver, we consider four different cases for the proof of the theorem for the first polar condition;

Case 1. Consider v, the mutated vertex. By definition,

$$N^T(v) := \sum_{a \,|\, t(a)=v} N(s(a)) = \sum_{a \,|\, s(a)=v} N(t(a))$$

mutation of Q gives the quiver $\tilde{Q}$ with arrows reversed at v.

$$\begin{aligned} \tilde{N}^T(v)_L &:= \sum_{b \in \tilde{Q} \,|\, t(b)=v} N(s(b)) \\ &= \sum_{a \in Q \,|\, s(a)=v} N(t(a)) \\ &= N^T(v) \end{aligned}$$

$$\begin{aligned} \tilde{N}^T(v)_R &:= \sum_{b \in \tilde{Q} \,|\, s(b)=v} N(t(b)) \\ &= \sum_{a \in Q \,|\, t(a)=v} N(s(a)) \\ &= N^T(v) \end{aligned}$$

At the mutated vertex v we have,

$$\tilde{N}^T(v) = \tilde{N}^T(v)_L = \tilde{N}^T(v)_R = N^T(v)$$

Case 2. Consider a vertex $u \in Q$ with no arrows joining it with v,

$$\begin{aligned} \tilde{N}^T(u)_L &:= \sum_{b \in \tilde{Q} \,|\, t(b)=u} N(s(b)) \\ &= \sum_{a \in Q \,|\, t(a)=u} N(s(a)) \\ &= N^T(u) \end{aligned}$$

$$\begin{aligned} \tilde{N}^T(u)_R &:= \sum_{b \in \tilde{Q} \,|\, s(b)=u} N(t(b)) \\ &= \sum_{a \in Q \,|\, s(a)=u} N(t(a)) \\ &= N^T(u) \end{aligned}$$

which illustrates equality as well as no change to $N^T(u)$, i.e. for a vertex u with no arrows joining it to v, we have after mutation of the quiver at v

$$\tilde{N}^T(u) = \tilde{N}^T(u)_L = \tilde{N}^T(u)_R = N^T(u) \tag{6.2.1}$$

Case 3. Consider a vertex $u \in Q$ with arrows $a \in Q_1$ such that $a|s(a) = u,\ t(a) = v$. We rewrite $N^T(u)$ to take to consideration this case. Thus

$$N^T(u) = \sum_{a \in Q \,|\, t(a)=u} N(s(a)) = \sum_{a \in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} N(v) + \sum_{a \in Q \left| \substack{s(a)=u \\ t(a)\neq v} \right.} N(t(a)) \tag{6.2.2}$$

Arrows incident to v from u are reversed in the unreduced mutated quiver Q', while any paths $a_2a_1 \in Q$ with $s(a_1) = u$, $t(a_1) = v = s(a_2)$ give arrows $[a_2a_1] \in Q'$. Define a non-empty subset $W \subset Q'_0$ as the set of target vertices for the arrows $[a_2a_1]$. Elements of this set are vertices w_i, $i \in \mathbb{N}$, hence $t(a_2) = w_i$ for some i.

$$\begin{aligned}
N'^T(u)_L &:= \sum_{b \in Q' \,|\, t(b)=u} \tilde{N}(s(b)) \\
&= \sum_{b \in Q' \left| \substack{t(b)=u \\ s(b)=v} \right.} \tilde{N}(v) + \sum_{b \in Q' \left| \substack{t(b)=u \\ s(b)\notin W} \right.} N(s(b)) + \sum_{b \in Q' \left| \substack{t(b)=u \\ s(b)\in W} \right.} N(s(b)) \\
&= \sum_{a \in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \tilde{N}(v) + \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)\notin W} \right.} N(s(a)) + \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)\in W} \right.} N(s(a)) \\
&= \sum_{a \in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \tilde{N}(v) + \sum_{a \in Q | t(a)=u} N(s(a)) \\
&= \sum_{a \in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \tilde{N}(v) + N^T(u)
\end{aligned} \tag{6.2.3}$$

$$
\begin{aligned}
N'^T(u)_R &:= \sum_{b\in Q' \,|\, s(b)=u} \tilde{N}(t(b)) \\
&= \sum_{b\in Q' \left| \substack{s(b)=u \\ t(b)\neq v} \right.} N(t(b)) + \sum_{[a_2a_1]\in Q' \left| \substack{s(a_1)=u \\ t(a_1)=v=s(a_2)} \right.} N(t(a_2)) \\
&= \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)\neq v} \right.} N(t(a)) + \sum_{a_1\in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \sum_{a_2 | s(a_2)=v} N(t(a_2)) \\
&= N^T(u) - \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} N(v) + \sum_{a_1\in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \sum_{a_2 | s(a_2)=v} N(t(a_2)) \text{.......... by (6.2.2)} \\
&= N^T(u) - \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} N(v) + \sum_{a_1\in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} N^T(v) \\
&= N^T(u) + \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} (N^T(v) - N(v)) \\
&= N^T(u) + \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \tilde{N}(v) \qquad (6.2.4)
\end{aligned}
$$

From (6.2.3) and (6.2.4), $N'^T(u)_L = N'^T(u)_R$ for the unreduced quiver Q'. We show that equality holds after reduction of Q' to a reduced quiver $\tilde{Q}$.

$$
\begin{aligned}
N'^T(u)_L &:= \sum_{b\in Q' \,|\, t(b)=u} \tilde{N}(s(b)) \\
&= \sum_{b\in Q' \left| \substack{t(b)=u \\ s(b)=v} \right.} \tilde{N}(v) + \sum_{b\in Q' \left| \substack{t(b)=u \\ s(b)\notin W} \right.} N(s(b)) + \sum_{b\in Q' \left| \substack{t(b)=u \\ s(b)\in W} \right.} N(s(b)) \qquad (6.2.5)
\end{aligned}
$$

$$N'^T(u)_R := \sum_{b\in Q' \,|\, s(b)=u} \tilde{N}(t(b))$$

$$= \sum_{b\in Q' \left| \substack{s(b)=u \\ t(b)\neq v} \right.} N(t(b)) + \sum_{[a_2a_1]\in Q' \left| \substack{s(a_1)=u \\ t(a_1)=v=s(a_2)} \right.} N(t(a_2)) \tag{6.2.6}$$

From the last summations in (6.2.5) and (6.2.6), $s(b) \in W$ and $t(a_2) \in W$. If $s(b) = w_i$ and $t(a_2) = w_i$ for some $w_i \in W$, then we have a 2-cycle in Q'. Reduction of Q' eliminates these trivial elements to give a quiver $\tilde{Q}$. Let $w_i \in W$, we define χ_{w_i} as

$$\chi_{w_i} = \sum_{[a_2a_1]\in Q' \left| \substack{s(a_1)=u \\ t(a_1)=v=s(a_2)} \right.} N(w_i) - \sum_{b\in Q' \left| \substack{t(b)=u \\ s(b)=w_i} \right.} N(w_i) \tag{6.2.7}$$

The elements $\chi_{w_j} < 0$ consists of arrows b while $\chi_{w_i} > 0$ consists of the composite arrows $[a_2a_1]$ that were left after reduction of the quiver. $\chi_{w_i} = 0$ implies that all new arrows cancel out with old ones. We now state $\tilde{N}^T(u)_L$ and $\tilde{N}^T(u)_R$ for the reduced quiver as;

$$\tilde{N}^T(u)_L = \sum_{b\in \tilde{Q} \left| \substack{t(b)=u \\ s(b)=v} \right.} \tilde{N}(v) + \sum_{b\in \tilde{Q} \left| \substack{t(b)=u \\ s(b)\notin W} \right.} N(s(b)) - \chi_{w_j} (for\ \chi_{w_j} \leq 0)$$

(6.2.8)

$$\tilde{N}^T(u)_R = \sum_{b\in \tilde{Q} \left| \substack{s(b)=u \\ t(b)\neq v} \right.} N(t(b)) + \chi_{w_i} (for\ \chi_{w_i} \geq 0)$$

where $i \neq j$. $N'^T(u)_L$ and $N'^T(u)_R$ are reduced by an equal amount, maintaining equality, that is, $\tilde{N}^T(u) = \tilde{N}^T(u)_L = \tilde{N}^T(u)_R$.

Case 4. Now consider a vertex u in Q with arrows $a|t(a) = u,\ s(a) = v$, the calculation for $\tilde{N}^T(u)$ is similar to the calculation above only that the direction of arrows joining u and v is reversed. Arrows target to u from v are reversed in the unreduced mutated quiver Q', while any path $a_1a_2 \in Q$ with $t(a_1) = u,\ s(a_1) = v = t(a_2)$ gives an arrow $[a_1a_2] \in Q'$. Define a non-empty subset $W \subset Q'_0$ as the set of vertices which are a source of arrows $[a_1a_2]$. Elements of this set are $w_k,\ k \in \mathbb{N}$, hence $s(a_2) = w_k$ for some k.

$$\begin{aligned} N'^T(u)_L &= \sum_{b\in Q' \mid t(b)=u} \tilde{N}(s(b)) \\ &= \sum_{b\in Q' \left| \substack{t(b)=u \\ s(b)\neq v} \right.} N(s(b)) + \sum_{[a_1a_2]\in Q' \left| \substack{t(a_1)=u \\ s(a_1)=v=t(a_2)} \right.} N(s(a_2)) \end{aligned} \tag{6.2.9}$$

$$\begin{aligned} N'^T(u)_R &= \sum_{b\in Q' \mid s(b)=u} \tilde{N}(t(b)) \\ &= \sum_{b\in Q' \left| \substack{s(b)=u \\ t(b)=v} \right.} \tilde{N}(v) + \sum_{b\in Q' \left| \substack{s(b)=u \\ t(b)\notin W} \right.} N(t(b)) + \sum_{b\in Q' \left| \substack{s(b)=u \\ t(b)\in W} \right.} N(s(b)) \end{aligned} \tag{6.2.10}$$

Calculations for equality are similar to those in Case 3. From the last summations in (6.2.9) and (6.2.10), $t(b) \in W$ and $s(a_2) \in W$. If $t(b) = w_k$ and $s(a_2) = w_k$ for some $w_k \in W$, then we have a 2-cycle in Q'. Reduction of Q' eliminates these trivial elements to give a quiver $\tilde{Q}$. Let $w_k \in W$, we define χ_{w_k} as

$$\chi_{w_k} = \sum_{[a_1 a_2] \in Q' \big| \substack{t(a_1)=u \\ s(a_1)=v=t(a_2)}} N(w_k) - \sum_{b \in Q' \big| \substack{s(b)=u \\ t(b)=w_k}} N(w_k)$$

In the reduced quiver, either $\chi_{w_k} \geq 0$ or $\chi_{w_k} \leq 0$ for any $w_k \in W$, but not both. We now state $\tilde{N}^T(u)_L$ and $\tilde{N}^T(u)_R$ for the reduced quiver as;

$$\begin{aligned}
\tilde{N}^T(u)_L &= \sum_{b \in \tilde{Q} \big| \substack{t(b)=u \\ s(b) \neq v}} N(s(b)) + \chi_{w_k}(for\ \chi_{w_k} \geq 0) \\
\tilde{N}^T(u)_R &= \sum_{b \in \tilde{Q} \big| \substack{s(b)=u \\ t(b)=v}} \tilde{N}(v) + \sum_{b \in \tilde{Q} \big| \substack{s(b)=u \\ t(b) \notin W}} N(t(b)) - \chi_{w_l}(for\ \chi_{w_l} \leq 0)
\end{aligned}$$

where $k \neq l$. The elements $\chi_{w_k}(for\ \chi_{w_k} \geq 0)$ and $\chi_{w_l}(for\ \chi_{w_l} \leq 0)$ reduces $N'^T(u)_L$ and $N'^T(u)_R$ by an equal amount. Equality is hence maintained after reduction ie $\tilde{N}^T(u) = \tilde{N}^T(u)_L = \tilde{N}^T(u)_R$.

In all the four cases, $\tilde{N}^T(v) = \tilde{N}^T(v)_L = \tilde{N}^T(v)_R$ for any vertex v of the quiver after mutation. This implies that for any vertex v in the mutated quiver $\tilde{Q}$,

$$\tilde{N}^T(v) = \sum_{b \in \tilde{Q} | t(b)=v} \tilde{N}(s(b)) = \sum_{b \in \tilde{Q} | s(b)=v} \tilde{N}(t(b))$$

Remark 6.2. If a quiver satisfies the first polar condition, its mutants will always satisfy the first polar condition.

To prove the theorem for the second polar condition, we consider four different cases.

Case 1. Consider the mutated vertex v. Let $(Q, \mathcal{S}, N, \theta)$ be a polar quiver, for any $v \in Q_0$ totals

$$\theta^L(v) := \sum_{a \in Q \,|\, t(a)=v} \theta(a) N(s(a)) \tag{6.2.11}$$

$$\theta^R(v) := \sum_{a \in Q \,|\, s(a)=v} \theta(a) N(t(a)) \tag{6.2.12}$$

are defined such that their sum,

$$\theta^L(v) + \theta^R(v) = 2\pi(N^T(v) - N(v))$$

Mutation of the quiver Q at v alters these angles. In the mutated quiver $\tilde{Q}$, these angles are be given by;

$$
\begin{aligned}
\tilde{\theta}^L(v) &= \sum_{b\in\tilde{Q}\,|\,t(b)=v} \tilde{\theta}(b)N(s(b)) \\
&= \sum_{b\in\tilde{Q}\,|\,t(b)=v} (\tilde{\theta}(v) - \theta(s(b)))_{mod\ 2\pi} N(s(b)) \\
&= \sum_{b\in\tilde{Q}\,|\,t(b)=v} \left(\theta(v) + \frac{\theta^R(v)}{\tilde{N}(v)} - \theta(s(b))\right)_{mod\ 2\pi} N(s(b)) \text{ .... by (5.2.2)} \\
&= \sum_{b\in\tilde{Q}\,|\,t(b)=v} (\theta(v) - \theta(s(b)))_{mod\ 2\pi} N(s(b)) + \sum_{b\in\tilde{Q}\,|\,t(b)=v} \frac{\theta^R(v)}{\tilde{N}(v)} N(s(b)) \\
&= -\sum_{b\in\tilde{Q}\,|\,t(b)=v} (\theta(s(b)) - \theta(v))_{mod\ 2\pi} N(s(b)) + \frac{\theta^R(v)}{\tilde{N}(v)} \sum_{b\in\tilde{Q}\,|\,t(b)=v} N(s(b)) \\
&= -\sum_{a\in Q\,|\,s(a)=v} (\theta(t(a)) - \theta(v))_{mod\ 2\pi} N(t(a)) + \frac{\theta^R(v)}{\tilde{N}(v)} \tilde{N}^T(v) \\
&= -\sum_{a\in Q\,|\,s(a)=v} \theta(a)N(t(a)) + \frac{\theta^R(v)}{\tilde{N}(v)} \tilde{N}^T(v) \text{ .... and by (6.2.12)} \\
&= -\theta^R(v) + \frac{\theta^R(v)}{\tilde{N}(v)} \tilde{N}^T(v)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\theta}^R(v) &= \sum_{b\in\tilde{Q}\,|\,s(b)=v} \tilde{\theta}(b)N(t(b)) \\
&= \sum_{b\in\tilde{Q}\,|\,s(b)=v} (\theta(t(b)) - \tilde{\theta}(v))_{mod\, 2\pi} N(t(b)) \\
&= \sum_{b\,|\,s(b)=v} \left(\theta(t(b)) - \theta(v) + \frac{\theta^L(v)}{\tilde{N}(v)}\right)_{mod\, 2\pi} N(t(b)) \text{ .... by (5.2.2)} \\
&= \sum_{b\in\tilde{Q}\,|\,s(b)=v} (\theta(t(b)) - \theta(v))_{mod\, 2\pi} N(t(b)) + \sum_{b\in\tilde{Q}\,|\,s(b)=v} \frac{\theta^L(v)}{\tilde{N}(v)} N(t(b)) \\
&= -\sum_{b\in\tilde{Q}\,|\,s(b)=v} (\theta(v) - \theta(t(b)))N(t(b)) + \frac{\theta^L(v)}{\tilde{N}(v)} \sum_{b\in\tilde{Q}\,|\,s(b)=v} N(t(b)) \\
&= -\sum_{a\in Q\,|\,t(a)=v} (\theta(v) - \theta(s(a)))N(s(a)) + \frac{\theta^L(v)}{\tilde{N}(v)} \tilde{N}^T(v) \\
&= -\sum_{a\in Q\,|\,t(a)=v} \theta(a)N(s(a)) + \frac{\theta^L(v)}{\tilde{N}(v)} \tilde{N}^T(v) \text{ .... and by (6.2.11)} \\
&= -\theta^L(v) + \frac{\theta^L(v)}{\tilde{N}(v)} \tilde{N}^T(v)
\end{aligned}
$$

Adding the two angles we have,

$$
\begin{aligned}
\tilde{\theta}^L(v) + \tilde{\theta}^R(v) &= -\theta^R(v) + \frac{\theta^R(v)}{\tilde{N}(v)}\tilde{N}^T(v) - \theta^L(v) + \frac{\theta^L(v)}{\tilde{N}(v)}\tilde{N}^T(v) \\
&= \left(\theta^L(v) + \theta^R(v)\right) \frac{\tilde{N}^T(v)}{\tilde{N}(v)} - \left(\theta^L(v) + \theta^R(v)\right) \\
&= 2\pi\tilde{N}(v)\frac{\tilde{N}^T(v)}{\tilde{N}(v)} - 2\pi\tilde{N}(v) \text{ .... by (4.2.2) and (5.2.1)} \\
&= 2\pi\left(\tilde{N}^T(v) - \tilde{N}(v)\right)
\end{aligned}
$$

Case 2. Consider u, a vertex in Q with no arrows joining it to v, mutation at v will have no affect on the angles at u as shown below,

$$\begin{aligned}\tilde{\theta}^L(u) &= \sum_{b\in\tilde{Q}\,|\,t(b)=u} \theta(b)N(s(b)) \\ &= \sum_{a\in Q\,|\,t(a)=u} \theta(a)N(s(a)) \\ &= \theta^L(u)\end{aligned}$$

$$\begin{aligned}\tilde{\theta}^R(u) &= \sum_{b\in\tilde{Q}\,|\,s(b)=u} \theta(b)N(t(b)) \\ &= \sum_{a\in Q\,|\,s(a)=u} \theta(a)N(t(a)) \\ &= \theta^R(u)\end{aligned}$$

adding the two angles,

$$\begin{aligned}\tilde{\theta}^L(u)+\tilde{\theta}^R(u) &= \theta^L(u)+\theta^R(u) \\ &= 2\pi\left(N^T(u)-N(u)\right)\end{aligned}$$

Since there are no arrows joining u and v, from Case 2 of the proof in the first condition, $N^T(u)=\tilde{N}^T(u)$. By the definition of mutation of a polar quiver at a vertex v, $\tilde{N}(u)=N(u)$ for any vertex $u\neq v$. Hence we can write the sum of the angles as;

$$\tilde{\theta}^L(u) + \tilde{\theta}^R(u) = 2\pi\left(\tilde{N}^T(u) - \tilde{N}(u)\right)$$

Case 3. Consider u, a vertex in Q with arrows $a|s(a) = u,\ t(a) = v$. Angles at u are given by,

$$\begin{aligned}
\theta^L(u) &= \sum_{a\in Q|t(a)=u} \theta(a)N(s(a)) \\
\theta^R(u) &= \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)=v}\right.} \theta(a)N(v) + \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)\neq v}\right.} \theta(a)N(t(a))
\end{aligned}$$

where, by definition,

$$\begin{aligned}
\theta^L(u) + \theta^R(u) &= 2\pi(N^T(u) - N(u)) \\
&= 2\pi\tilde{N}(u)
\end{aligned}$$

Arrows target to v from u are reversed in the unreduced mutated quiver Q', while any path $a_2a_1 \in Q$ with $s(a_1) = u,\ t(a_1) = v = s(a_2)$ gives an arrow $[a_2a_1] \in Q'$. Define a non-empty subset $W \subset Q'_0$ as the set of vertices on which arrows $[a_2a_1]$ are target. Elements of this set are vertices $w_i,\ i \in \mathbb{N}$, hence $t(a_2) = w_i$ for some i. We calculate the angles at u for the reduced quiver $\tilde{Q}$, where the big brackets account for the reduction, $i \neq j$. In the big bracket, let $s(b) = w_j$ and $t(a_2) = w_j$ for some j.

$$\tilde{\theta}^L(u) = \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)=v}\right.} \tilde{\theta}(b)\tilde{N}(v) + \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)\notin W}\right.} \theta(b)N(s(b)) +$$

$$\left(\sum_{b\in Q'\left|\substack{t(b)=u\\ s(b)=w_j}\right.} \theta(b)N(w_j) - \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} \theta(b)N(w_j) \right)$$

In the big bracket below, let $s(b) = w_i$ and $t(a_2) = w_i$ for some i

$$\tilde{\theta}^R(u) = \left(\sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} \theta([a_2a_1])N(w_i) - \sum_{b\in Q'\left|\substack{t(b)=u\\ s(b)=w_i}\right.} \theta([a_2a_1])N(w_i) \right) +$$

$$\sum_{b\in\tilde{Q}\left|\substack{s(b)=u\\ t(b)\neq v}\right.} \theta(b)N(t(b))$$

Let

$$B_1 = \left(\sum_{b\in Q'\left|\substack{t(b)=u\\ s(b)=w_j}\right.} \theta(b)N(s(w_j)) - \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} \theta(b)N(w_j) \right)$$

and

$$B_2 = \left(\sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} \theta([a_2a_1])N(w_i) - \sum_{b\in Q'\left|\substack{t(b)=u\\ s(b)=w_i}\right.} \theta([a_2a_1])N(w_i) \right)$$

B_1 consists of arrows from w_j to u that were left after reduction of the quiver Q' while B_2 consists of the composite arrows $[a_2a_1]$, corresponding to the paths $a_2a_1 \in Q$ through v, that were left after reduction of Q'. Reduction of the quiver quotients out any two equivalent paths between two vertices but opposite in direction.

We intend to calculate for $\tilde{\theta}^L(u) + \tilde{\theta}^R(u)$. To do this, we first evaluate B_1, B_2 and find their sum. The result of this calculation will help in the calculation for $\tilde{\theta}^L(u) + \tilde{\theta}^R(u)$.

$$\begin{aligned} B_1 &= \sum_{b \in Q' \left| \substack{t(b)=u \\ s(b)=w_j} \right.} \theta(b) N(w_j) - \sum_{[a_2a_1] \in Q' \left| \substack{s(a_1)=u \\ t(a_1)=v=s(a_2)} \right.} \theta(b) N(w_j) \\ &= \sum_{b \in \tilde{Q} \left| \substack{t(b)=u \\ s(b)=w_j} \right.} \theta(b) N(w_j) - \sum_{[a_2a_1] \in \tilde{Q} \left| \substack{s(a_1)=u \\ t(a_1)=v=s(a_2)} \right.} \left(2\pi - \theta([a_2a_1])\right) N(w_j) \end{aligned}$$

$\theta(b)_{mod\ 2\pi} = (2\pi - \theta([a_2a_1]))_{mod\ 2\pi}$ since b and $[a_2a_1]$ are two arrows between w_j and u but in opposite directions. We drop the *mod* 2π since $(2\pi - \theta([a_2a_1])) > 0$.

$$B_1 = \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)=w_j}\right.} \theta(a)N(w_j) - 2\pi \sum_{a_2a_1\in Q\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} N(w_j) + \sum_{a_2a_1\in Q\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} (\theta(a_2)+\theta(a_1))N(w_j) \text{ .... by lemma (4.5)}$$

$$= \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)=w_j}\right.} \theta(a)N(w_j) - 2\pi \sum_{a_2a_1\in Q\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} N(w_j) + \sum_{a_1\in Q\left|\substack{s(a_1)=u\\ t(a_1)=v}\right.} \sum_{a_2|s(a_2)=v} \theta(a_1)N(w_j) + \sum_{a_1\in Q\left|\substack{s(a_1)=u\\ t(a_1)=v}\right.} \sum_{a_2|s(a_2)=v} \theta(a_2)N(w_j)$$

We split the sum to a double sum. This enables us to sum over individual arrows in the path $a_2a_1 \in Q$.

$$B_2 = \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} \theta([a_2a_1])N(w_i) - \sum_{b\in Q'\left|\substack{t(b)=u\\ s(b)=w_i}\right.} \theta([a_2a_1])N(w_i)$$

$$= \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} (\theta(a_2)+\theta(a_1))\,N(w_i) - \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)=w_i}\right.} (2\pi-\theta(a))N(w_i)$$

Since b and $[a_2a_1]$ are two arrows between w_j and u but in opposite directions, we have $\theta([a_2a_1]) = (2\pi - \theta(a))_{mod\,2\pi}$. We drop the *mod* 2π since $(2\pi - \theta(a)) > 0$. We also split the first summand to a double sum, so that we can sum over individual arrows in the path $a_2a_1 \in Q$.

$$B_2 = \sum_{a_1 \in Q \left| \begin{smallmatrix} s(a_1)=u \\ t(a_1)=v \end{smallmatrix} \right.} \sum_{a_2 | s(a_2)=v} \theta(a_1)N(w_i) + \sum_{a_1 \in Q \left| \begin{smallmatrix} s(a_1)=u \\ t(a_1)=v \end{smallmatrix} \right.} \sum_{a_2 | s(a_2)=v} \theta(a_2)N(w_i) -$$

$$2\pi \sum_{a \in Q | \begin{smallmatrix} t(a)=u \\ s(a)=w_i \end{smallmatrix}} N(w_i) + \sum_{a \in Q | \begin{smallmatrix} t(a)=u \\ s(a)=w_i \end{smallmatrix}} \theta(a)N(w_i)$$

We now evaluate the sum of B_1 and B_2.

$$\begin{aligned}
B_1 + B_2 = & \sum_{a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \sum_{a_2 | s(a_2)=v} \theta(a_1) N(w_j) + \sum_{a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \sum_{a_2 | s(a_2)=v} \theta(a_1) N(w_i) + \\
& \sum_{a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \sum_{a_2 | s(a_2)=v} \theta(a_2) N(w_j) + \sum_{a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \sum_{a_2 | s(a_2)=v} \theta(a_2) N(w_i) + \\
& \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)=w_j} \right.} \theta(a) N(w_j) + \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)=w_i} \right.} \theta(a) N(w_i) - \\
& 2\pi \left(\sum_{a_2 a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v=s(a_2)} \right.} N(w_j) + \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)=w_i} \right.} N(w_i) \right) \\
= & \sum_{a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \sum_{a_2 | s(a_2)=v} \theta(a_1) N(t(a_2)) + \sum_{a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \sum_{a_2 | s(a_2)=v} \theta(a_2) N(t(a_2)) + \\
& \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)\in W} \right.} \theta(a) N(s(a)) + 2\pi \left(\sum_{a_2 a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v=s(a_2)} \right.} N(w_j) + \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)=w_i} \right.} N(w_i) \right) \\
= & \sum_{a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \theta(a_1) N^T(v) + \sum_{a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v} \right.} \theta^R(v) + \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)\in W} \right.} \theta(a) N(s(a)) - \\
& 2\pi \left(\sum_{a_2 a_1 \in Q \left| \substack{s(a_1)=u \\ t(a_1)=v=s(a_2)} \right.} N(w_j) + \sum_{a \in Q \left| \substack{t(a)=u \\ s(a)=w_i} \right.} N(w_i) \right)
\end{aligned} \tag{6.2.13}$$

We now get back to the calculations for $\tilde{\theta}^L(u)$ and $\tilde{\theta}^R(u)$.

$$\begin{aligned}
\tilde{\theta}^L(u) &= \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)=v}\right.} \tilde{\theta}(b)\tilde{N}(v) + \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)\notin W}\right.} \theta(b)N(s(b)) + B_1 \\
&= \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)=v}\right.} \left(\theta(u) - \tilde{\theta}(v)\right)_{mod\ 2\pi} \tilde{N}(v) + \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)\notin W}\right.} \theta(b)N(s(b)) + B_1 \\
&= \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)=v}\right.} \left(\theta(u) - \theta(v) + \frac{\theta^L(v)}{\tilde{N}(v)}\right)_{mod\ 2\pi} \tilde{N}(v) + \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)\notin W}\right.} \theta(b)N(s(b)) + B_1 \\
&= \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)=v}\right.} \left(\frac{\theta^L(v)}{\tilde{N}(v)} - (\theta(v) - \theta(u))\right)_{mod\ 2\pi} \tilde{N}(v) + \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)\notin W}\right.} \theta(a)N(s(a)) + B_1 \\
&= \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)=v}\right.} \left(\frac{\theta^L(v)}{\tilde{N}(v)} - \theta(a)\right)_{mod\ 2\pi} \tilde{N}(v) + \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)\notin W}\right.} \theta(a)N(s(a)) + B_1
\end{aligned}$$

By the assumption of the theorem, we have $\frac{\theta^L(v)}{\tilde{N}(v)} > \theta(a) \quad \forall a|s(a) =$

$u,\ t(a) = v$. This enables us to open the bracket as,

$$\begin{aligned}
\tilde{\theta}^L(u) &= \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \left(\frac{\theta^L(v)}{\tilde{N}(v)} - \theta(a) \right) \tilde{N}(v) + \sum_{a\in Q \left| \substack{t(a)=u \\ s(a)\notin W} \right.} \theta(a) N(s(a)) + B_1 \\
&= \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \theta^L(v) - \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \theta(a)\tilde{N}(v) + \sum_{a\in Q \left| \substack{t(a)=u \\ s(a)\notin W} \right.} \theta(a) N(s(a)) + B_1 \\
&= \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \theta^L(v) - \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \theta(a) \left(N^T(v) - N(v) \right) + \sum_{a\in Q \left| \substack{t(a)=u \\ s(a)\notin W} \right.} \theta(a) N(s(a)) + B_1 \\
&= \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \theta^L(v) + \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \theta(a) N(v) - \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)=v} \right.} \theta(a) N^T(v) + \\
&\quad \sum_{a\in Q \left| \substack{t(a)=u \\ s(a)\notin W} \right.} \theta(a) N(s(a)) + B_1
\end{aligned}$$

$$\begin{aligned}
\tilde{\theta}^R(u) &= \sum_{b\in \tilde{Q} \left| \substack{s(b)=u \\ t(b)\neq v} \right.} \theta(b) N(t(b)) + B_2 \\
&= \sum_{a\in Q \left| \substack{s(a)=u \\ t(a)\neq v} \right.} \theta(a) N(t(a)) + B_2
\end{aligned}$$

Adding the two angles gives,

$$
\begin{aligned}
\tilde{\theta}^L(u)+\tilde{\theta}^R(u) &= \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)=v}\right.} \theta(a)N(v) + \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)\neq v}\right.} \theta(a)N(t(a)) + \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)=v}\right.} \theta^L(v) - \\
&\quad \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)=v}\right.} \theta(a)N^T(v) + \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)\notin W}\right.} \theta(a)N(s(a)) + B_1 + B_2 \\
&= \theta^R(u) + \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)=v}\right.} \theta^L(v) - \sum_{a\in Q\left|\substack{s(a)=u\\ t(a)=v}\right.} \theta(a)N^T(v) + \\
&\quad \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)\notin W}\right.} \theta(a)N(s(a)) + B_1 + B_2
\end{aligned}
$$

by Equation 6.2.13,

$$
\begin{aligned}
\tilde{\theta}^L(u)+\tilde{\theta}^R(u) &= \theta^R(u)-\sum_{a\in Q\left|\substack{s(a)=u\\t(a)=v}\right.}\theta(a)N^T(v)+\sum_{a\in Q\left|\substack{s(a)=u\\t(a)=v}\right.}\theta(a)N^T(v)+\\
&\sum_{a\in Q\left|\substack{t(a)=u\\s(a)\notin W}\right.}\theta(a)N(s(a))+\sum_{a\in Q\left|\substack{t(a)=u\\s(a)\in W}\right.}\theta(a)N(s(a))+\sum_{a\in Q\left|\substack{s(a)=u\\t(a)=v}\right.}\theta^L(v)+\\
&\sum_{a\in Q\left|\substack{s(a)=u\\t(a)=v}\right.}\theta^R(v)-2\pi\left(\sum_{a_2a_1\in Q\left|\substack{s(a_1)=u\\t(a_1)=v=s(a_2)}\right.}N(w_j)+\sum_{a\in Q\left|\substack{t(a)=u\\s(a)=w_i}\right.}N(w_i)\right)\\
&= \theta^R(u)+\sum_{a\in Q|t(a)=u}\theta(a)N(s(a))+\sum_{a\in Q\left|\substack{s(a)=u\\t(a)=v}\right.}\left(\theta^L(v)+\theta^R(v)\right)-\\
&2\pi\left(\sum_{a_2a_1\in Q\left|\substack{s(a_1)=u\\t(a_1)=v=s(a_2)}\right.}N(w_j)+\sum_{a\in Q\left|\substack{t(a)=u\\s(a)=w_i}\right.}N(w_i)\right)\\
&= \theta^R(u)+\theta^L(u)+2\pi\sum_{a\in Q\left|\substack{s(a)=u\\t(a)=v}\right.}\tilde{N}(v)-\\
&2\pi\left(\sum_{a_2a_1\in Q\left|\substack{s(a_1)=u\\t(a_1)=v=s(a_2)}\right.}N(w_j)+\sum_{a\in Q\left|\substack{t(a)=u\\s(a)=w_i}\right.}N(w_i)\right)\\
&= 2\pi\tilde{N}(u)+2\pi\sum_{a\in Q\left|\substack{s(a)=u\\t(a)=v}\right.}\tilde{N}(v)-\\
&2\pi\left(\sum_{a_2a_1\in Q\left|\substack{s(a_1)=u\\t(a_1)=v=s(a_2)}\right.}N(w_j)+\sum_{a\in Q\left|\substack{t(a)=u\\s(a)=w_i}\right.}N(w_i)\right) \qquad (6.2.14)
\end{aligned}
$$

and from equation (6.2.8) which we have rewritten below,

$$\tilde{N}^T(u) = \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)=v}\right.} \tilde{N}(v) + \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)\notin W}\right.} N(s(b)) - \chi_{w_j}(for\ \chi_{w_j} \leq 0)$$

we have

$$\begin{aligned}
\tilde{N}^T(u) &= \sum_{b\in\tilde{Q}\left|\substack{s(b)=u\\ t(b)=v}\right.} \tilde{N}(v) + \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)\notin W}\right.} N(s(b)) - \chi_{w_j}(for\ \chi_{w_j} \leq 0) \\
&= \sum_{b\in\tilde{Q}\left|\substack{s(b)=u\\ t(b)=v}\right.} \tilde{N}(v) + \sum_{b\in\tilde{Q}\left|\substack{t(b)=u\\ s(b)\notin W}\right.} N(s(b)) + \sum_{b\in Q'|\substack{t(b)=u\\ s(b)=w_j}} N(w_j) - \\
&\quad \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} N(w_j) \\
&= \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)=v}\right.} \tilde{N}(v) + \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)\notin W}\right.} N(s(a)) + \sum_{a\in Q|\substack{t(a)=u\\ s(a)=w_j}} N(w_j) - \\
&\quad \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} N(w_j)
\end{aligned}$$

From which we get,

$$\sum_{a\in Q\left|\substack{t(a)=u\\ s(a)=v}\right.} \tilde{N}(v) = \tilde{N}^T(u) - \sum_{a\in Q\left|\substack{t(a)=u\\ s(a)\notin W}\right.} N(s(a)) - \sum_{a\in Q|\substack{t(a)=u\\ s(a)=w_j}} N(w_j) + \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u\\ t(a_1)=v=s(a_2)}\right.} N(w_j)$$

Substituting this result into (6.2.14),

$$\begin{aligned}
\tilde{\theta}^L(u) + \tilde{\theta}^R(u) &= 2\pi\tilde{N}(u) + 2\pi\tilde{N}^T(u) - 2\pi \sum_{a\in Q\left|\substack{t(a)=u \\ s(a)\notin W}\right.} N(s(a)) - \\
&\quad 2\pi \sum_{a\in Q\left|\substack{t(a)=u \\ s(a)=w_j}\right.} N(w_j) - 2\pi \sum_{a\in Q\left|\substack{t(a)=u \\ s(a)=w_i}\right.} N(w_i) + \\
&\quad 2\pi \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u \\ t(a_1)=v=s(a_2)}\right.} N(w_j) - 2\pi \sum_{[a_2a_1]\in Q'\left|\substack{s(a_1)=u \\ t(a_1)=v=s(a_2)}\right.} N(w_j) \\
&= 2\pi\tilde{N}(u) + 2\pi\tilde{N}^T(u) - 2\pi \sum_{a\in Q\left|\substack{t(a)=u \\ s(a)\notin W}\right.} N(s(a)) - 2\pi \sum_{a\in Q\left|\substack{t(a)=u \\ s(a)\in W}\right.} N(s(a)) \\
&= 2\pi\tilde{N}(u) + 2\pi\tilde{N}^T(u) - 2\pi \sum_{a\in Q|t(a)=u} N(s(a)) \\
&= 2\pi\left(\tilde{N}(u) + \tilde{N}^T(u) - N^T(u)\right) \\
&= 2\pi\left(\tilde{N}(u) + \tilde{N}^T(u) - \tilde{N}(u) - N(u)\right) \quad \text{.......... by (5.2.1)} \\
&= 2\pi\left(\tilde{N}^T(u) - N(u)\right)
\end{aligned}$$

From the definition of mutation of a polar quiver at a vertex v, $\tilde{N}(u) = N(u)$ for any vertex $u \neq v$. Our equation becomes,

$$\tilde{\theta}^L(u) + \tilde{\theta}^R(u) = 2\pi\left(\tilde{N}^T(u) - \tilde{N}(u)\right)$$

Case 4. Lastly we consider u, a vertex in Q with arrows $a|t(a) = u,\ s(a) = v$. Calculations for $\tilde{\theta}^L(u)$ and $\tilde{\theta}^R(u)$ in this case are similar to the calculations in case 3 above, only that arrows between u and v are reversed. Addition of these angles gives the same result as in the case above.

In all the four cases, if the conditions of the theorem are satisfied, mutation of a quiver Q will give a quiver $\tilde{Q}$ after reduction with the angles $\tilde{\theta}^L(v)$ and $\tilde{\theta}^R(v)$, at any vertex v of the quiver, such that;

$$\tilde{\theta}^L(v) + \tilde{\theta}^R(v) = 2\pi\left(\tilde{N}^T(v) - \tilde{N}(v)\right)$$

We show that $\tilde{N}^T(v) > \tilde{N}(v)$ for any vertex v in the mutated quadruple $(\tilde{Q}, \tilde{S}, \tilde{N}, \tilde{\theta})$. For any vertex v in the mutated quiver $\tilde{Q}$ we have

$$\tilde{\theta}^L(v) + \tilde{\theta}^R(v) = 2\pi\left(\tilde{N}^T(v) - \tilde{N}(v).\right)$$

$\tilde{\theta}^L(v) > 0$ and $\tilde{\theta}^R(v) > 0$ implying that $2\pi\left(\tilde{N}^T(v) - \tilde{N}(v).\right) > 0$. This can only be true if $\tilde{N}^T(v) > \tilde{N}(v)$.

We now prove the theorem for the third and last condition. Let Q be a polar quiver, then for any cyclic element $a_1 \dots a_n \in \mathcal{S}$, $n > 2$ and $\sum_i \theta(a_i) = 2\pi$. Let a_i and a_{i+1} be arrows in Q such that $t(a_{i+1}) = v = s(a_i)$. We will have a new potential $\mathcal{S}'$ for the unreduced quiver Q' resulting from the mutation of Q at v given by;

$$\mathcal{S}' = [\mathcal{S}] + \Delta_v$$

where $[\mathcal{S}]$ is the potential for Q with the paths $a_i a_{i+1} \in Q$ through v composed to arrows $[a_i a_{i+1}]$ in Q' and Δ_v comprises of cyclic elements $[a_i a_{i+1}] a_{i+1}^* a_i^*$ resulting from the composite arrows $[a_i a_{i+1}]$. For any cyclic element $a_1 \dots [a_i a_{i+1}] \dots a_n \in [\mathcal{S}]$, $n > 2$ and $\sum_i \theta(a_i) = 2\pi$. Since $\theta([a_i a_{i+1}]) = \theta(a_i) + \theta(a_{i+1})$. We check for cyclic elements $[a_i a_{i+1}] a_{i+1}^* a_i^*$

in Δ_v,

$$\begin{aligned}\theta([a_i a_{i+1}]) &= \theta(a_i) + \theta(a_{i+1}) \\ &= (\theta(t(a_i)) - \theta(v))_{mod\,2\pi} + (\theta(v) - \theta(s(a_{i+1})))_{mod\,2\pi}\end{aligned}$$

$$\begin{aligned}\theta(a_{i+1}^* a_i^*) &= \theta(a_i^*) + \theta(a_{i+1}^*) \\ &= (\tilde{\theta}(v) - \theta(s(a_i^*)))_{mod\,2\pi} + (\theta(t(a_{i+1}^*)) - \tilde{\theta}(v))_{mod\,2\pi} \\ &= \left(\theta(v) - \theta(s(a_i^*)) + \frac{\theta^R(v)}{\tilde{N}(v)}\right)_{mod\,2\pi} + \left(\theta(t(a_{i+1}^*)) - \theta(v) + \frac{\theta^L(v)}{\tilde{N}(v)}\right)_{mod\,2\pi} \\ &= \left(\frac{\theta^R(v)}{\tilde{N}(v)} - (\theta(s(a_i^*)) - \theta(v))\right)_{mod\,2\pi} + \left(\frac{\theta^L(v)}{\tilde{N}(v)} - (\theta(v) - \theta(t(a_{i+1}^*)))\right)_{mod\,2\pi} \\ &= \left(\frac{\theta^R(v)}{\tilde{N}(v)} - (\theta(t(a_i)) - \theta(v))\right)_{mod\,2\pi} + \left(\frac{\theta^L(v)}{\tilde{N}(v)} - (\theta(v) - \theta(s(a_{i+1})))\right)_{mod\,2\pi} \\ &= \left(\frac{\theta^R(v)}{\tilde{N}(v)} - \theta(a_i)\right)_{mod\,2\pi} + \left(\frac{\theta^L(v)}{\tilde{N}(v)} - \theta(a_{i+1})\right)_{mod\,2\pi} \qquad (6.2.15)\end{aligned}$$

By the assumption of the theorem, $\frac{\theta^L(v)}{\tilde{N}(v)} > \theta(a_{i+1})$ and $\frac{\theta^R(v)}{\tilde{N}(v)} > \theta(a_i)$ which implies each of the brackets in (6.2.15) is positive, and

$$\begin{aligned}\theta([a_i a_{i+1}] a_{i+1}^* a_i^*) &= \theta(a_i) + \theta(a_{i+1}) + \left(\frac{\theta^R(v)}{\tilde{N}(v)} - \theta(a_i)\right) + \left(\frac{\theta^L(v)}{\tilde{N}(v)} - \theta(a_{i+1})\right) \\ &= \frac{\theta^R(v)}{\tilde{N}(v)} + \frac{\theta^L(v)}{\tilde{N}(v)} \\ &= 2\pi\end{aligned}$$

S' is the potential for the unreduced quiver Q'. There might be arrows in cyclic elements of $[S]$ which are in the opposite direction to the new composite arrows $[a_i a_{i+1}]$ hence giving rise to 2-cycles. Reduction of Q' eliminates 2-cycles while replacing these arrows with equivalent paths (paths with same starting and target vertices as the arrows). By lemma (4.5) the angles of the arrows and their equivalent paths are equal. Reduction of Q' does not alter the angles of cyclic elements of S'. For each cyclic element $a_1 \dots a_n \in \tilde{S}$, the potential for the reduced quiver $\tilde{Q}$, $n > 2$ and $\sum_i \theta(a_i) = 2\pi$.

Suppose $\frac{\theta^L(v)}{\tilde{N}(v)} < \theta(a_{i+1})$, then

$$\begin{aligned} \theta(a^*_{i+1}a^*_i) &= \left(\frac{\theta^L(v)}{\tilde{N}(v)} - \theta(a_{i+1})\right)_{mod\ 2\pi} + \left(\frac{\theta^R(v)}{\tilde{N}(v)} - \theta(a_i)\right)_{mod\ 2\pi} \\ &= \left(2\pi + \frac{\theta^L(v)}{\tilde{N}(v)} - \theta(a_{i+1})\right) + \left(\frac{\theta^R(v)}{\tilde{N}(v)} - \theta(a_i)\right) \end{aligned}$$

In which case

$$\begin{aligned} \theta([a_i a_{i+1}]a^*_{i+1}a^*_i) &= \theta(a_i) + \theta(a_{i+1}) + \left(2\pi + \frac{\theta^L(v)}{\tilde{N}(v)} - \theta(a_{i+1})\right) + \left(\frac{\theta^R(v)}{\tilde{N}(v)} - \theta(a_i)\right) \\ &= 2\pi + \frac{\theta^L(v)}{\tilde{N}(v)} + \frac{\theta^R(v)}{\tilde{N}(v)} \\ &= 4\pi \end{aligned}$$

This contradicts the third condition for polar quivers, and hence completes the proof.

□

The following lemmas are an immediate consequence of the theorem.

Lemma 6.3. *Let $(Q, \mathcal{S}, N, \theta)$ be a polar quiver. If for a vertex v, $\tilde{N}(v) = 1$ then the mutated quadruple $(\tilde{Q}, \tilde{\mathcal{S}}, \tilde{N}, \tilde{\theta})$ is a polar quiver.*

Proof. Let $\tilde{N}(v) = 1$, the sums $\theta^L(v)$ and $\theta^L(v)$ are strictly greater than each of their components. By theorem 6.1, the mutated quadruple $(\tilde{Q}, \tilde{\mathcal{S}}, \tilde{N}, \tilde{\theta})$ is a polar quiver. □

Lemma 6.4. *Let $(Q, \mathcal{S}, N, \theta)$ be a polar quiver. If, for a vertex v, there exists θ and θ' such that $\theta(a) = \theta$ for all $a \,|\, t(a) = v$ and $\theta(b) = \theta'$ for all $b \,|\, s(b) = v$ then the mutated quadruple $(\tilde{Q}, \tilde{\mathcal{S}}, \tilde{N}, \tilde{\theta})$ is a polar quiver.*

Proof. If $\theta(a)$ is identical for all $a \,|\, t(a) = v$ then

$$\theta^L(v) = \theta(a) \sum_{a \,|\, t(a)=v} N(s(a)) = \theta(a) N^T(v)$$

.

$$\frac{\theta^L(v)}{\tilde{N}(v)} = \theta(a) \frac{N^T(v)}{N^T(v) - N(v)} > \theta(a).$$

Similarly for identical $\theta(b)$ for all $b \,|\, s(b) = v$,

$$\theta^R(v) = \theta(b) \sum_{b \,|\, s(b)=v} N(t(v)) = \theta(b) N^T(v)$$

.

$$\frac{\theta^R(v)}{\tilde{N}(v)} = \theta(b) \frac{N^T(v)}{N^T(v) - N(v)} > \theta(b)$$

. By theorem 6.1, the mutated quadruple $(\tilde{Q}, \tilde{\mathcal{S}}, \tilde{N}, \tilde{\theta})$ is a polar quiver.

□

6.3 Examples highlighting the importance of the main theorem

In this section, we revisit Examples 5.3 and 5.4 from chapter 5. We show that the quiver whose mutants break some of the polar conditions fails to satisfy the theorem. We also show that the quiver whose mutants satisfy the polar conditions fulfills the requirements of the theorem.

Example 6.5. In this example, we have a polar quiver which fails to satisfy the conditions of theorem 6.1. This is the quiver in Example 5.3 shown below.

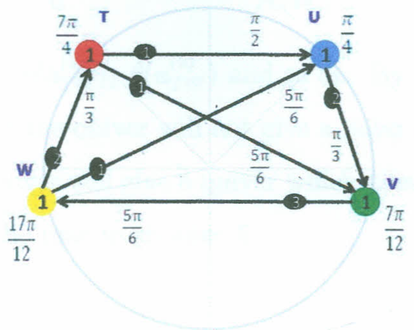


Figure 6.3.1: Diagram of P^2 quiver blown up at one point. It is a polar quiver, but not any of its mutants.

In Example 5.3, the mutant of this quiver at W failed to satisfy the second and third polar conditions. We show that this quiver fails to satisfy the conditions of the theorem.

From the polar quiver above, we have;

$$\frac{\theta^L(T)}{\tilde{N}(T)} = \frac{2\pi}{3} > \frac{\pi}{3} = \theta(a_{TW}^{(1,2)}) \qquad \frac{\theta^R(T)}{\tilde{N}(T)} = \frac{4\pi}{3} \begin{cases} > \frac{\pi}{2} = \theta(a_{UT}), \\ > \frac{5\pi}{6} = \theta(a_{VT}) \end{cases}$$

$$\frac{\theta^L(U)}{\tilde{N}(U)} = \frac{4\pi}{3} \begin{cases} > \frac{\pi}{2} = \theta(a_{UT}), \\ < \frac{5\pi}{6} = \theta(a_{VT}) \end{cases} \qquad \frac{\theta^R(U)}{\tilde{N}(U)} = \frac{2\pi}{3} > \frac{\pi}{3} = \theta(a_{VU}^{(1,2)})$$

$$\frac{\theta^L(V)}{\tilde{N}(V)} = \frac{3\pi}{4} \begin{cases} > \frac{\pi}{3} = \theta(a_{VU}^{(1,2)}), \\ < \frac{5\pi}{6} = \theta(a_{VT}) \end{cases} \qquad \frac{\theta^R(V)}{\tilde{N}(V)} = \frac{5\pi}{4} > \frac{5\pi}{6} = \theta(a_{WV}^{(1,2,3)}).$$

$$\frac{\theta^L(W)}{\tilde{N}(W)} = \frac{5\pi}{4} > \frac{5\pi}{6} = \theta(a_{WV}^{(1,2,3)}) \qquad \frac{\theta^R(W)}{\tilde{N}(W)} = \frac{3\pi}{4} \begin{cases} > \frac{\pi}{3} = \theta(a_{TW}^{(1,2)}), \\ < \frac{5\pi}{6} = \theta(a_{UW}) \end{cases}$$

where $\theta(a_{TW}^{(1,2)}) = \theta(a_{TW}^{(1)}),\ \theta(a_{TW}^{(2)})$ and so on. By Theorem 6.1, mutation at any vertex of the quiver will not give a polar quiver. Mutation at any vertex of this quiver will give a quiver which doesn't satisfy the polar properties, as was the case in chapter 5.

Example 6.6. Here we revisit the quiver in Example 5.4 shown below.

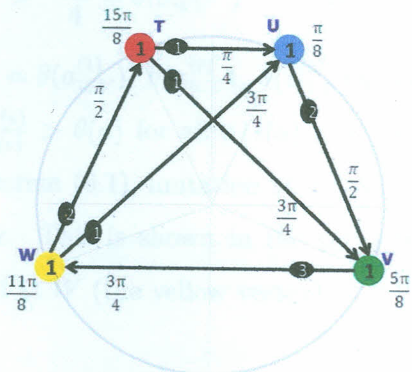


Figure 6.3.2: P^2 quiver blown up at one point. This is a polar quiver, so are its mutants.

As shown in example 5.4, mutation of this quiver gives a polar quiver. We show that this quiver satisfy the conditions of theorem 6.1. From the polar quiver above, we have;

$$\frac{\theta^L(T)}{\tilde{N}(T)} = \pi \; > \; \frac{\pi}{2} = \theta(a_{TW}^{(1,2)}) \qquad \frac{\theta^R(T)}{\tilde{N}(T)} = \pi > \begin{cases} \frac{\pi}{4} & = & \theta(a_{UT}), \\ \frac{3\pi}{4} & = & \theta(a_{VT}) \end{cases}$$

$$\frac{\theta^L(U)}{\tilde{N}(U)} = \pi > \begin{cases} \frac{\pi}{4} & = & \theta(a_{UT}), \\ \frac{3\pi}{4} & = & \theta(a_{VT}) \end{cases} \quad \frac{\theta^R(U)}{\tilde{N}(U)} = \pi \; > \; \frac{\pi}{2} = \theta(a_{VU}^{(1,2)})$$

$$\frac{\theta^L(V)}{\tilde{N}(V)} = \frac{7\pi}{8} > \begin{cases} \frac{\pi}{2} & = & \theta(a_{VU}^{(1,2)}), \\ \frac{3\pi}{4} & = & \theta(a_{VT}) \end{cases} \quad \frac{\theta^R(V)}{\tilde{N}(V)} = \frac{9\pi}{8} \; > \; \frac{3\pi}{4} = \theta(a_{WV}^{(1,2,3)})$$

$$\frac{\theta^L(W)}{\tilde{N}(W)} = \frac{9\pi}{8} > \frac{3\pi}{4} = \theta(a_{WV}^{(1,2,3)}) \qquad \frac{\theta^R(W)}{\tilde{N}(W)} = \frac{7\pi}{8} > \begin{cases} \frac{\pi}{2} & = \theta(a_{TW}^{(1,2)}), \\ \frac{3\pi}{4} & = \theta(a_{UW}) \end{cases}$$

where $\theta(a_{WV}^{(1,2,3)}) = \theta(a_{WV}^{(1)}),\ \theta(a_{WV}^{(2)}),\ \theta(a_{WV}^{(3)})$ and so on. For any vertex v of the quiver, $\frac{\theta^L(v)}{\tilde{N}(v)} > \theta(a)$ for all $a \,|\, t(a) = v$ and $\frac{\theta^R(v)}{\tilde{N}(v)} > \theta(b)$ for all $b \,|\, t(b) = v$. By theorem (6.1), mutation at any vertex of this quiver will give a polar quiver. This is shown in the previous chapter where the quiver was mutated at W (the yellow vertex).

Chapter 7

Summary and recommendations

We introduced the concept of polar quivers and their mutations. Theorem 6.1 gives the conditions under which mutation of a polar quiver gives a polar quiver. Two interesting questions are;

1. Is there a general way of assigning polar co-ordinates to quivers with potential so that they satisfy the polar conditions?
2. Under what conditions is mutation an operation on polar quivers?

The examples we used are taken from the work of Stern [18] and relate to geometry. With these examples, we have some idea of how to answer the two questions using the geometry. It is not clear whether or not we will be able to generalize these ideas.

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